Homework #3

1. Determine what the following sequences converge to and then check if they converge linearly or quadratically:

   (a) $\{p_n\}_{n=1}^{\infty}$ with $p_n = n^{-10}$.

   (b) $\{p_n\}_{n=0}^{\infty}$ with $p_n = 1 + 10^{-n}$.

   (c) $\{p_n\}_{n=0}^{\infty}$ with $p_n = 5 + 3^{-2^n}$.

2. Consider the sequence with terms $p_n = 2^{-n^2}$. Show this sequence is linearly convergent with asymptotic error constant 0 but not convergent for any order $\alpha > 1$.

3. Find the multiplicity of the zero at $x = 1$ for the following functions:

   (a) $f(x) = (x-1)^2 e^{x-1}$.

   (b) $f(x) = \sin^2(x-1)$.

   (c) $f(x) = (x-1)(1-e^{x-1})$.

4. Consider $f(x) = x^k$ for $k > 1$. We are looking for the root of $f(x)$.

   (a) Simplify the expression for the approximations p_n generated using Newton's method and given $p_0 \neq 0$.

   (b) Show the sequence converges only linearly to the root.

   (c) Show the first approximation of Newton's method on $h(x) = f(x)/f'(x)$ hits the exact root for any $p_0 \neq 0$.

   (d) Show the first approximation of the method with

   $$p_{n+1} = p_n - k\frac{f(p_n)}{f'(p_n)}$$

   hits the exact root for any $p_0 \neq 0$.

5. Consider $f(x) = 1 - e^x$. We are looking for the simple root of $f(x)$.

   (a) Find p_n for $n = 1, 2, 3$ using Newton's method on $f(x)$ when $p_0 = 0.1$.

   (b) Compute $|p - p_{n+1}|/|p - p_n|$ for $n = 0, 1, 2$ using knowledge of the exact root location $p = 0$. Does this look like it is converging to 0?

   (c) Compute $|p - p_{n+1}|/|p - p_n|^2$ for $n = 0, 1, 2$ using knowledge of the exact root location $p = 0$. What number does it look like it is converging to?

6. Repeat the previous problem using Newton's method on $h(x) = f(x)/f'(x)$.

7. Consider $f(x) = x(1 - e^x)$. We are looking for the root of multiplicity 2 of $f(x)$.

(a) Find p_n for $n = 1, 2, 3$ using Newton's method on $f(x)$ when $p_0 = 0.1$.

(b) Compute $|p - p_{n+1}|/|p - p_n|$ for $n = 0, 1, 2$ using knowledge of the exact root location $p = 0$. What number does it look like it is converging to?

(c) Compute $|p - p_{n+1}|/|p - p_n|^2$ for $n = 0, 1, 2$ using knowledge of the exact root location $p = 0$. Does this look like it is going to infinity?

8. Consider $f(x) = x(1 - e^x)$. We are looking for the root of multiplicity 2 of $f(x)$.

(a) Find p_n for $n = 1, 2, 3$ using Newton's method on $h(x) = f(x)/f'(x)$ when $p_0 = 0.1$.

(b) Compute $|p - p_{n+1}|/|p - p_n|^2$ for $n = 0, 1, 2$ using knowledge of the exact root location $p = 0$. What number does it look like it is converging to?

(c) Find p_n for $n = 1, 2, 3$ using the method with

$$p_{n+1} = p_n - k\frac{f(p_n)}{f'(p_n)}$$

when $p_0 = 0.1$.

(d) Compute $|p - p_{n+1}|/|p - p_n|^2$ for $n = 0, 1, 2$ using knowledge of the exact root location $p = 0$. What number does it look like it is converging to?