Math 172 Final

June 16, 2006

- Please put your name, ID number, and sign and date.
- There are 8 problems worth a total of 200 points.
- **You must show your work to receive credit**.

Print Name: ______________________________

Student ID: ______________________________

Signature and Date: ______________________________

Problem	Score
1	/25
2	/25
3	/25
4	/25
5	/25

Problem	Score
6	/25
7	/25
8	/25
Total	/200

1. (25 pts) Write down the PDE the following approximation schemes are consistent with:

(a) $$\frac{U_k^{n+1} - U_k^n}{\Delta t} = 10\frac{U_{k+1}^n - 2U_k^n + U_{k-1}^n}{\Delta x^2} + (U_k^n)^2.$$

(b) $$\frac{U_k^{n+1} - U_k^n}{\Delta t} + 3\frac{U_k^n - U_{k-1}^n}{\Delta x} = 0.$$

(c) $$\frac{U_k^{n+1} - 0.5(U_{k+1}^n + U_{k-1}^n)}{\Delta t} - 3\frac{U_{k+1}^{n+1} - U_{k-1}^{n+1}}{2\Delta x} = \frac{U_{k+1}^{n+1} - 2U_k^{n+1} + U_{k-1}^{n+1}}{\Delta x^2}.$$

(d) $$\frac{U_k^{n+1} - U_k^n}{\Delta t} + \frac{U_{k+1}^n - U_{k-1}^n}{2\Delta x} - \frac{\Delta t}{2}\frac{U_{k+1}^n - 2U_k^n + U_{k-1}^n}{\Delta x^2} = 0.$$

(e) $$\frac{U_k^{n+1} - U_k^n}{\Delta t} = \frac{1}{2}\left[\frac{U_{k+1}^{n+1} - 2U_k^{n+1} + U_{k-1}^{n+1}}{\Delta x^2} + \frac{U_{k+1}^n - 2U_k^n + U_{k-1}^n}{\Delta x^2}\right] + 1.$$

(f) $$\frac{U_{k+1,l} - 2U_{k,l} + U_{k-1,l}}{\Delta x^2} + \frac{U_{k,l+1} - 2U_{k,l} + U_{k,l-1}}{\Delta x^2} - U_{k,l} = \sin x_k \sin y_l.$$

2. (25 pts) Consider the grid with $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$, $x_3 = 3/2$ over the interval $[0, 3/2]$. Suppose $U_k^0 = 2x_k + 1$ for all k.

   (a) Find U_k^2 for all k for the approximation scheme $\dfrac{U_k^{n+1} - U_k^n}{\Delta t} + \dfrac{U_k^n - U_{k-1}^n}{\Delta x} = 0$ where $U_0^n = 1$ for all n and $\Delta t = \Delta x/2$.

(b) Find U_k^1 for all k for the approximation scheme $\dfrac{U_k^{n+1} - U_k^n}{\Delta t} = \dfrac{U_{k+1}^{n+1} - 2U_k^{n+1} + U_{k-1}^{n+1}}{\Delta x^2}$. where $U_0^n = 1, U_3^n = 4$ for all n and $\Delta t = \Delta x$.

3. (25 pts) Use energy estimates to show the problem

$$\begin{cases} u_t - u_x = u_{xx}, & x \in [a,b], t \in [0,T] \\ u(x,0) = u_0(x) & x \in [a,b] \\ u(a,t) = u(b,t) = 0 & t \in [0,T] \end{cases}$$

is well-posed.

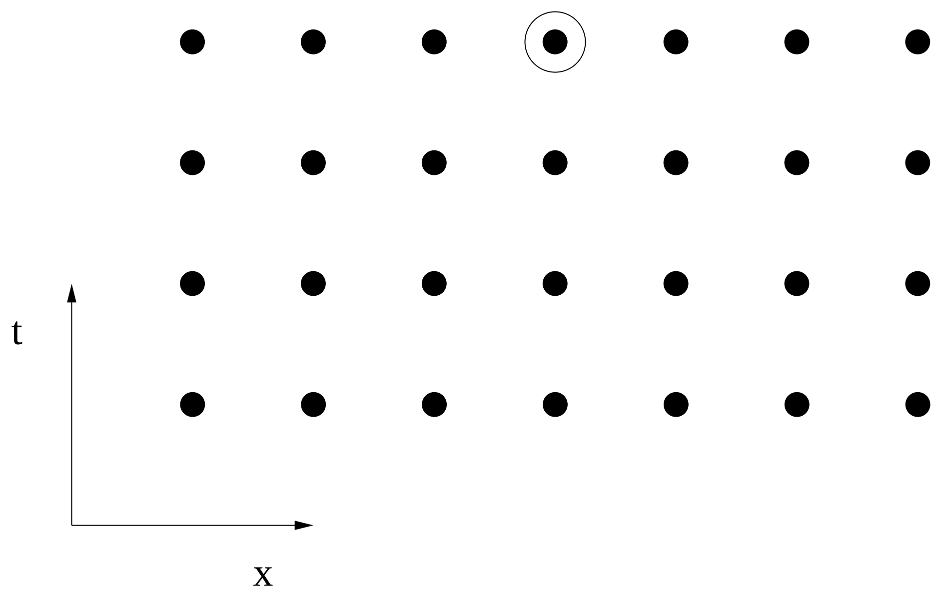
4. (25 pts) Find conditions on Δt so that the approximation scheme

$$\frac{U_k^{n+1} - U_k^n}{\Delta t} = 5\frac{U_{k+1}^n - 2U_k^n + U_{k-1}^n}{\Delta x^2}$$

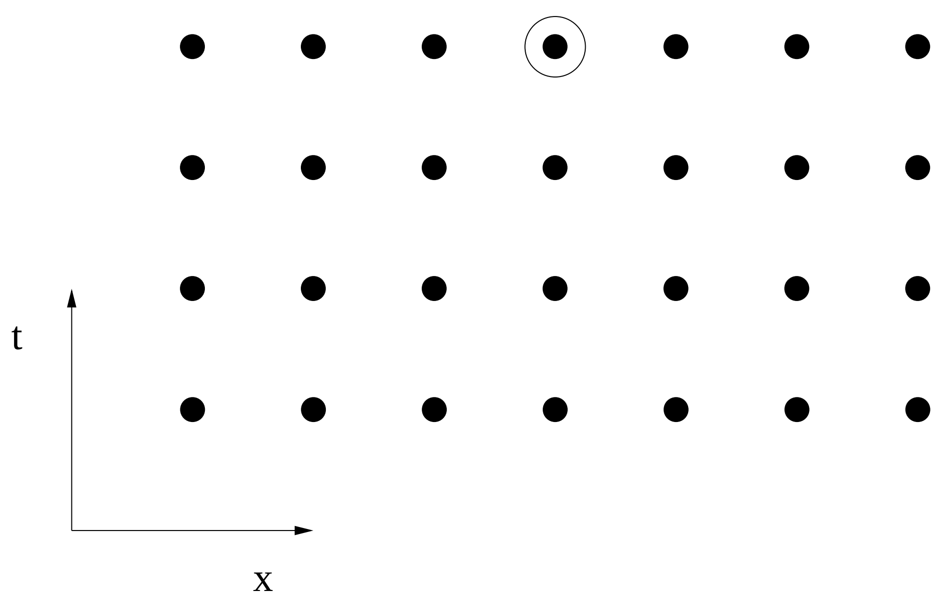
will be stable.

5. (25 pts) Given the following portion of a grid over x-t space and the circled gridpoint of interest, circle the gridpoints that are in the domain of dependence of that gridpoint for the following approximation schemes:

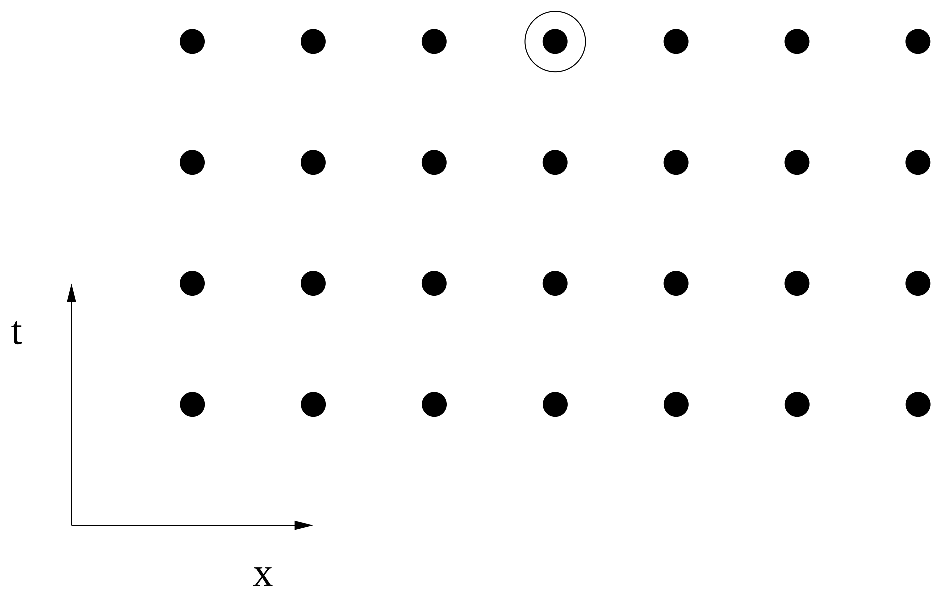
(a) $\dfrac{U_k^{n+1} - U_k^n}{\Delta t} = \dfrac{U_{k+1}^n - 2U_k^n + U_{k-1}^n}{\Delta x^2}.$



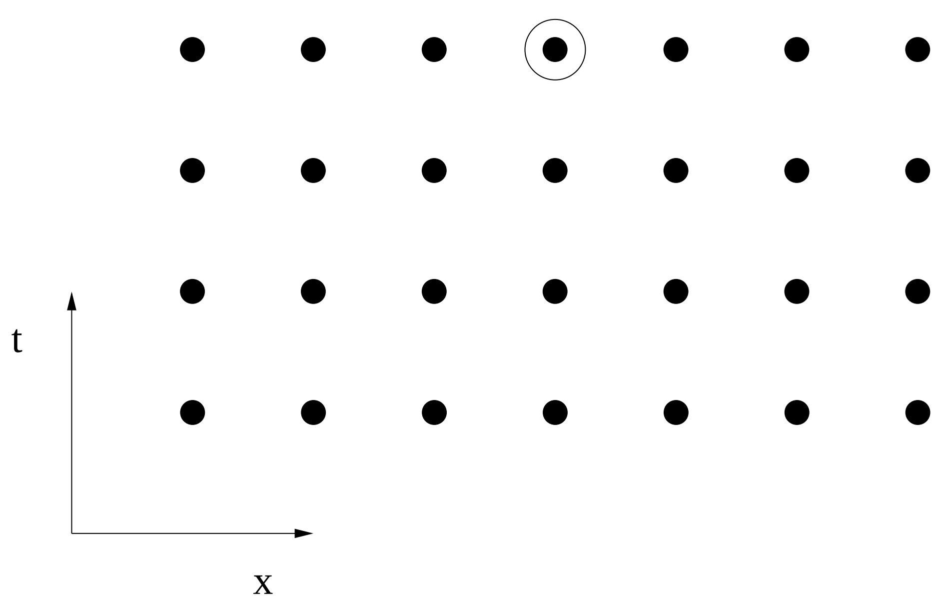
(b) $\dfrac{U_k^{n+1} - U_k^n}{\Delta t} + \dfrac{U_k^n - U_{k-1}^n}{\Delta x} = 0.$



(c) $\frac{U_k^{n+1} - 0.5(U_{k+1}^n + U_{k-1}^n)}{\Delta t} - \frac{U_{k+1}^n - U_{k-1}^n}{2\Delta x} = 0.$



(d) $\frac{U_k^{n+1} - U_k^n}{\Delta t} = \frac{U_{k+1}^{n+1} - 2U_k^{n+1} + U_{k-1}^{n+1}}{\Delta x^2}.$



6. (25 pts)

(a) Consider the grid with

$$a = x_0 < x_1 < \ldots < x_N = b$$

over the interval $[-1, 1]$. Let ϕ_j be the basis function that has value 1 at x_j. Calculate all possible values of

$$\int_a^b \phi_j' \phi_k' \, dx$$

for $1 \leq j, k \leq N - 1$ in terms of $h_l = x_{l+1} - x_l$.

(b) Consider the grid with $x_0 = -1$, $x_1 = 1/2$, $x_2 = 3/4$, $x_3 = -1/2$, $x_4 = -1/4$, $x_5 = 1$ over the interval $[-1, 1]$. Let ϕ_j be the basis function that has value 1 at x_j and let

$$u_h = \alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3 + \alpha_4\phi_4$$

be the approximate solution of the finite element method for the problem

$$\begin{cases} u''(x) = 1, & x \in [-1, 1] \\ u(-1) = u(1) = 0. \end{cases}$$

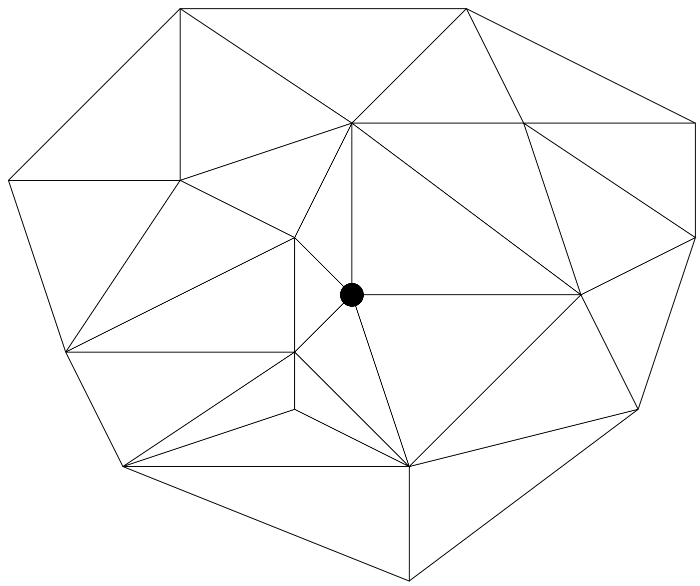
Write down the linear system of equations $A\vec{\alpha} = \vec{b}$, where $\vec{\alpha} = [\alpha_1, \alpha_2, \alpha_3, \alpha_4]^t$.

7. (25 pts) Find the Ritz and Galerkin formulations of the problem

$$\begin{cases} u''(x) - 3u(x) = 2\cos x, & x \in [0, \pi] \\ u(0) = u(\pi) = 0. \end{cases}$$

8. (25 pts) Given a portion of a triangulation and the basis function ϕ_j whose value is 1 at the denoted vertex:

   (a) Shade in the area where ϕ_j is non-zero.



   (b) Circle the vertices whose basis functions ϕ_k (ϕ_k has value 1 at that vertex) have non-zero values of

$$-\int_\Omega \nabla\phi_j \cdot \nabla\phi_k \, dV.$$