Mathematics Education For The Gifted Elementary School Student*

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Any form of instruction rests, either explicitly or implicitly, on a philosophy or attitude toward the subject on the part of those designing curricula and teaching it. Standard school mathematics instruction evolved at a time when mathematics played a less vital role in society than at present. This article attempts to make as sharp a contrast as possible between two sets of philosophies and attitudes.

I. Standard Instruction
   A. Underlying Philosophical Basis
      1. Mathematics is an essentially static collection of facts, methods, rules, etc.
      2. These facts, methods, rules, etc. were created by geniuses. Mere mortals cannot or need not understand their genesis.
      3. Mathematics instruction is a procedure for transferring a package of information from teacher to student.
      4. Students learn mainly by drill and practice in the skills involved.
      5. The main goal of mathematics instruction is to produce students who can solve problems. In the context of standard instruction this means that they:
         a. classify the problem,
         b. recall the method appropriate for solving it,
         c. correctly apply the method.
   B. Pedagogical Consequences
      1. Mathematics is broken into a succession of small pieces (units or skills).
      2. The units are presented to a student in an order dictated primarily by difficulty and logical sequence. Instruction consists of cycles:
         a. introduction to the skill, fact, or method,
         b. practice or drill,
         c. testing to determine if mastery has occurred.
II. Nonstandard Instruction
   A. Underlying Philosophical Basis
      1. Mathematics is more a subject of ideas than a subject of facts.
      2. One can only understand an idea when one has, in some measure, thought of it oneself.
      3. Ideas cannot be transmitted or transferred from the mind of the teacher to the mind of the student. Understanding must be created. "Ideas must be born in the student's mind and the teacher can act only as midwife" (Socrates).
      4. People learn by being placed in an environment that stimulates, encourages, and supports their thinking.
      5. The main goal of mathematics instruction is to produce students who can solve problems. In the context of nonstandard instruction, this means that they:
         a. analyze the problem,
         b. use their understanding of mathematical ideas to devise one or more plans of attack,
         c. continue to work interactively with the problem until a method is devised to solve it.
   B. Pedagogical Consequences
      1. One of the main roles for the teacher is to structure the learning environment so that it is conducive to the development of the students' thinking ability.
      2. The learner must assume responsibility for learning. The teacher acts in a supporting role.
      3. Students cannot learn to solve problems if they are not exposed to problems to solve.
      4. A teacher must attend to the development of successful attitudes and personality characteristics. A proper mental set is required for successful creative work which, among other things, includes an inquisitive spirit and high tolerance for frustration. Conversely, mental blocks are quite real and can prevent otherwise capable students from dealing successfully with mathematics.
      5. The most successful way to learn mathematics is to participate in its creation. Students must be shown that this is a possibility.
      6. Mathematics is learned through cycles consisting of:
         a. experience,
         b. abstraction of concepts derived from experiences,
         c. symbolic representation of concepts,

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d. internalization of the concepts by experience with their use and symbolic representation,
e. (on higher levels) construction of theories to organize the concepts.

7. During the exploratory phase of a cycle a low-pressure environment is mandatory. Creative processes are not enhanced in a high-pressure environment. In particular, testing during the exploratory phase will create an antipathetic environment.
8. New ideas and concepts are best learned in the context of a student's present ideas and interests. It should be possible for students to learn in this fashion even if it produces unevenness or illogical sequencing of subject matter.
9. A great deal is to be gained by encouraging interaction between students as they generate ideas.

These lists are not intended to be exhaustive but rather to reveal the underlying basis of nonstandard instruction. The contrast between the resulting styles of instruction may perhaps be made clearer by an analogy:

> Consider the task of finding our way from place to place in a town. One way of doing this is to ask someone familiar with the town to provide a list of directions. If we have, let's say, four places to go to and from we will be supplied with 12 sets of directions allowing us to go from any one of these places to any other ("go down two blocks, turn left, etc...."). The more places we want to go the larger the set of directions will be.
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> Now imagine a new person coming to town. Although a newcomer, this person seems to have an uncanny ability to get from place to place. To someone whose conception of getting around town involves memorizing directions this seems to be an almost impossible feat. In fact, though, the newcomer has a map and telephone directory. He approaches the task in an entirely different way. With his approach he will be able to get around not only this town but any other...and without the mental effort required by memorizing directions.

The differences in approach to the task of getting around a town are similar to the differences in the way successful and unsuccessful students approach the learning of mathematics. More than any innate intellectual ability, successful and unsuccessful students can be distinguished by the methods they use to learn. The broadest distinction exists between a successful, *active* approach to learning mathematics and an unsuccessful, *passive* one. Successful students actively involve themselves in the ideas of the subject. They provide experiences for themselves which allow them to encounter the major ideas, they reflect upon their experiences and generate concepts from them. They set themselves tasks and projects, ask themselves questions. Unsuccessful students tend to approach mathematics passively. Their main learning technique is rote memorization. To say that their approach is passive is not to imply a lack of diligence: they will spend hours going again and again through pages in a book or problems on a page trying to memorize what is there.

The differences in approach between successful and unsuccessful students could themselves form the subject of an article. Differences appear in almost every aspect of learning. In reading a mathematical book, for example, successful students go through the material longitudinally; skipping back and forth, gradually penetrating more and more deeply (something like peeling an onion layer by layer). Unsuccessful students read laterally. They will not look at page two until they have mastered page one. Curiosity is a driving force for successful students. They, like professional mathematicians, tend to organize their learning around questions they would like to answer, problems they would like to solve, projects they would like to do. Their activities enable them to develop insight and intuition. They can retain mathematical information more readily because their involvement with the subject provides a kind of lattice or framework into which the information fits.

The approach to mathematics which we have discussed in connection with nonstandard instruction exemplifies some of what we know about the learning style of successful mathematicians. Nonstandard instruction attempts to provide the environment and support needed for the development of a successful approach to the subject. It is our contention that standard instruction teaches students to approach mathematics in an inherently unsuccessful way. Schools tend to overstress the information content of the subject and to emphasize rote learning. They are trying to teach students to find their way around a large and complicated mathematical world with an approach adequate only for getting around a very small town. Recent trends in education (back to basics, accountability, mastery learning, diagnostic/prescriptive programs, etc.) are further entrenching a commitment to a mode of instruction that will not adequately equip students to function in an increasingly mathematical world.

Educating Gifted Students

While it is doubtful that any student should be taught mathematics in the standard way, such a method is completely inappropriate for gifted and talented students. Talented students are capable of learning a great deal of mathematics by (selectively) recreating it. Presumably they have greater potential for developing the ability to be creative and productive in mathematics and allied areas. A style of instruction so clearly designed for consumers rather than producers, which actively stifles curiosity, insight, imagination, and creativity, which provides a misleading image of the nature of the subject and the way to learn it is not destined to produce people who have the kind of ability which we would hope to develop in an instructional program for gifted students. Those who succeed in learning mathematics well most often seem to do it in spite of the way they're taught rather than because of it. Some children develop highly efficient learning strategies at an early age and maintain them in the face of instruction which teaches them to do otherwise. Under the belief that standard instruction offers a method of approach to mathematics which not only embodies an unsuccessful learning strategy but is also directed at the wrong ends, acceleration does not offer any promise as a way of handling gifted students: there simply is no virtue in doing the wrong things fast. An adequate program requires a redesign of the mode of instruction and a resetting of its goals.

The idea that traditional schoolwork in all subjects overemphasizes a content-oriented approach has received attention by some workers in the area of gifted education. It should be noted that standard instruction reduces mathematics to a subject in which only fairly low level cognitive skills are required. Even when standard instruction deals with problem solving, it is in a way which requires the classification of a problem and the application of a standard method (usually learned by rote). Students who are successful with the approach embodied in standard instruction can be expected to be very good at applying routine methods to routine problems. They may also be expected to perform well on standardized tests which, by and large, are designed to be harmonious with this form of instruction. It will be in the area of creative problem solving that they are found to be deficient. They are at a loss to deal with problems that do not easily fit one of the routines they have learned.

Creativity In Mathematics

A growing recognition of the importance of creativity has led to the development of alternate forms of instruction, particularly for gifted students, which attempt to deal more effectively with higher order cognitive processes. Renzulli (1977) has analyzed some of the current programs for gifted students and provides a model for the development of what he calls "defensible programs for the gifted and talented." Although the ideas presented in this paper were developed without awareness of Renzulli's work, some of the suggestions match so nearly the ideas that he advances that his work will be referenced for the underlying justification and critique. The goal was to design a program which focuses more on the development of the intellect than on the content of the subject.

Since mathematics is a fast-growing field it is particularly important to develop a student's ability to think mathematically rather than to increase his or her store of mathematical information. One failing of some of the programs that deal with the development of mental abilities is summarized by Renzulli in a cartoon captioned, "Oh, we're not learning anything in school yet. We're just developing our higher mental processes." The idea that the development of mental processes can somehow be accomplished in isolation from content is faulty. The program advocated here uses the content of mathematics as the raw material or tool for the development of mental ability. The idea, in a nutshell, is to have children learn mathematics by becoming mathematicians. The information available on the working methods of creative mathematicians is used to help create the conditions necessary for this to occur.

Extensive knowledge of all that is involved, emotionally and intellectually, in becoming creative in a field like mathematics is not essential for the development of a suitable instructional program. We can teach students to ride a bicycle without knowing every detail of the whole complex of psychological, mental, and physical abilities which must be developed by providing the support and environment for them to teach themselves. The task of teaching someone to be creative in mathematics is quite analogous. In fact, it is the student who ultimately teaches him/herself. The teacher merely provides the support that seems necessary for this to happen. Also, as with bicycle riding, the aim is to have the students engage in the activity completely independently as soon as possible. Children differ in their readiness for and their interest in engaging in mathematics. Our experiences have indicated that it is fairly difficult to predict, using the data available from testing, class performance, etc., which students will be most responsive. Generally the most successful students are doing well in their schoolwork, but are not necessarily at the very top of their class. There have been some surprises (one student who was in the lowest math group in normal schoolwork showed a great deal of creative potential in branches of mathematics which did not draw upon his inadequate arithmetic background). It does seem that the fourth through sixth grade is the appropriate place to begin expecting students to assume control of their own learning.

The Math Course

The Math Course, offered through the adult education program at the University of California at San Diego, was developed to provide gifted elementary school students with supplementary mathematics instruction. The course meets for 10 weeks, once each week for three hours. Enrollment is limited to 15 students, accompanied by their parents. The original reason for including parents in the course was logistic: it provided an answer to the question of what they were going to do for three hours while their children were taking the course. After the first session, however, it became clear that the inclusion of parents had some important consequences, and that the course needed to be directed as much at them as at their children.

The Math Course is a course in teaching oneself mathematics successfully. The children who attend the course are provided with some of what they need to begin working independently in mathematics, and the parents are taught how they can play a supporting role in their child's intellectual development. Mathematics is a subject commonly misunderstood, and the idea that it can be an arena for creative work is novel for most adults. Parents also need to understand the rationale for the approach taken in the course and why it does not primarily focus on increasing their child's store of mathematical information.

The atmosphere or environment provided in the course is perhaps more important than any of the specific activities used. From the start, an environment is established in which students feel free to explore. Children are told at the beginning that there will be no tests and no grades. Testing and grading not only puts the students under pressure, which interferes with creative learning, but it reinforces a value system (which many of them already bring with them) in which one works for external rewards rather than internal satisfaction. This is not to say, however, that the students' ideas are not evaluated; it is more that they are not judged. In keeping with the ultimate goal of the course students are expected to turn in (at the next to last meeting of the class) a report on a self-selected independent project. The reports are read and commented upon (but not graded) and are presented to the rest of the class at the last meeting.

Subject Matter

Subject matter for the course is selected from areas of mainstream mathematics and their applications which have been found to be accessible to elementary school students. The specific areas used have varied from class to class. Some of the fields represented are:

1. Geometry — straight-edge and compasses constructions, areas and volumes, pi (what it is and how it can be calculated), coordinate geometry, dimensionality (including dimensions higher than 3), equations of lines.
2. Probability and Statistics — collecting, analyzing, and presenting data, sampling experiments, the influence of sample size on accuracy, observation of chance events, comparing observations with the calculation of probabilities.
3. Numbers and Number Theory — numbers and numeration, devising numeration systems and computational algorithms for these systems (the relationship between numeration and computation algorithms), special numbers or classes of numbers (primes, arithmetic and geometric sequences and series, Fibonacci numbers), negative numbers, and decimals.
4. Functions and their Graphs.
5. Topology and Combinatorics.
6. Scientific and Business Applications — the law of the pendulum, curve fitting, finding functional relationships, interest, percentage, discount, inflation, profit, the relationship between selling price and demand, etc.

The subject matter is most often presented in the form of projects, problems, games, or computer programs. In each case activities are selected in which students will encounter some of the main ideas of the subject in question. A variety of formats are used. In one case, for example, students were divided in groups of four. Each group was presented with a collection of problems which involved adjusting "payoffs" for some two-player chance games to make the game "fair." No prior discussion of probability or fairness was conducted, and the object was to have students work from whatever ideas they already had to a concept of probability. Students were told to read over the problems themselves and then begin discussing them with others in their group. The rules were that each group must unanimously decide on an answer to each question, and that they could argue but not use physical violence to achieve unanimity. When all groups indicated that they had completed their work, they reported the results to the class. If any disagreements arose, spokespersons were asked to present justifications for their group's position. If the disagreement still persisted they were asked to devise some means (e.g., an experiment that could be performed) to resolve the disagreement.

Use of Computers

One of the integral parts of the math course is an introduction to computer programming. In a search for good problem-solving activities for elementary school students we found that the act of writing original computer programs provides an excellent opportunity for students to exercise and develop creativity, imagination, and reasoning skill. The computer is also an effective learning tool which can be used in the exploration of many branches of mathematics. To learn computer programming effectively one must have prolonged access to a computer. Programmable calculators were used in the course. A programmable calculator is actually a low-cost, special purpose computer which only deals with numbers, but has many of the features of other computers. In addition to their low cost they have several other advantages. The instructions are on the keys and so can be entered into a program at the press of a button. This means that a lack of typing ability does not inhibit students from making full use of the machines. The programming language used by the calculators, moreover, is similar enough to computer languages that students have found it easy to make the transition if the opportunity arises, yet it has a simple enough grammar and syntax that students don't need to spend a great deal of time on the programming language and can proceed to the design of algorithms. The machines currently in use are HP-33E programmable calculators manufactured by Hewlett-Packard. The logic used in these machines, while originally designed for efficiency for adult users, has proved to have a simplicity that children can grasp.

The Learning Process

As the course progresses students are exposed to a large variety of mathematical topics and ideas. They are also experiencing mathematics as a subject of ideas and are using a variety of learning techniques. Teaching in this way provides a context for students to learn not only about mathematics but, often implicitly, about the process of learning itself.

Students in the course are frequently in a position of explaining their ideas to others and listening to others explain ideas to them. This has the effect of demonstrating the variety of approaches and processes which can be used to solve problems. It is extremely important to direct the attention of students away from "getting the right answer" (which is the focus of standard instruction) to problem-solving processes. At times, there are perfect illustrations of the positive value of mistakes. In one class doing the probability project discussed above, a group gave incorrect answers to several of the problems. When asked to explain their reasoning, however, they provided a clear exposition of the ideas behind probability. Their answer was wrong because of something they overlooked in a counting argument, which was easily fixed. Hopefully they went away from this experience with a basically positive feeling about their intellectual powers and an increased willingness to accept their own mistakes.

An unfortunate characteristic found in many gifted students is a tendency to be very hard and demanding on themselves. On the other hand, an important characteristic of a successful problem-solver is a basically nonjudgmental attitude towards him/herself. One way this is reflected is in a willingness to make mistakes. Gifted students are often accustomed to being right and have a hard time allowing themselves to be wrong. Computer programming is an excellent place for students to practice coping with initial failure. Computer programmers know that "debugging" (i.e., finding out why programs don't work as intended) is a prominent feature of computer programming. As soon as one begins writing programs beyond the most trivial level, the chances of their working correctly the first time go down considerably. Computer programmers also know that they learn more from their failures than from their successes. A willingness to make mistakes and learning how to benefit from them are part of the repertoire of a creative mathematician. Both the thrill of victory and the agony of defeat are essential experiences for someone who plans to succeed in a creative intellectual field. The price paid for always staying on safe ground is not going very far.

Guidelines For Establishing A Program

The following are some guidelines for others who might wish to begin a program similar to the Math Course. The first and most important is that the program should, if possible, be taught by someone actively working in mathematics. Because a teacher also communicates attitudes, and a feeling for or about the subject, it is more probable that a practitioner of an area will be able to convey its spirit than someone whose involvement is second or third hand.

Teaching gifted students in mathematics requires a flexibility and knowledge base that a mathematician is more likely to have. In spite of advance planning, classes in progress often take unexpected turns. To follow these requires, in some cases, the development of curricula on the spot, which in turn draws on a store of knowledge about and experience with the subject. In addition, a professional mathematician may be more able to focus on the process of creating mathematics rather than concentrating exclusively on the information content, and is not as likely to be intimidated by gifted students.

The second guideline is to allow for fairly wide differences in ability and interest. There are so many things that can be done in mathematics and at so many different levels that each student can always find an individual niche. On the other hand, it is virtually impossible to find projects that are exactly appropriate for everyone. The structure of the course should not demand that each student does everything, just that each student does something.

The third guideline involves purpose and flexibility: the instructor, when developing activities, should have a clear idea of the purpose (or purposes) to be served and a rationale for believing that the activity will accomplish the purpose. While teaching, however, the instructor must be sensitive to feedback from the class so that activities can be modified if they do not work as expected.

A good source of inspiration for projects is a study of the history of mathematics. The sorts of questions that the discoverers of a mathematical concept were asking often turn out to be the best sort of questions to raise for students who are encountering the core ideas of the subject. George Polya (1962, 1969) has done extensive work in the area of problem solving and teaching. His work has not had the impact it should have on the teaching of mathematics, perhaps because of the general misunderstanding of the subject. However, to someone who is involved in mathematics and is trying to acquire some skills for working with children, his work offers a great deal of food for thought.

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