

## Homework #0

- Write the following equations as root finding problems  $f(x) = 0$ :
  - $x^5 - x^4 = 2x^3 - x^2 + 1$ .
  - $x \sin x = 1$ .
  - $x^2 - \sin x = xe^x$ .
- Write the equation  $x + e^x = \cos x$  as two different root finding problems  $f_1(x) = 0$  and  $f_2(x) = 0$ .
- Can the bisection method be used to find a root of the following functions using the following intervals? Why or why not?
  - $f(x) = \cos x + e^x$  with  $[0, \pi/2]$ .
  - $f(x) = x^3 + x + 1$  with  $[-1, 0]$ .
  - $f(x) = 1/x$  with  $[-1, 1]$ .
- Use the bisection method to generate the first 4 approximations of  $2\sqrt{2}$  by finding the positive root of  $x^2 - 8$  using the initial interval  $[2, 3]$ .
  - Find the bound of the absolute error of the final approximation (without knowing the exact root) and compare it to the exact absolute error.
- Consider  $f(x) = x(x - 1)(x + 2)$ , which has roots at  $x = 0, 1, -2$ . Which root does the bisection method approximate when using starting interval  $[-3, 2]$ ?
- Suppose  $f(x)$  is a given continuous function in  $[-1, 2]$  such that  $f(-1)$  and  $f(2)$  have different signs.
  - Bound the absolute error (without knowing the exact root) of the approximation generated after 30 iterations of the bisection method.
  - How many approximations of the bisection method need to be generated to achieve an absolute error less than  $10^{-11}$  to a root of  $f(x)$  in  $[-1, 2]$ ?
- Consider the modified bisection procedure of:
  - Start with interval  $[a, b]$  where  $f$  is continuous and  $f(a)$  and  $f(b)$  have opposite signs. Fix a value for  $0 < t < 1$ .
  - Form approximation  $c = (1 - t)a + tb \in (a, b)$ .
  - If  $f(c) = 0$  then exit procedure. Else if  $f(a)$  and  $f(c)$  have opposite signs, set  $b = c$ . Else if  $f(c)$  and  $f(b)$  have opposite signs, set  $a = c$ .
  - Repeat from Step 2.

Note if  $t$  is chosen to be  $1/2$ , this is exactly bisection method. Now let  $f(x) = x \sin x - 1$ .

- (a) Choose the interval  $[0, \pi/2]$  and  $t = 1/4$ . Use the modified procedure and generate the first 4 approximations.
  - (b) Bound the absolute error of the 4th approximation (without knowing the exact root).
  - (c) Use bounds to also determine how many approximations it will take of the modified bisection method to achieve an absolute error less than  $10^{-6}$  to a root?
8. Answer True or False for the following questions (you do not need to show work but write down your explanation if you are unsure):
- (a) The bisection method will always generate a sequence of approximations converging to a root of the continuous function  $f(x)$  using starting interval  $[a, b]$  if  $f(a)$  and  $f(b)$  have different signs.
  - (b) The bisection method only works when  $f(x)$  has exactly one root in the starting interval  $[a, b]$ .
  - (c) When there are multiple roots of  $f(x)$  in the starting interval  $[a, b]$ , the bisection method will approximate the smallest root.
  - (d) The absolute error of the  $n$ th approximation of the bisection method for  $f(x)$  in the starting interval  $[a, b]$  is  $\leq 1/2^n$ .