Algebra Qual Prep: Summer, 2oo7.
Practice Qual 2

February 3, 2008

1. State and prove the Cayley-Hamilton Theorem.

2. Let $G = S_5$.

   (a) What are the elements of order 3? How many are there?

   (b) How many 3-Sylow subgroups are there in G?

   (c) How many elements of order 8 are there?

   (d) Describe the conjugacy classes in G. How many elements are in each class?

   (e) Describe all the normal subgroups of G.

3. Let $R \subseteq M_2(\mathbb{Q})$ be the subring of upper triangular matrices. With justification, find all right and left ideals. Then, find all two-sided ideals.

4. Let R be a commutative ring with 1 and suppose that P and Q are prime ideals with $P + Q = 1$.

   (a) Prove R/P is a domain.

   (b) Prove $R/(P \cap Q) \cong R/P \times R/Q$.

   (c) Can $R/(P \cap Q)$ be a domain?

   (d) Can $P \cap Q$ be a prime ideal?

   (e) Explain why the product of two fields is not a field.

5. (a) If I is a right ideal of a ring R with identity and B a left R-module, then there is a group isomorphism $R/I \otimes_R B \cong B/IB$, where IB is the subgroup of B generated by all elements rb with $r \in I, b \in B$.

   (b) If R is commutative and I, J ideals of R, then there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$.

6. Let p be a prime number, and let $F \subseteq K$ be fields with K Galois over F and $[K : F] = p^n$. Prove that there is a field L with $F \subseteq L \subseteq K$ and $[L : F] = p$, and that every such L is Galois over F.