Homework #5

1. Draw the stencils for the following approximation equations.

   (a) \[ \frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} \]

   (b) \[ \frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - U_{j}^n}{\Delta x} \]

   (c) \[ \frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j}^n - U_{j-1}^n}{\Delta x} \]

   (d) \[ \frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^{n+1} - U_{j-1}^n}{2\Delta x} \]

2. Draw the domains of dependence on a small grid for the following approximation schemes.

   (a) Using \( \Delta t = \Delta x \) in the grid,

   \[ \frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} \]

   (b) Using \( \Delta t = \Delta x/2 \) in the grid,

   \[ \frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - U_{j}^n}{\Delta x} \]

   (c) Using \( \Delta t = 2\Delta x \) in the grid,

   \[ \frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j}^n - U_{j-1}^n}{\Delta x} \]

   (d) Using \( \Delta t = \Delta x \) in the grid,

   \[ \frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^{n+1} - U_{j-1}^n}{2\Delta x} \]

3. Using domains of dependence, argue that the following approximation schemes will not be convergent for the PDE

   \[ u_t + 2u_x = 0 \]

   with \( u(x, 0) = u_0(x) \) given and whose solution is \( u(x, t) = u_0(x - 2t) \).
(a) \[
\frac{U_{j}^{n+1} - U_{j}^{n}}{\Delta t} + \frac{U_{j+1}^{n} - U_{j}^{n}}{\Delta x} = 0
\]
with \(\Delta t = \Delta x/2\).

(b) \[
\frac{U_{j}^{n+1} - U_{j}^{n}}{\Delta t} + \frac{U_{j}^{n} - U_{j-1}^{n}}{\Delta x} = 0
\]
with \(\Delta t = \Delta x\).

4. Consider the approximation scheme
\[
\frac{U_{j}^{n+1} - U_{j}^{n}}{\Delta t} - \frac{U_{j+1}^{n} - U_{j}^{n}}{\Delta x} = 0
\]
for the PDE
\[u_t - u_x = 0\]
with \(\Delta t = \Delta x/2\). This approximation scheme is convergent.

(a) Write and simplify the expression for \(U_{j}^{n+1}\) in terms of \(U_{j}^{n}\) and \(U_{j+1}^{n}\).

(b) Write \(U_{0}^{n+1}\) in terms of \(U_{j}^{n}\), for \(j = 0, 1\).

(c) Write \(U_{0}^{n+1}\) in terms of \(U_{j}^{n-1}\), for \(j = 0, 1, 2\).

(d) Write \(U_{0}^{n+1}\) in terms of \(U_{j}^{n-2}\), for \(j = 0, 1, 2, 3\).

(e) What do you guess is the coefficient of \(U_{0}^{0}\) when \(U_{0}^{n+1}\) is written in terms of \(U_{j}^{0}\), for \(j = 0, 1, \ldots, n + 1\)? What does this coefficient go to when \(n \to \infty\)?

5. Consider the PDE
\[u_t = u_{xx}\]
for \(x \in [0, 1]\) and \(t \in [0, 1]\). Consider the initial condition \(u(x, 0) = \cos 2\pi x\) and the boundary conditions \(u(0, t) = u(1, t) = 1\). Let \(U_{j}^{n}\) be our approximation over the grid using \(\Delta x = 1/3\) and \(\Delta t = 1/3\), with \(j = 0, 1, 2, 3\) and \(n = 0, 1, 2, 3\).

(a) Write down the approximation scheme using first order backward differencing in time and second order central differencing in space at the point \((x_j, t_n)\).

(b) Draw the stencil for this approximation scheme.

(c) Use the stencil for this approximation scheme to calculate \(U_{j}^{1}\) for \(j = 1, 2\).

6. Consider the PDE
\[u_t + u_x = 0\]
for \(x \in [0, 1]\) and \(t \in [0, 1]\). Consider the initial condition \(u(x, 0) = \cos 2\pi x\) and the boundary condition \(u(0, t) = 1\). Let \(U_{j}^{n}\) be our approximation over the grid using \(\Delta x = 0.2\) and \(\Delta t = 0.2\), with \(j = 0, 1, 2, 3, 4, 5\) and \(n = 0, 1, 2, 3, 4, 5\).
(a) Write down the approximation scheme using first order forward differencing in
time and first order backward differencing in space at the point $(x_j, t_n)$.

(b) Draw the stencil for this approximation scheme.

(c) Use the approximation scheme to calculate $U_j^1$ for $j = 1, 2, 3, 4$.

7. Write a Matlab program that can solve problem #5 using instead periodic boundary
conditions, $\Delta t = \Delta x$, and total number of subdivisions $J = 50$ in space for 50 time
steps. Plot the final approximation.

8. Write a Matlab program that can solve problem #6 using instead $\Delta t = \Delta x/2$ and total
number of subdivisions $J = 50$ in space for 50 time steps. Plot the final approximation.

9. Write a Matlab program that uses first order forward differencing in time and second
order central differencing in space to solve problem #6 with instead periodic boundary
conditions, $\Delta t = \Delta x$, and total number of subdivisions $J = 50$ in space. Look at the
approximation at a few time steps. Is this approximation scheme convergent? Does
the domain of dependence argument contradict this?