Algebra Qual Prep: Summer, 2oo7.
Practice Qual One

February 3, 2008

1. State and prove the Singular Value Decomposition Theorem.

2. If G is a finite group then define the *exponent* of G to be the least common multiple of the orders of the elements in G. Prove that the exponent of G is the order of G if and only if every Sylow subgroup of G is cyclic.

3. Which of the following pairs of rings are isomorphic? (Prove your assertions.)

   (a) $2\mathbb{Z}$ and $3\mathbb{Z}$ (thought of as subrings of $\mathbb{Z}$).

   (b) $\mathbb{Q}[\frac{5+\sqrt{2}}{4}]$ and $\mathbb{Q}[x]/(x^2-2)$

4. Give examples of the following (and prove that they are examples):

   (a) a non-commutative ring;

   (b) a commutative ring that is not an integral domain;

   (c) an integral domain that is not a unique factorization domain;

   (d) a UFD that is not a principal ideal domain;

   (e) a PID that is not a Euclidean domain;

   (f) a finite dimensional field extension of $\mathbb{Q}$ that is not Galois.

   (g) a finite dimensional field extension of $\mathbb{F}_p$ that is Galois.

   (h) a finite dimensional field extension of $\mathbb{F}_p(x)$ that is not Galois.

5. Suppose L is a splitting field of some polynomial over F and $p(x) \in F[x]$ is an irreducible polynomial. Show that if $p(x)$ has one root in L, then all other roots also lie in L.