Homework #6

1. Consider the approximation scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{U_j^n - U_{j-1}^n}{\Delta x} = 0$$

for the transport equation $u_t + u_x = 0$. Choosing $\Delta t = \Delta x$ and given that this approximation scheme is stable, explain why the approximation in the two-norm is first order accurate.

2. Consider the approximation scheme

$$\frac{U_j^n - U_j^{n-1}}{\Delta t} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}$$

for the heat equation $u_t = u_{xx}$. Choosing $\Delta t = \Delta x$ and given that this approximation scheme is stable, explain why the approximation in the two-norm is first order accurate.

3. Plug in $U_j^n = g^n e^{ij\theta}$ and solve for g in the approximation scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + 4\frac{U_j^n - U_{j-1}^n}{\Delta x} = 0.$$

4. Plug in $U_j^n = g^n e^{ij\theta}$ and solve for g in the approximation scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + 4\frac{U_{j+1}^n - U_j^n}{\Delta x} = 0.$$

5. Plug in $U_j^n = g^n e^{ij\theta}$ and solve for g in the approximation scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + 4\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0.$$

6. Consider the approximation scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = 5\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}$$

for the heat equation $u_t = 5u_{xx}$.

   (a) Plug in $U_j^n = g^n e^{ij\theta}$ and solve for g.

   (b) Considering the condition that $\Delta t = C_0 \Delta x^2$, find which C_0 allow the approximation scheme to be stable and thus convergent.

   (c) Explain why the approximation in the two-norm is second order accurate.

7. Consider the approximation scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0.$$

for the transport equation $u_t + u_x = 0$.

(a) Plug in $U_j^n = g^n e^{ij\theta}$ and solve for g.

(b) Considering the condition that $\Delta t = C_0 \Delta x$, show there is no $C_0 > 0$ that allows the approximation scheme to be stable and thus convergent.

8. Consider the approximation scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = 5\frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Delta x^2}$$

for the heat equation $u_t = 5u_{xx}$.

(a) Plug in $U_j^n = g^n e^{ij\theta}$ and solve for g.

(b) Considering the condition that $\Delta t = C_0 \Delta x$, show all $C_0 > 0$ allow the approximation scheme to be stable and thus convergent.