Algebra Qual Prep: Summer, 2oo7.
Ring Theory and Module Theory Problems

February 3, 2008

Note: All of our rings contain 1.

1. Prove or disprove: an ideal of a ring R contains a unit if and only if it is the whole ring.

2. Let $I_1, \ldots, I_n$ be ideals in an integral domain. Prove that if $I_1 \cap \ldots \cap I_n = 0$ then at least one of the I_i must be (0). Give an example where the conclusion fails for an intersection of infinitely many ideals. Then show that if the intersection of infinitely many nonzero ideals $\{I_1, I_2, I_3, \ldots\}$ is, in fact, (0) then $I_k \cap I_{k+1} \cap I_{k+2} \ldots$, is (0) as well.

3. Prove that the characteristic of an integral domain is either 0 or a prime.

4. * Prove that every integral domain can be embedded in a field.

5. * Let R be an integral domain and prove that if $R[x]$ is a PID then R is a field. (Hint: let $a \in R$ with $a \neq 0$ and $a \notin R^* and consider (a, x) < R[x]$.)

6. Let R be a commutative ring and let $f = r_n x^n + \ldots + r_0$ be a zero divisor in $R[x]$. Prove that there is a nonzero $s \in R$ such that $r_n s = r_{n-1} s = \ldots = r_0 s = 0$.

7. Let R be a commutative ring and I an ideal of R. The radical of an ideal I is $\sqrt{I} = \{x \in R | x^n \in I$ for some $n \in \mathbb{N}\}$. I is said to be radical if $I = \sqrt{I}$. Prove that if I is maximal then I is radical.

8. * Let R be a local ring (ie, R has a unique maximal ideal, M). Prove that every element of $R \setminus M$ is a unit. Prove conversely that if R is a ring in which the units form an ideal, R must be local.

9. Suppose R is an integral domain. Prove that if the following two conditions hold then R is a PID:

   (a) any two nonzero elements $a, b \in R$ have a gcd that can be written $ax + yb$ for some $x, y \in R$.

   (b) if $a_1, a_2, \ldots a_n$ are nonzero element of R satisfying $a_{i+1} | a_i$ for all i then there is a positive N for which a_n is a unit times a_N for all $n \geq N$.

10. Prove that if R is a PID and D is a multiplicatively closed subset of R then $D^{-1}R$ is a PID.

11. Let R be a commutative ring, and let I be a nonzero ideal of R. Prove that I is free as an R-module if and only if I is a principal ideal generated by a nonzero divisor of R.

12. Let R be a ring. Of course, R is free as an R-module, with base $\{1\}$. For $a, b \in R$, prove that $\{a, b\}$ is a base of R as a (left) R-module if and only if there exist $r, s \in R$ such that $ar = 1$, $as = 0$, $br = 0$ and $bs = 1$, and $ra + sb = 1$.

13. * Prove that $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$ has order gcd(m, n).

14. Suppose R is a local commutative ring and that A and B are finitely generated R-modules. Prove that if $A \otimes_R B = (0)$ then either $A = (0)$ or $B = (0)$. Give an example where this doesn't hold.

15. Let R be a ring and denote by $R[[t]]$ the formal power series over R. Prove that an element $f(t) \in R[[t]]$, say $f(t) = a_0 + a_1 t + a_2 t^2 \cdots$ is invertible if and only if a_0 is a unit in R. Determine all ideals of $R[[t]]$.

16. * Let k be a field and let P be a prime ideal in $k[x, y]$. Show that P is generated by two elements, by considering cases: $P \cap k[x] = (0)$ and $P \cap k[x] \neq (0)$.

17. * Calculate Spec$\mathbb{C}[z]$.

18. * Let A be an integral domain and let S be a multiplicative subset of A, so $S^{-1}A$ is the localization of A with respect to S.

    (a) Prove that any ideal in $S^{-1}A$ is of the form $S^{-1}I$ for some ideal $I \subseteq A$.

    (b) Prove that if $P \subseteq A$ is a prime ideal of A such that $P \cap S = \emptyset$ then $S^{-1}P \subseteq S^{-1}A$ is a prime ideal.

    (c) Prove that the natural map Spec$(S^{-1}A) \to$ Spec(A) is injective or give a counterexample.

19. If R is Noetherian, and I is an ideal of R, then R/I is Noetherian.

20. Prove that every submodule of a finitely generated module is finitely generated. Now, suppose N is a submodule of M. Prove that if N and M/N are finitely generated then so is M.

21. Let R be a Noetherian ring and S a multiplicative subset. Prove that $S^{-1}R$ is Noetherian.

22. Assume R is commutative. Show an R-module M is irreducible if and only if M is isomorphic (as an R-module) is to R/I where I is a maximal ideal of R. Determine all irreducible $\mathbb{Z}$-modules.

23. * If M is a module over a ring R and $\chi : M \to M$ is a homomorphism of modules, prove χ is either 0 or an isomorphism. Deduce that if M is irreducible then End$_R(M)$ is a division ring.

24. Let R be a commutative ring. The (left-) annihilator of an R-module V is the set $I = \{r \in R | rV = 0\}$. Prove that I is an ideal of R. What is the annihilator of the $\mathbb{Z}$-module $\mathbb{Z}/(2) \times \mathbb{Z}/(3) \times \mathbb{Z}/(4)$? What about the $\mathbb{Z}$-module $\mathbb{Z}$?

25. Let I be an ideal of a ring R. Prove that I is a free R-module if and only if it is a principal ideal generated by an element that is not a zero-divisor in R.

26. * Prove that a nonzero commutative ring R such that every finitely generated R-module is free is, in fact, a field. (Is that converse true?)

27. Classify finitely generated modules over the ring $\mathbb{C}[\epsilon]$ where $\epsilon^2 = 0$.

28. Prove that for R-modules A, B and M, we have $\mathrm{Hom}_R(A \times B, M) \cong \mathrm{Hom}_R(A, M) \times \mathrm{Hom}_R(B, M)$.

29. If R is a commutative ring and F a free R-module of finite rank. Prove the following isomorphism: $\mathrm{Hom}_R(F, R) \cong F$. Now let M be an arbitrary R-module and show the following:

    (a) $\mathrm{Hom}_R(R, M) \cong M$ as left R-modules.

    (b) $\mathrm{Hom}_R(F, M) \cong M \times M \cdots \times M$ (n times).

30. An element $e \in R$ is called *central idempotent* if $e^2 = e$ and $er = re$ for all $r \in R$. If e is central idempotent in R, prove that $M = eM \oplus (1 - e)M$.

31. Let R be a PID, and let A, B, and C be finitely generated R-modules. Prove that if $A \oplus C \cong B \oplus C$ then $A \cong B$. (Note: The assumed isomorphism from $A \oplus C$ to $B \oplus C$ needn't map $(0) \oplus C$ to $(0) \oplus C$. You will need to use the structure theory for modules over a PID. Here, the elementary divisors are more convenient invariants to work with that the invariant factors.)