Algebra Qual Prep: Summer, 2oo8.
Practice Qual 2

August 21, 2008

1. Show that any commutative semisimple left artinian ring is a direct product of fields.

2. Let G be a nonabelian group of order p^3. Show that the center of G has order p. Show that G has a normal subgroup of order p^2, and prove such a group exists.

3. Let $S = M_2(R)$ be the set of matrices over a commutative ring. Show that $(AB - BA)^2 \in Z(S)$ for any $A, B \in S$. (Hint: use Cayley-Hamilton.)

4. Let R be a commutative ring with 1. Prove every maximal ideal in $R[x]$ contains a regular element (non-zero-divisor). (Hint: x is not a zero divisor.)

5. Let A and B be vector spaces over a field, k. Define $\mathrm{Hom}_k(A, B)$ to be the set of all k-linear transformations from A to B and set $A^* = \mathrm{Hom}_k(A, k)$. Prove that $A^* \otimes_k B \cong \mathrm{Hom}_k(A, B)$, and further that $A^* \otimes_k A \cong k$. Now, if we identify $A^* \otimes_k A \cong \mathrm{Hom}_k(A, A)$, then the isomorphism in part (b) can be given by the trace of the linear transformation.

6. Let p be a prime number, and let $F \subseteq K$ be fields with K Galois over F and $[K : F] = p^n$. Prove that there is a field L with $F \subseteq L \subseteq K$ and $[L : F] = p$, and that every such L is Galois over F.