Homework #3

1. Find an approximation scheme for $f'(a)$ using the stencil $a-h, a, a+h$. What is the order of the absolute error?

2. Let $f(x) = \sin x$. So $f''(x) = -\sin x$. Consider second order central differencing,

$$f''(a) \approx \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

   Choose $a = 1$.

   (a) Make a table with the approximated values, exact values, and absolute errors when $h = 0.1$, 0.05, 0.025, 0.0125, 0.00625, are used.

   (b) Let $E(h)$ denote the absolute error. Find $E(h)/E(h/2)$ for $h = 0.1$, 0.05, 0.025, 0.0125. What would you guess to be

$$\lim_{h\to 0} \frac{E(h)}{E(h/2)}?$$

   (c) How does this agree with the fact that the absolute error should be $\mathcal{O}(h^2)$?

3. Find an approximation scheme for $f''(a)$ using the stencil $a, a+h, a+2h$. What is the order of the absolute error?

4. Consider an equally spaced grid with stepsize $h = 0.2$ over the interval $[0, 1]$. For each of the following functions, write down the discretized function as a vector.

   (a) $u(x) = x^2$.

   (b) $u(x) = \sin \pi x$.

   (c) $u(x) = e^x$.

5. Using the fact that

$$\begin{aligned} u_i &= u(x_i) \\ \tfrac{u_{i+1}-u_i}{h} &\approx u'(x_i) \\ \tfrac{u_i-u_{i-1}}{h} &\approx u'(x_i) \\ \tfrac{u_{i+1}-u_{i-1}}{2h} &\approx u'(x_i) \\ \tfrac{u_{i+1}-2u_i+u_{i-1}}{h^2} &\approx u''(x_i) \end{aligned}$$

   Determine what the following expressions are approximating as $h \to 0$.

   (a) $4\left(\frac{u_{i+1}-u_{i-1}}{2h}\right)\left(\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}\right) - 3u_i\left(\frac{u_i-u_{i-1}}{h}\right) + x_i^3 - 2x_i + 1$.

   (b) $2\left(\frac{u_{i+1}-u_i}{h}\right)^2 + \frac{1}{2}\left[\frac{u_{i+1}-u_i}{h} + \frac{u_i-u_{i-1}}{h}\right] - \sin x_i$.

   (c) $-\frac{\sqrt{x_i}}{u_i} + h\left(\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}\right) + \frac{u_{i-1}+u_{i+1}}{2}$.

6. Suppose we are considering periodic functions in $[0, 2\pi]$ discretized on a grid with stepsize $\pi/2$. Write down the matrix corresponding to the following operations on when the discretized functions are viewed as vectors.

   (a) first order forward differencing to approximate the first derivative.

   (b) first order backward differencing to approximate the first derivative.

   (c) second order central differencing to approximate the first derivative.

   (d) second order central differencing to approximate the second derivative.

7. Suppose we are considering periodic functions in $[0, 2\pi]$ discretized on a grid with stepsize $\pi/2$. Write down the matrix corresponding to the following operations on when the discretized functions are viewed as vectors.

   (a) first order forward differencing to approximate the first derivative and second order central differencing to approximate the second derivative in $u'' - u'$.

   (b) first order backward differencing to approximate the first derivative and second order central differencing to approximate the second derivative in $u'' + u' + 2u$.

   (c) second order central differencing to approximate the first derivative and second order central differencing to approximate the second derivative in $u'' + u' - u^2$.

8. Consider the ODE

$$\begin{aligned} u' &= -3u + t \quad t \in [1, 2] \\ u(1) &= -1 \end{aligned}$$

   (a) Use Euler's method:

$$U_{i+1} = U_i + h(-3U_i + t_i)$$

   with stepsize $h = 0.2$ to write down the discretized function approximating the solution u as a vector.

   (b) Count the number of additions/subtractions you had to perform in part (a).

9. Download "eulerproblem.m" from the course website. You can run the program in Matlab, for example, by

   >> eulerproblem(5)

   and try other numbers instead of 5. Explain what the program is doing. Also, by continually halving the stepsize, for example, running

   >> eulerproblem(10)

   >> eulerproblem(20)

   and so on, write down the resulting absolute errors and guess its order.