Homework #9

1. Use the Gram-Schmidt process to find the degree 0 to 3 orthogonal polynomials in the interval $[-1, 1]$ with respect to weight $w \equiv 1$ (the Legendre polynomials).

2. Write $p(x) = x^3 + x + 1$ in $[-1, 1]$ as a linear combination of Legendre polynomials

$$p(x) = a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x)$$

   by multiplying both sides by $\phi_j(x)$, integrating, and solving for a_j.

3. Let $f(x) = x^4 + x$ in $[-1, 1]$ and consider the weight function $w \equiv 1$. Use Legendre polynomials for the following:
   (a) Find the best degree 1 polynomial in the least squares sense.
   (b) Find the best degree 2 polynomial in the least squares sense.
   (c) Find the best degree 3 polynomial in the least squares sense.

4. Use the Gram-Schmidt process to find the degree 0 to 1 orthogonal polynomials in the interval $[1, 2]$ with respect to weight $w(x) = x$. Then use these polynomials to construct the best fitting line for the function $f(x) = x^2 + 2x - 3$ in the interval $[1, 2]$.

5. Use the zeros of the degree three monic Chebyshev polynomial to construct an interpolating polynomial of degree two for the following functions in the interval $(-1, 1)$. Also, bound the maximum error of the approximation in $(-1, 1)$:
   (a) $f(x) = e^x$.
   (b) $f(x) = \sin x$.

6. Use the zeros of the degree four monic Chebyshev polynomial to construct an interpolating polynomial of degree three for the following functions in the interval $(-1, 1)$. Also, bound the maximum error of the approximation in $(-1, 1)$:
   (a) $f(x) = e^x$.
   (b) $f(x) = \sin x$.

7. Let

$$p(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

   in the interval $(-1, 1)$.
   (a) Reduce the degree of the polynomial by one using the degree 3 monic Chebyshev polynomial. What is the bound on the maximum error of this approximation in $(-1, 1)$?
   (b) Further reduce the degree of the resulting polynomial by one using the degree 2 monic Chebyshev polynomial. What is the bound on the maximum error of this approximation in $(-1, 1)$?