

Algebra Qual Prep: Summer, 2007.

Linear Algebra and Group Theory Problems

February 3, 2008

1. * Prove that the algebraic multiplicity is never less than its geometric multiplicity.
2. Show that if $A \in M_n$ and x^*Ax is real for all $x \in \mathbb{C}^n$ then A is Hermitian.
3. Given $A \in M_{mn}$ with $\text{rank}(A) = \min\{m, n\}$, find the *minimum* value of the ratio $\frac{\|Ax\|_2}{\|x\|_2}$ over all $x \neq 0$. At which value of x is the minimum achieved?
4. Prove that if A is normal then A is Hermitian if and only if $\sigma(A) \subset \mathbb{R}$. Give an example of a non-Hermitian matrix with real eigenvalues.
5. Characterize all diagonal real orthogonal matrices.
6. If $U \in M_n$ is unitary show that \bar{U} , U^T and U^* are unitary as well.
7. * Show that commuting nilpotent matrices, A and B , can be simultaneously triangularized.
8. Show that two nilpotent matrices are similar implies that they have the same index of nilpotency. (Recall, k is the index of nilpotency of A if A^k is the smallest power of A equal to the zero matrix).
9. * Prove that two matrices, A and B , are simultaneously diagonalizable if and only if they are diagonalizable and commute.
10. Let $A \in M_n$ with $\lambda_i \in \sigma(A)$. Prove that if $\text{trace}(A^*A) = \sum_i^n |\lambda_i|^2$ then A is unitarily diagonalizable.
11. * Given $A \in M_n(\mathbb{C})$, use the Schur decomposition to show that for every $\epsilon > 0$, there exists a diagonalizable matrix B such that $\|A - B\|_2 \leq \epsilon$. Use this fact to show that the set of diagonalizable matrices is dense in $M_N(\mathbb{C})$ and that the Jordan Canonical form is not a continuous function of its entries.
12. Horn and Johnson, p180, #1; p181, #4; p267, #7; p268, prove Theorem 5.3.2.
13. If G is a group of even order, prove it contains an element of order 2.

14. If G is a group and H is a subgroup of index two then H is normal in G .
15. All subgroups of the quaternion group are normal.
16. * Find the center of the following groups:
 - \mathbb{Z}_n
 - S_n
 - D_n
 - $Gl_n(\mathbb{C})$
17. * If G is a finite group and $G/Z(G)$ is cyclic then G is Abelian.
18. Let G and H be cyclic groups of order m and n , respectively. Categorize the homomorphisms from G to H .
19. Prove that $\mathbb{Q} \times \mathbb{Q}$ is not cyclic.
20. * Prove that a group of order 15 is cyclic.
21. * Let H and K be subgroup of a group G . Prove that HK is a subgroup if and only if $HK = KH$.
22. Let M and N be normal subgroups of a group G with $G = MN$. Prove that $G/M \cap N \cong G/M \times G/N$.
23. If p is prime, prove every group of order p^2 is Abelian.
24. Let G be a group and N a normal subgroup. Show that G is solvable if and only if G/N and N are both solvable.
25. * Let p and q be primes. Show that groups of size p^n , pq , and p^2q are solvable.
26. If G is a group of order 231 then show that the 11-Sylow subgroup is normal.
27. How many Abelian groups of order 2^43^6 are there?
28. Show that $\mathbb{Z}_p^\times \cong \mathbb{Z}/(p-1)\mathbb{Z}$.