Homework #7

1. (a) Suppose we want a degree d piecewise polynomial that interpolates data points $(x_0, f(x_0))$, $(x_1, f(x_1))$,..., $(x_n, f(x_n))$ and whose first to fourth derivatives are continuous. What is the smallest d that can be used?

   (b) How many further conditions can we prescribe for this choice of d so that there will be a unique solution?

2. Let

$$S(x) = \begin{cases} S_0(x) = a + bx + cx^2 + dx^3, & \text{if } 0 \le x \le 2, \\ S_1(x) = A + Bx + Cx^2 + Dx^3, & \text{if } 2 \le x \le 4, \end{cases}$$

be a natural cubic spline on $[0, 4]$ interpolating the data points $(0, 3), (2, 4), (4, 2)$. Write down the system of linear equations for the unknowns a, b, c, d, A, B, C, D (but do not solve).

3. Let

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3, & \text{if } 0 \le x \le 1, \\ S_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3, & \text{if } 1 \le x \le 2, \end{cases}$$

be a natural cubic spline on $[0, 2]$. Find the constants b, c, d.

4. Let

$$S(x) = \begin{cases} S_0(x) = a + bx + cx^2 + dx^3, & \text{if } 0 \le x \le 1, \\ S_1(x) = 1 + (x-1) + (x-1)^2 + (x-1)^3, & \text{if } 1 \le x \le 2, \end{cases}$$

be a clamped cubic spline on $[0, 2]$ with boundary conditions $S'(0) = -1$ and $S'(2) = 6$. Find a, b, c, d.

5. Consider the underlying function $f(x) = \sin x$ and nodes $x_0, x_1, \ldots, x_n \in [0, 2\pi]$ satisfying $x_j = 2\pi j/n$. Using error bounds, determine n such that the absolute error between $f(x)$ and the clamped cubic spline interpolant $S(x)$ of $f(x)$ using the given nodes will be less than 10^{-10}.

6. Answer True or False for the following questions. You do not need to show your work but if you are not sure, write an explanation.

   (a) Suppose $S(x)$ is the clamped cubic spline for data coming from the underlying function $f(x)$ in the interval $[0, 3]$ with nodes at $x = 0, 1, 2, 3$. Suppose $H(x)$ is the piecewise cubic Hermite interpolating polynomial for data also coming from $f(x)$ in the interval $[0, 3]$ with nodes at $x = 0, 1, 2, 3$. Then $S(x) = H(x)$.

   (b) Suppose $S(x)$ is the clamped cubic spline for data coming from the underlying function $f(x)$ in the interval $[0, 3]$ with nodes at $x = 0, 1, 2, 3$. Suppose $H(x)$ is the piecewise cubic Hermite interpolating polynomial for data coming from $S(x)$, the clamped cubic spline, in the interval $[0, 3]$ with nodes at $x = 0, 1, 2, 3$. Then $S(x) = H(x)$.

(c) Suppose $S(x)$ is the natural cubic spline for data coming from the underlying function $f(x)$ with nodes at $x = 0, 2, 4, 6$ and $T(x)$ is the natural cubic spline coming from $f(x)$ with nodes at $x = 0, 2, 4, 6, 8$. Then $S(x) = T(x)$ in $[0, 6]$.

(d) Suppose $H(x)$ is the piecewise cubic Hermite interpolating polynomial for data coming from the underlying function $f(x)$ with nodes at $x = 0, 2, 4, 6$ and $J(x)$ is the piecewise cubic Hermite interpolating polynomial for data coming from $f(x)$ with nodes at $x = 0, 2, 4, 6, 8$. Then $H(x) = J(x)$ in $[0, 6]$.

(e) Suppose $S(x)$ is the natural cubic spline for data coming from the underlying function $f(x)$ with nodes at $x = 0, 2, 4, 6$ and $T(x)$ is the natural cubic spline for data coming from $f(x)$ with nodes at $x = 0, 2, 4, 5$. Then $S(x) = T(x)$ in $[0, 2]$.

(f) Suppose $H(x)$ is the piecewise cubic Hermite interpolating polynomial for data coming from the underlying function $f(x)$ with nodes at $x = 0, 2, 4, 6$ and $J(x)$ is the piecewise cubic Hermite interpolating polynomial for data coming from $f(x)$ with nodes at $x = 0, 2, 4, 5$. Then $H(x) = J(x)$ in $[0, 2]$.

7. Enter Matlab and type:

```
>> help spline
>> help ppval
```

Read these instructions on how the two functions work. Then enter:

```
>> x = 0:.1:1;
>> y = [0.95 0.23 0.6 0.48 0.89 0.76 0.45 0.01 0.82 0.44 0.61];
```

These are the data points. Then enter:

```
>> cs = spline(x,[0 y 0]);
>> plot(x,y,'o')
>> hold on
>> xfine = 0:.005:1;
>> plot(xfine,ppval(cs,xfine))
>> axis([0 1 -3 3])
>> hold off
```

This is the clamped cubic spline with zero derivative at endpoints. Print this plot out and turn it in.