Math 100B: Abstract Algebra II (UC San Diego, winter 2018)
Homework 5: due Wednesday, February 14 at 6pm

Reminder: no homework due February 21, as we have a midterm that day.

(1) Artin, chapter 11, exercise 4.3.
(2) Artin, chapter 11, exercise 5.6.
(3) Artin, chapter 11, exercise 6.4.
(4) Artin, chapter 11, exercise 7.1.
(5) Artin, chapter 11, exercise 7.2.
(6) Artin, chapter 11, exercise 8.3.
(7) Artin, chapter 11, exercise M.2.

(8) Let $R$ be an integral domain.
   (a) Prove that a monic polynomial of degree $n$ over $R$ has at most $n$ distinct roots in $R$. (Hint: reduce immediately to the case of a field.)
   (b) Suppose that $R$ is infinite. Let $P \in R[x_1, \ldots, x_n]$ be an element such that $P(\alpha_1, \ldots, \alpha_n) = 0$ for all $\alpha_1, \ldots, \alpha_n \in R$. Prove that $p = 0$. (Hint: induct on $n$.)
   (c) Prove that (b) fails for $R = \mathbb{F}_p$, $n = 1$. (Hint: one option is to use the little Fermat theorem.)

(9) Let $n$ be a positive integer and define the ring $R = \mathbb{Z}[A_{ij} : i, j = 1, \ldots, n]$. Let $A$ be the $n \times n$ matrix over $R$ with $i, j$-entry $A_{ij}$.
   (a) Let $P(x)$ be the characteristic polynomial of $A$. Using the Cayley-Hamilton theorem over $\mathbb{C}$ and the previous problem, prove that $P(A) = 0$.
   (b) Using substitution, prove the Cayley-Hamilton theorem over an arbitrary ring.

(10) Let $R$ be a ring. Let $n$ be a positive integer. Define the elementary symmetric polynomials $e_1, \ldots, e_n \in R[x_1, \ldots, x_n]$ by the formula
        $$(1 + x_1T)\cdots(1 + x_nT) = 1 + e_1T + \cdots + e_nT^n.$$  
        (For example, $e_1 = x_1 + \cdots + x_n$ and $e_n = x_1 \cdots x_n$.) Prove that the homomorphism
        $\psi : R[y_1, \ldots, y_n] \to R[x_1, \ldots, x_n]$ given by the formula
        $$\psi \left( \sum_{i_1, \ldots, i_n} c_{i_1, \ldots, i_n} y_1^{i_1} \cdots y_n^{i_n} \right) = \sum_{i_1, \ldots, i_n} c_{i_1, \ldots, i_n} e_1^{i_1} \cdots e_n^{i_n}$$ 
        is a bijection. (Hint: proceed by induction on $n$, using the substitution map $R[x_1, \ldots, x_n] \to R[x_1, \ldots, x_{n-1}]$ taking $x_n$ to 0. If you get stuck, see Theorem 16.1.6.)