

## Homework #2

1. Let

$$f(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ -\sqrt{-x}, & \text{if } x < 0. \end{cases}$$

- (a) Generate Newton's method's  $x_1, x_2, x_3$  in terms of  $x_0 > 0$ .
  - (b) Will Newton's method converge for any  $x_0 \neq 0$ ? Why does this not violate the theorem on the convergence of Newton's method if the initial guess is close enough to the root?
2. Consider  $f(x) = \sin x$  and its root at  $x = 0$ . Prove Newton's method converges to this root for any  $x_0 \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ .
3. Determine what the following sequences converge to and then check if they converge linearly or quadratically:
- (a)  $\{x_n\}_{n=1}^{\infty}$  with  $x_n = n^{-10}$ .
  - (b)  $\{x_n\}_{n=0}^{\infty}$  with  $x_n = 1 + 10^{-n}$ .
  - (c)  $\{x_n\}_{n=0}^{\infty}$  with  $x_n = 5 + 3^{-2^n}$ .
  - (d) Newton's method's sequence of approximations on  $f(x) = x^k$  with  $k > 1$  using  $x_0 \neq 0$ .
4. Let  $\{x_n\}_{n=0}^{\infty}$  be a sequence of approximations that converges to  $r$ .
- (a) Prove if  $x_n$  has order of convergence at least  $\alpha$ , then it also has order of convergence at least  $\beta$ , where  $0 \leq \beta < \alpha$ .
  - (b) Prove if  $\lim_{n \rightarrow \infty} \frac{|x_{n+1} - r|}{|x_n - r|^\alpha}$  exists and is nonzero for some  $\alpha \geq 0$ , then  $x_n$  does not have order of convergence at least  $\beta$ , where  $\beta > \alpha$ .
5. Consider  $f(x) = 1 - e^x$ . We are looking for the root of  $f(x)$ .
- (a) Find  $x_1, x_2, x_3$  using Newton's method on  $f(x)$  when  $x_0 = 0.1$ .
  - (b) Compute  $|r - x_{n+1}|/|r - x_n|$  and  $|r - x_{n+1}|/|r - x_n|^2$  for  $n = 0, 1, 2$  using knowledge of the exact root location  $r = 0$ . Then decide if Newton's method looks linearly and/or quadratically convergent in this case?
6. Consider  $f(x) = x(1 - e^x)$ .
- (a) Verify that  $f(0) = 0$  and  $f'(0) = 0$ .
  - (b) Using Newton's method with  $x_0 = 0.1$ , compute  $|r - x_{n+1}|/|r - x_n|^2$  when  $r = 0$  for  $n = 0, 1, 2$ . Does Newton's method look like it is quadratically convergent in this case?

Now consider the variations

- Variation #1: Applying Newton's method instead to the root finding problem  $h(x) = 0$ , where  $h(x) = f(x)/f'(x)$ .
- Variation #2 using  $s$ : Modifying Newton's method's approximations to

$$x_{n+1} = x_n - s \frac{f(x_n)}{f'(x_n)}.$$

Continue with the following parts:

- Using Variation #1 with  $x_0 = 0.1$ , compute  $|r - x_{n+1}|/|r - x_n|^2$  when  $r = 0$  for  $n = 0, 1, 2$ . Does this variation look like it is quadratically convergent in this case?
  - Using Variation #2 with  $s = 2$  and  $x_0 = 0.1$ , compute  $|r - x_{n+1}|/|r - x_n|^2$  when  $r = 0$  for  $n = 0, 1, 2$ . Does this variation look like it is quadratically convergent in this case?
- Write down the formula for the Secant method with initial guesses  $x_0 = a$  and  $x_1 = b$  and simplify to show it is the same as the formula for the Secant method with initial guesses  $x_0 = b$  and  $x_1 = a$ .
  - Use the Secant method to generate the approximations  $x_2, x_3, x_4$  of  $2\sqrt{2}$  by finding the positive root of  $x^2 - 8$  using  $x_0 = 2, x_1 = 3$ . Also write down the absolute error of the last approximation.
  - (Matlab) Consider the hybrid method that comes from modifying the bisection method so that approximations and the interval cutting location are both chosen at the approximation of Secant method:  $b - f(b)(b - a)/(f(b) - f(a))$ . Write a Matlab function that takes as input  $x$  and outputs  $f(x)$ , when given a rootfinding problem  $f(x) = 0$ . Name the file "hw2f.m". Then write a Matlab function that takes as input a starting interval endpoints  $a, b$  and number of iterations  $N$  and outputs the hybrid method's approximation  $c_N$ . Name the file "hw2hybrid.m" and be sure that inside it calls "hw2f" to get values of  $f$ .
    - Turn in your programs for the case  $f(x) = \cos x - x$ .
    - Write down your results when the starting interval endpoints are  $a = 0, b = \pi/2$  and  $N = 1, 5, 10, 20$ .