**Homework #2**

1. Let $f(x) = \sin x$. So $f'(x) = \cos x$. Consider the approximation scheme

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}.$$

Choose $a = \pi/8$.

(a) Make a table with the approximated values, exact values, and absolute errors when $h = 0.1$, $0.05$, $0.025$, $0.0125$, $0.00625$ are used.

(b) Let $E(h)$ denote the absolute error. Find $E(h)/E(h/2)$ for $h = 0.1$, $0.05$, $0.025$, $0.0125$. What would you guess to be

$$\lim_{h\to 0} \frac{E(h)}{E(h/2)}?$$

(c) How does this agree with the fact that the absolute error should be $\mathcal{O}(h)$?

2. Suppose $|f''(x)| \le M$ for all $x$. Use Taylor series to show that the absolute error of the approximation

$$f'(a) \approx \frac{f(a) - f(a-h)}{h}.$$

is bounded by $Mh/2$.

3. Let $f(x) = e^x \sin(\pi x)$. So $f'(x) = e^x(\sin \pi x + \pi \cos \pi x)$. Consider the approximation scheme

$$f'(a) \approx \frac{f(a) - f(a-h)}{h}.$$

Choose $a = 0.2$.

(a) Make a table with the approximated values, exact values, and absolute errors when $h = 1, 0.1$, $0.01$, $0.001$, $0.0001$, are used.

(b) Let $E(h)$ denote the absolute error. Find $E(h)/E(h/10)$ for $h = 1, 0.1$, $0.01$, $0.001$. What would you guess to be

$$\lim_{h\to 0} \frac{E(h)}{E(h/10)}?$$

(c) How does this agree with the fact that the absolute error should be $\mathcal{O}(h)$?

4. Let $E(h)$ denote absolute error of some approximation of the derivative $f'(a)$ using stepsize $h$. Consider the table:

| $h$ | 1/64 | 1/128 | 1/256 | 1/512 | 1/1024 |
|---|---|---|---|---|---|
| $E(h)$ | 1.2112 | 0.1515 | 0.019 | 0.00238 | 0.0002976 |

What would you guess to be the order of this approximation scheme? (In other words, this approximation is $\mathcal{O}(h^p)$ for what $p$?)

5. Let $f(x) = \cos(x^2)$. So $f'(x) = -2x \sin(x^2)$. Consider the approximation scheme

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}.$$

Choose $a = -3$.

(a) Make a table with the approximated values, exact values, and absolute errors when $h = 0.1$, $0.05$, $0.025$, $0.0125$, $0.00625$ are used.

(b) Let $E(h)$ denote the absolute error. Find $E(h)/E(h/2)$ for $h = 0.1$, $0.05$, $0.025$, $0.0125$. What would you guess to be

$$\lim_{h \to 0} \frac{E(h)}{E(h/2)}?$$

(c) How does this agree with the fact that the absolute error should be $\mathcal{O}(h^2)$?

6. Let $f(x) = x^2 + 3x + 3$. So $f'(x) = 2x + 3$. Consider the approximation scheme

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}.$$

Choose $a = 1$.

(a) Make a table with the approximated values, exact values, and absolute errors when $h = 2$, $1$, $0.5$, $0.25$ are used.

(b) Use the fact that the absolute error of the approximation scheme is

$$\frac{h^2}{12}|f'''(\xi_1) + f'''(\xi_2)|,$$

for some $\xi_1$ and $\xi_2$, to explain why the numbers in part (a) are so small.

7. Consider the table of data for the function $f(x)$:

| $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 1 | 1.008 | 1.064 | 1.216 | 1.512 | 2 |

Make a table for the approximate values of $f'(x)$ at the same $x$ locations using

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}$$

whenever possible and otherwise

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

or

$$f'(a) \approx \frac{f(a) - f(a-h)}{h}.$$

8. Download "approxsinederiv.m" from the course website. You can run the program in Matlab, for example, by

   ```
   >> approxsinederiv(10);
   ```

   and try other numbers instead of 10 (for example, 20 or 100). By looking at the code and its results, explain what the program is doing.