

Algebra Qual Prep: Summer, 2008.

Module Theory Exam

August 21, 2008

1. Prove that every submodule of a finitely generated free module is finitely generated. Now, suppose N is a submodule of M . Prove that if N and M/N are finitely generated then so is M .
2. Show that \mathbb{Q} is not a free \mathbb{Z} -module. (Hint: Show that any two non-trivial submodules of \mathbb{Q} intersect non-trivially. Conclude that \mathbb{Q} would have to be rank 1 and hence isomorphic to \mathbb{Z} . Explain why this is a contradiction.)
3. Recall that an element m of a (left) R -module M is called *torsion* if $rm = 0$ for some nonzero $r \in R$. Let R be an integral domain and suppose M is a left R -module. Prove that $T(M)$, the set of torsion elements of M , is a submodule of M and that $M/T(M)$ is torsion-free.
4. Prove that over any ring R , a simple R -module is cyclic. Further, show that if R is a commutative ring then any simple module over R is of the form R/I for a maximal ideal I of R .
5. Let R be a ring and suppose M is a left R -module with DCC on submodules. If $\theta : M \rightarrow M$ is an R -module homomorphism such that $\ker(\theta) = (0)$, prove that θ is surjective. (Hint: Look at submodules of the form $\text{im}(\theta^n)$.)