

## Homework #1

- Can the bisection method be used to find a root of the following functions using the following intervals? Why or why not?
  - $f(x) = \cos x + e^x$  with  $[0, \pi/2]$ .
  - $f(x) = x^3 + x + 1$  with  $[-1, 0]$ .
  - $f(x) = 1/x$  with  $[-1, 1]$ .
  - $f(x) = \begin{cases} -x - 1, & x \leq 0 \\ x - 1, & x > 0 \end{cases}$  with  $[-2, 1/2]$ .
- Consider  $f(x) = x(x - 1)(x + 2)$ , which has roots at  $x = 0, 1, -2$ . Determine which root the bisection method approximates when using the starting interval  $[-3, 2]$ ?
- Suppose  $f(x)$  is a given continuous function in  $[-1, 4]$  such that  $f(-1)$  and  $f(4)$  have different signs.
  - Bound the absolute error for the approximation  $c_{30}$  generated after 30 iterations of the bisection method.
  - Also use the bound on absolute error to determine how many iterations of the bisection method need to be taken to achieve an absolute error less than  $10^{-11}$  for a root of  $f(x)$  in  $[-1, 4]$ ?
- Suppose we modify the bisection method into these three variations:
  - Variation #1: Approximations are chosen at the midpoint of the interval. The interval is cut into two at the location  $(2a + b)/3$ .
  - Variation #2: Approximations are chosen at the location  $(2a + b)/3$ . The interval is cut into two at the midpoint of the interval.
  - Variation #3: Approximations are chosen at the location  $(2a + b)/3$ . The interval is cut into two at the location  $(2a + b)/3$ .

Answer the following questions about these variations:

- Bound the absolute errors of the  $n$ th iteration for each variation when the starting interval is  $[a, b]$ . Which variation has the best bound?
  - Calculate the first 2 approximations  $c_0, c_1$  for each variation when  $f(x) = \cos x - x$  with starting interval  $[0, \pi/2]$ .
- Use the bisection method to generate the first 4 approximations  $c_0, c_1, c_2, c_3$  of  $2\sqrt{2}$  by finding the positive root of  $x^2 - 8$  using the starting interval  $[2, 3]$ .
    - Find the bound of the absolute error for the final approximation and verify that the actual absolute error satisfies this bound.

- (c) Use Newton's method to generate  $x_1, x_2, x_3$  starting with  $x_0 = 3$ . Compare the absolute error of  $x_3$  to that of  $c_3$ .
6. Consider the problem of finding the point on the graph of  $y = x^3$  closest to the point  $(3, -1)$ .
- (a) Write down the expression for  $d(x) =$  the square of the distance from  $(x, x^3)$  to  $(3, -1)$ .
- (b) Minimize  $d(x)$  by finding the, in this case, unique critical point: approximating the solution of  $f(x) = d'(x) = 0$  using Newton's method to generate  $p_2$  when  $p_0 = 2$ .
7. Consider the equation  $xy^2 + \cos y = x$  that implicitly defines  $y$  as a function of  $x$ :  $y = y(x)$ . Approximate  $y(2)$  using Newton's method for 2 iterations using initial guess  $y_0 = 1$ .
8. Answer True or False for the following questions (you do not need to show work but write down your explanation if you are unsure):
- (a) The bisection method will always generate a sequence of approximations converging to a root of the continuous function  $f(x)$  using starting interval  $[a, b]$  if  $f(a)$  and  $f(b)$  have different signs.
- (b) The bisection method only works when  $f(x)$  has exactly one root in the starting interval  $[a, b]$ .
- (c) When there are multiple roots of  $f(x)$  in the starting interval  $[a, b]$ , the bisection method will approximate the smallest root.
- (d) Newton's method gets the exact root after one iteration when  $f$  is a line with nonzero slope.
- (e) When the function  $f(x)$  has multiple roots, when Newton's method converges, it converges to the root closest to the initial guess used.
9. (Matlab) Write a Matlab function that takes as input  $x$  and outputs  $g(x)$ , where we will define  $g$  a little later. Name the file "hw1g.m". Then write a Matlab function that takes as input initial guess  $x_0$  and number of iterations  $N$  and outputs the  $x_N$  following the iterative formula  $x_{n+1} = g(x_n)$ . Call "hw1g" to get values of  $g$  and name the file "hw1iterative.m".
- (a) Turn in your programs for Newton's method on  $f(x) = x^2 - 8$ .
- (b) Write down your results when  $x_0 = 2$  and  $N = 5, 10, 20$  and when  $x_0 = 3$  and  $N = 5, 10, 20$ .