Homework #5

1. Draw the stencils for the following approximation equations.

   (a)
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}.$$

   (b)
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - U_j^n}{\Delta x}.$$

   (c)
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_j^n - U_{j-1}^n}{\Delta x}.$$

   (d)
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x}.$$

2. Draw the domains of dependence on a small grid for the following approximation schemes.

   (a) Using $\Delta t = \Delta x$ in the grid,
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}.$$

   (b) Using $\Delta t = \Delta x/2$ in the grid,
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - U_j^n}{\Delta x}.$$

   (c) Using $\Delta t = 2\Delta x$ in the grid,
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_j^n - U_{j-1}^n}{\Delta x}.$$

   (d) Using $\Delta t = \Delta x$ in the grid,
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x}.$$

3. Using domains of dependence, argue that the following approximation schemes will not be convergent for the PDE
$$u_t + 2u_x = 0$$
with $u(x, 0) = u_0(x)$ given and whose solution is $u(x, t) = u_0(x - 2t)$.

(a)

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + 2\frac{U_{j+1}^n - U_j^n}{\Delta x} = 0$$

with $\Delta t = \Delta x/2$.

(b)

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + 2\frac{U_j^n - U_{j-1}^n}{\Delta x} = 0$$

with $\Delta t = \Delta x$.

4. Consider the approximation scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} - \frac{U_{j+1}^n - U_j^n}{\Delta x} = 0$$

for the PDE

$$u_t - u_x = 0$$

with $\Delta t = \Delta x/2$. This approximation scheme is convergent.

(a) Write and simplify the expression for U_j^{n+1} in terms of U_j^n and U_{j+1}^n.

(b) Write U_0^{n+1} in terms of U_j^n, for $j = 0, 1$.

(c) Write U_0^{n+1} in terms of U_j^{n-1}, for $j = 0, 1, 2$.

(d) Write U_0^{n+1} in terms of U_j^{n-2}, for $j = 0, 1, 2, 3$.

(e) What do you guess is the coefficient of U_0^0 when U_0^{n+1} is written in terms of U_j^0, for $j = 0, 1, \ldots, n+1$? What does this coefficient go to when $n \to \infty$?

5. Consider the PDE

$$u_t = u_{xx}$$

for $x \in [0, 1]$ and $t \in [0, 1]$. Consider the initial condition $u(x, 0) = \cos 2\pi x$ and the boundary conditions $u(0, t) = u(1, t) = 1$. Let U_j^n be our approximation over the grid using $\Delta x = 1/3$ and $\Delta t = 1/3$, with $j = 0, 1, 2, 3$ and $n = 0, 1, 2, 3$.

(a) Write down the approximation scheme using first order backward differencing in time and second order central differencing in space at the point (x_j, t_n).

(b) Draw the stencil for this approximation scheme.

(c) Use the approximation scheme to calculate U_j^1 for $j = 1, 2$.

6. Consider the PDE

$$u_t + u_x = 0$$

for $x \in [0, 1]$ and $t \in [0, 1]$. Consider the initial condition $u(x, 0) = \cos 2\pi x$ and the boundary condition $u(0, t) = 1$. Let U_j^n be our approximation over the grid using $\Delta x = 0.2$ and $\Delta t = 0.2$, with $j = 0, 1, 2, 3, 4, 5$ and $n = 0, 1, 2, 3, 4, 5$.

(a) Write down the approximation scheme using first order forward differencing in time and first order backward differencing in space at the point (x_j, t_n).

(b) Draw the stencil for this approximation scheme.

(c) Use the approximation scheme to calculate U_j^1 for $j = 1, 2, 3, 4$.

7. Write a Matlab program that can solve problem #5 using instead periodic boundary conditions, $\Delta t = \Delta x$, and total number of subdivisions $J = 50$ in space for 50 time steps. Plot the final approximation.

8. Write a Matlab program that can solve problem #6 using instead $\Delta t = \Delta x/2$ and total number of subdivisions $J = 50$ in space for 50 time steps. Plot the final approximation.

9. Write a Matlab program that uses first order forward differencing in time and second order central differencing in space to solve problem #6 with instead periodic boundary conditions, $\Delta t = \Delta x$, and total number of subdivisions $J = 50$ in space. Look at the approximation at a few time steps. Is this approximation scheme convergent? Does the domain of dependence argument contradict this?