Algebra Qual Prep: Summer, 2oo7.
Practice Qual 2

August 23, 2007

Note: n is a natural number and p a prime throughout.

1. State and prove Schur's Theorem on the decomposition of a matrix. Now, suppose that A and B are in M_n and satisfy $AB = BA$. Assume that A is Hermitian with distinct eigenvalues. Show that if Q is the unitary matrix associated with a Schur decomposition of A, then $Q^H BQ$ is diagonal.

2. State and prove the Rayleigh-Ritz Theorem.

3. Prove that the characteristic polynomial of a linear transformation T on a finite dimensional vector space is independent of the basis chosen.

4. Prove that if A is Hermitian, then the number of nonzero eigenvalues of A is equal to the rank of A. Give an example of a non-Hermitian matrix for which this does not hold

5. Suppose G is a finite group of order divisible by p. Consider the set S of p-tuples of elements of G with product equal to the identity. Define an action of $\mathbb{Z}_p$ on this set and use this to explain why G contains an element of order p.

6. Suppose that G is a finitely generated free abelian group and H is a subgroup.

   (a) Is H a free group?

   (b) Now suppose $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and H is generated by the elements

   $$(1, 0, 0, 0, 0), (2, 2, 0, 0, 0), (3, 3, 3, 0, 0), (4, 4, 4, 4, 0), (5, 5, 5, 5, 5).$$

   Describe the invariant factors and elementary divisors for G/H.

7. Let G be a group of order p^n. Show that any nontrivial normal subgroup intersects the center of G nontrivially.

8. Let G be a group of order p^4. Suppose G contains an element of order p^3, but none of order p^4.

   (a) Determine a presentation of G by generators and relations, and give a careful proof that G has this presentation

(b) Determine $|\mathrm{Aut}(G)|$.

9. Let G be a finite solvable group whose center contains a p-Sylow subgroup, P. Show that $G = PQ$ for some unique subgroup Q of G with $P \cap Q = \{e\}$.

10. (a) Prove that there exists a nonabelian group of order 21

    (b) Give a representation of this group by generators and relations.

    (c) Prove that this is the unique (up to isomorphism) nonabelian group of order 21.

11. Let G be a finite p-group of order p^n. Prove that G has a normal subgroup of order p^i for every $i < n$.

12. Prove that a finite p-group has a nontrivial center

13. Let G be a p-group of order at least p^2. Prove that p divides the order of $\mathrm{Aut}(G)$. (Hint: First consider the case with G nonabelian. Also, recall that $\mathrm{Aut}(\mathbb{Z} \times \mathbb{Z})$ has order $(p^2 - 1)(p^2 - p)$.)

14. Let $G = \langle x, y | x^2 = e, y^2 = e\rangle$. Prove that G is nonabelian and has infinite order.

15. Let I, J be ideals in a commutative ring R with 1. Define $K = \{r | rJ \subseteq I\}$. Show that K is an ideal. Now, if R is a PID, so $I = (i)$ and $J = (j)$, give a formula for a generator $k \in K$.

16. Let I and J be two-sided ideals of the ring R. Prove that if $I \cap J = 0$ and if the rings R/I and R/J are left Noetherian, then R is left Noetherian.

17. Let R be a Noetherian commutative ring and suppose $f = \sum_{i=0}^{\infty} r_i x^i$ in the formal power series ring $R[[x]]$. Prove that f is nilpotent if and only if each r_i is nilpotent.

18. Describe as fully as you can finite rings with no nilpotent elements.

19. Let R be an integral domain with quotient field K and let $c \in R$ with $c \neq 0$. Prove that the following conditions are equivalent:

    (a) $R[c^{-1}] = K$

    (b) Every nonzero prime ideal of R contains c

    (c) Every nonzero ideal of R contains a power of c

20. Let A, B, C be modules over a commutative ring R.

    (a) Prove that the set $\mathcal{L}(A, B; C)$ of all R-bilinear maps $A \times B \to C$ is an R-module with $(f + g_)a, b) = f(a, b) + g(a, b)$ and $(rf)(a, b) = rf(a, b)$.

    (b) Show that each of the following R-modules is isomorphic to $\mathcal{L}(A, B; C)$.

        i. $\mathrm{Hom}_R(A \otimes_R B, C)$

        ii. $\mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C))$

iii. $\mathrm{Hom}_R(B, \mathrm{Hom}_R(A, C))$

21. Describe up to isomorphism all the $\mathbb{R}[x]$-module structures one might put on a 3-dimensional real vector space extending the fixed $\mathbb{R}$ action.

22. Let $R = \mathbb{C}[x, y]$ and let I be the ideal of R generated by the two elements $xy - 1$ and $x^2 + y$.

    (a) Prove or disprove that I is a free R-module.

    (b) Prove or disprove that I is a maximal ideal.

23. Let F be any field with characteristic not equal to 2. Set $R = F[x, y]/(x^2 + y^2 - 1)$ and $M = (\overline{x}, \overline{y} - 1)$ an ideal of R.

    (a) Prove that R is an integral domain

    (b) Prove that M is a maximal ideal of R

    (c) Let $S = R - M$ and R_S the localization of R with respect to S. Prove that M_S is a principal ideal of R_S.

24. Let R be a PID and $F = R \oplus \cdots \oplus R = R^n$ be a free R-module. Suppose an element $\alpha = (a_1, \ldots, \alpha_n) \in F$ is a unimodular row, i.e. there is no prime element $p \in R$ such that $p|a_i$ for all i.

    (a) Prove that $F/R\alpha$ is a torsion free R-module.

    (b) Prove that $F \cong R\alpha \oplus (F/R\alpha)$

    (c) Prove that there are $\beta_2, \ldots, \beta_n \in F$ such that $\alpha, \beta_2, \ldots, \beta_n$ form a base of F as a free R-module.

    (d) Prove that there is a matrix A in $M_n(R)$ such that A has an inverse and the top row of A is the unimodular row α.

25. Let K be a subfield of the finite field F with $|K| = q = p^r$ and $|F| = q^n$. Prove that F is the splitting field over K of a separable polynomial.

26. Consider $K \subseteq L \subseteq F$ fields with all extensions finite and Galois. Let $f(x) \in K[x]$ be an irreducible polynomial having a root in F. What can we say about the factorization of $f(x)$ in $L[x]$? Do the irreducible factors have the same degree? Prove your assertion.

27. Let K be a finite field with $|K| = p^n$. Prove that every element of K has a unique p^{th} root.

28. Recall that a perfect field of characteristic p is one for which the Frobenius map $Fr : x \mapsto x^p$ is onto. Let K be a perfect field and F an algebraic extension. Show that F is perfect.

29. Let F be a finite field of order p^2. Prove that 5 is a square in F.

30. Suppose $F \subseteq K$ are fields of characteristic 0. Suppose every $\alpha \in K$ is a root of a polynomial in $F[x]$ of degree less than or equal to n. Prove that $[K : F] \leq n$. Give an example to show that this does not always hold in characteristic p.

31. Let $F \subseteq L \subseteq K$ be fields with $[K : F] < \infty$ and K Galois over F. Let $G = Gal(K/F)$, H be the subgroup of G corresponding to L, and N be the normalizer of H in G.

    (a) Prove that $N = \{\tau \in G | \tau(L) = L\}$

    (b) Prove that $Gal(L/F) \cong N/H$

32. Let $F \subseteq K$ be fields with K Galois over F and $[K : F] < \infty$. Prove that if L is any field containing F, then the compositum KL is Galois over L and that $Gal(KL/L) \cong Gal(K/K \cap L)$

Give examples of the following:

1. A nonsolvable group other than A_n or S_n

2. Fields $F \subseteq K$ with K purely inseparable over F and $[K : F] = 9$.

3. Fields $F \subseteq K$ with K Galois over F and $Gal(K/F) \cong A_6$

4. An integral domain with exactly three prime ideals

5. An integral domain which is not a UFD

6. A subfield of $\mathbb{C}$ of index 2 other than $\mathbb{R}$.

7. Fields $F \subseteq K$ with $[K : F] < \infty$ and K neither separable nor purely inseparable over F.

8. Modules A, A', B, B' over a commutative ring R with injective R-homomorphisms $i : A \to A'$ and $j : B \to B'$ stuch that $i \otimes j : A \otimes_R B \to A' \otimes_R B'$ is not injective.

9. An infinite simple group

10. A finite commutative ring R of order 50 and only one prime ideal.