Algebra Qual Prep: Summer, 2oo7.
Field Theory Exam

August 16, 2007

1. Let R be an integral domain containing a field F as a subring and which is finite-dimensional as a vector space over F. Prove that R is a field.

2. Let $F \subseteq K$ be fields of characteristic different from 2 and with $[K : F] = 2$. Show that K is in fact of the form $F(\sqrt{\alpha})$ for some $\alpha \in K \setminus K^2$

3. Suppose $F \subseteq K$ are fields and that K_1 and K_2 are two algebraic extensions of F contained in K. Show that $[K_1K_2 : F] \leq [K_1 : F][K_2 : F]$. Show further that if gcd$([K_1 : F], [K_2, F]) = 1$ then equality holds.

4. Suppose F is a field with $[F(\alpha) : F]$ odd. Show that $F(\alpha) = F(\alpha^2)$.

5. Let ϕ denote the Frobenius map $x \mapsto x^p$ on the finite fields $\mathbb{F}_{p^n}$. Prove first that $\phi \in \text{Aut}(\mathbb{F}_{p^n})$ and, second, that ϕ has order n in $\text{Aut}(\mathbb{F}_{p^n})$.