Homework #4

1. Consider the ODE

$$u' = -3u + t \quad t \in [1,2]$$
$$u(1) = -1$$

   (a) Use Backward Euler's method:

$$\frac{U_{i+1} - U_i}{h} = -3U_{i+1} + t_{i+1}$$

   with stepsize $h = 0.2$ to write down the discretized function approximating the solution u as a vector.

2. Draw the grid where $x \in [-1,1]$ and $t \in [0,1]$ with stepsizes $\Delta x = 0.2$ and $\Delta t = 0.1$ and circle the points corresponding to (x_0, t_4), (x_3, t_5), (x_7, t_4), and (x_9, t_6).

3. Lay down a grid for $x \in [-1,1]$ and $t \in [0,1]$ with stepsizes $\Delta x = 0.2$ and $\Delta t = 0.1$. For each of the following functions $u(x,t)$, find the values of u_0^4, u_3^5, u_7^4, and u_9^6.

   (a) $u(x,t) = 2x - t$.

   (b) $u(x,t) = 1 + x^2 + t^2$.

   (c) $u(x,t) = e^t \sin(\pi x)$.

4. Let $u(x,t) = \sqrt{x^2 + t^2}$. Approximate the following derivatives.

   (a) $u_x(1,2)$ using first order forward differencing with stepsize 0.01.

   (b) $u_x(1,2)$ using first order backward differencing with stepsize 0.01.

   (c) $u_x(1,2)$ using second order central differencing with stepsize 0.01.

   (d) $u_{xx}(1,2)$ using second order central differencing with stepsize 0.01.

5. Let $u(x,t) = \sqrt{x^2 + t^2}$. Approximate the following derivatives.

   (a) $u_t(1,2)$ using first order forward differencing with stepsize 0.01.

   (b) $u_t(1,2)$ using first order backward differencing with stepsize 0.01.

   (c) $u_t(1,2)$ using second order central differencing with stepsize 0.01.

   (d) $u_{tt}(1,2)$ using second order central differencing with stepsize 0.01.

6. Consider the discretized u_j^n corresponding to the function $u(x,t)$ at a point (x_j, t_n) of a grid. For the following PDE's in u at (x_j, t_n), find an approximate equation in terms of the discretized values of u_j^n.

   (For example, if for $u_t - u_x = x$, at (x_j, t_n) we may write down the approximate equation

$$\frac{u_j^n - u_j^{n-1}}{\Delta t} - \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = x_i$$

   using first order backward differencing for the time derivative and second order central differencing for the space derivative.)

(a) $u_t + 2u_x = u^2$.

(b) $u_t = 5u_{xx} + \sin x$.

(c) $u_{tt} = 4u_{xx} - e^x$.

7. Consider the PDE

$$u_t = u_{xx}$$

for $x \in [0,1]$ and $t \in [0,1]$. Consider the initial condition $u(x,0) = \cos 2\pi x$ and the boundary conditions $u(0,t) = u(1,t) = 1$. Let U_j^n be our approximation over the grid using $\Delta x = 0.2$ and $\Delta t = 0.02$, with $j = 0,1,2,3,4,5$ and $n = 0,1,\ldots,50$.

- What can we take for the values of U_j^0? What about U_0^n and U_J^n?
- Write down the approximation scheme using first order forward differencing in time and second order central differencing in space at the point (x_j, t_n).
- Calculate U_j^1 for $j = 1,2,3,4$.

8. Download "pdeproblem.m" from the course website. You can run the program in Matlab, for example, by

```
>> [x,U] = pdeproblem(5,20);
```

to get the approximation U to the PDE in the previous problem. You can plot U at different times by

```
>> plot(x,U(1,:))
>> plot(x,U(2,:))
```

or even

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>> plot(x,U(end,:))
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Run the program with a grid with 50 subdivisions for 1000 time steps and print out the plot of the solution at the end time.

9. Modify your Matlab program in the previous problem to use $\Delta t = \Delta x$ instead of $\Delta x^2/2$. Observe the graphs of U_j^n for $n = 0,1,2,\ldots,10$. What happens to the approximation?

10. Modify your Matlab program in the previous problem back to using $\Delta t = \Delta x^2/2$. Then use the approximation

$$\frac{U_{j+2}^n - 2U_{j+1}^n + U_j^n}{\Delta x^2}$$

for the space derivative u_{xx} (your result in HW#3, problem #3). Also set $U_j^n = 1$ whenever $j > N$. Observe the graphs of U_j^n for $n = 0,1,2,\ldots,10$. What happens to the approximation?