

Algebra Qual Prep: Summer, 2008.

Module Theory Problems

August 23, 2008

Unless otherwise stated, R is a ring with unity.

1. *Let R be a commutative ring, and let I be a nonzero ideal of R . Prove that I is free as an R -module if and only if I is a principal ideal generated by an element that is not a zero divisor of R .
2. * An element $e \in R$ is called *central idempotent* if $e^2 = e$ and $er = re$ for all $r \in R$. If e is central idempotent in R , prove that $M = eM \oplus (1 - e)M$.
3. * Prove that $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$ has order $\gcd(m, n)$.
4. Let R be a left Artinian ring, and let M be a simple R -module. Prove that $\text{Ann}_R(M)$ is a maximal two-sided ideal of R .
5. Let R be a commutative Noetherian ring (with 1) and let $M \neq (0)$ be a finitely generated R -module containing 1_R . For any $m \in M$, the *annihilator* of m is the set $\text{Ann}(m) = \{r \in R \mid rm = 0\}$. Since R is Noetherian, there is an ideal I of R which contains all annihilators of nonzero elements of M . Prove that I is a prime ideal of R . Now prove that there is a chain of submodules

$$(0) = M_n \subseteq \cdots \subseteq M_2 \subseteq M_1 \subseteq M_0 = M$$

of M , such that for each i there is a prime ideal P_i of R with $M_{i-1}/M_i \cong R/P_i$ (as R -modules).

6. If R is a commutative ring and F a free R -module of finite rank. Prove the following isomorphism: $\text{Hom}_R(F, R) \cong F$. Now let M be an arbitrary R -module and show the following:
 - (a) $\text{Hom}_R(R, M) \cong M$ as left R -modules.
 - (b) $\text{Hom}_R(F, M) \cong M \times M \cdots \times M$ (n times).
7. Suppose R is a local commutative ring and that A and B are finitely generated R -modules. Prove that if $A \otimes_R B = (0)$ then either $A = (0)$ or $B = (0)$. Give an example where this doesn't hold.

8. Let R be a ring, E a free R -module on five generators and ϕ an R -module endomorphism of E .
- Describe how ϕ can be represented by a matrix A .
 - Explain why ϕ is invertible if and only if the determinant of A is a unit in the ring R .
 - Assume that R is a principal ideal domain. Show that there are bases B and B' so that the matrix representing ϕ with respect to these bases is a diagonal matrix.
 - Explain why (c) is false if we require in addition that $B = B'$.
9. Assume R is commutative. Show an R -module M is irreducible if and only if M is isomorphic (as an R -module) to R/I where I is a maximal ideal of R . Determine all irreducible \mathbb{Z} -modules.
10. * If M and N are irreducible modules over a ring R and $\chi : M \rightarrow N$ is a homomorphism of modules, prove χ is either 0 or an isomorphism. Deduce that if M is irreducible $\text{End}_R(M)$ is a division ring.
11. Let R be a commutative ring. The (left-) annihilator of an R -module V is the set $I = \{r \in R \mid rV = 0\}$. Prove that I is an ideal of R . What is the annihilator of the \mathbb{Z} -module $\mathbb{Z}/(2) \times \mathbb{Z}/(3) \times \mathbb{Z}/(4)$? What about the \mathbb{Z} -module \mathbb{Z} ?
12. * Prove that a nonzero commutative ring R such that every finitely generated R -module is free is, in fact, a field. (Is that converse true?)
13. Classify finitely generated modules over the ring $\mathbb{C}[\epsilon]$ where $\epsilon^2 = 0$.
14. Prove that for R -modules A, B and M , we have $\text{Hom}_R(A \times B, M) \cong \text{Hom}_R(A, M) \times \text{Hom}_R(B, M)$.
15. Let R be a ring. Of course, R is free as an R -module, with base $\{1\}$. For $a, b \in R$, prove that $\{a, b\}$ is a base of R as a (left) R -module if and only if there exist $r, s \in R$ such that $ar = 1, as = 0, br = 0$ and $bs = 1$, and $ra + sb = 1$.
16. Let R be a PID, and let A, B , and C be finitely generated R -modules. Prove that if $A \oplus C \cong B \oplus C$ then $A \cong B$. (Note: The assumed isomorphism from $A \oplus C$ to $B \oplus C$ needn't map $(0) \oplus C$ to $(0) \oplus C$. You will need to use the structure theory for modules over a PID. Here, the elementary divisors are more convenient invariants to work with than the invariant factors.)