Homework #1

1. Will the bisection method find a root of the following functions using the following starting intervals? Why or why not?
   
   (a) \( f(x) = \cos x + e^x \) with \([0, \pi/2]\).
   
   (b) \( f(x) = x^3 + x + 1 \) with \([-1, 0]\).
   
   (c) \( f(x) = 1/x \) with \([-1, 7]\).
   
   (d) \( f(x) = \begin{cases} -x - 1, & x \leq 0 \\ x - 1, & x > 0 \end{cases} \) with \([-2, 1/2]\).

2. Consider \( f(x) = x(x - 1)(x + 2) \), which has roots at \( x = 0, 1, -2 \). Perform enough iterations to determine, definitively, which root the bisection method approximates when using the starting interval \([-3, 2]\).

3. Suppose \( f(x) \) is a given continuous function in \([-1, 4]\) such that \( f(-1) \) and \( f(4) \) have different signs and consider the bisection method on \( f(x) \) using starting interval \([-1, 4]\).
   
   (a) Bound the absolute error for the approximation \( c_{30} \) (Remember, we define \( c_0 = (a_0 + b_0)/2 \)).
   
   (b) Use the bound on absolute error to determine which \( c_n \) are guaranteed to have absolute errors less than \( 10^{-7} \).

4. Suppose “\( f \) has different sign at \( a \) and \( b \)” is defined to be true if and only if \( f(a)f(b) \leq 0 \).
   
   (a) Find \( c_n \) generated by the bisection method on \( f(x) = x^2 \) when the starting interval is \([0, b]\), for \( b > 0 \).
   
   (b) What does this sequence of approximations converge to and what is the order of convergence?

5. (a) Given the first two approximations, \( c_0, c_1 \), generated from a bisection method, determine the starting interval \([a_0, b_0]\) that was used.

   (b) Apply this to find the starting interval when \( c_0 = -0.2 \) and \( c_1 = 0.3 \).

6. Suppose we modify the bisection method into the following variation: for each step, with bracketing interval \([a, b]\), approximations are chosen at the location \((2a + b)/3\), but the interval is cut into two at the different location \((a + 3b)/4\).
   
   (a) Calculate the first 2 approximations \( c_0, c_1 \) for this variation when \( f(x) = \cos x - x \) with starting interval \([0, \pi/2]\).
   
   (b) Bound the absolute errors of the approximations \( c_n \) for a starting interval of length \( L \). Is this bound better or worse than that of bisection method?
   
   (c) How many function evaluations are used in generating \( c_n \)? How does this compare to the number used in bisection method?
7. (a) Use the bisection method to generate the approximations $c_0, c_1, c_2, c_3$ of $2\sqrt{2}$ by finding the positive root of $x^2 - 8$ using the starting interval $[2, 3]$.

(b) Find the bound of the absolute error for the final approximation and verify that the actual absolute error satisfies this bound.

8. Answer True or False for the following questions (you do not need to show work but write down your explanation if you are unsure). In each, $f : \mathbb{R} \to \mathbb{R}$ denotes a continuous function in $[a, b]$, with different signs at $a$ and $b$.

(a) The bisection method on $f(x)$ using starting interval $[a, b]$ will always generate a sequence of approximations converging to a root of $f(x)$.

(b) The bisection method only works when $f(x)$ has exactly one root in the starting interval $[a, b]$.

(c) When there are multiple roots of $f(x)$ in the starting interval $[a, b]$, the bisection method will approximate the root closest to the midpoint $(a + b)/2$.

(d) The sequence of approximations generated by the bisection method on $f(x)$ using starting interval $[a, b]$ is the same as the sequence generated by the bisection method on $g(f(x))$ using starting interval $[a, b]$, for any continuous $g : \mathbb{R} \to \mathbb{R}$ with values $g(x)$ that are negative for $x$ positive, and positive for $x$ negative.

(e) The bisection method on $g(x)$ using starting interval $[a, b]$ may breakdown if $g$ is a function that is continuous in $[a, b]$ but not defined outside of $[a, b]$.

9. (Matlab) Given $f(x)$, suppose “hw1f.m” is a function that takes as input the number $x$ and outputs $f(x)$.

(a) Write a Matlab function that inputs:
   - $a, b$: endpoints of starting interval $[a, b]$;
   - $N$: number of iterations to perform;

   and outputs $c_N$ using the variation of problem 6 on $f(x)$. Write or print out your function and turn it in.

(b) Apply your function to $f(x) = \sin x - 1/2$ with starting interval $[0, \pi/2]$ for 20 iterations. Write or print out the resulting approximation and and its absolute error (exact root is at $\arcsin(1/2) = \pi/6)$.