Homework #7

1. Consider $u_t = 2u_{xx}$ and $v_t = v_{xx}$ with $x \in [-1, 1]$ and Dirichlet boundary conditions $u(0,t) = u(1,t) = v(0,t) = v(1,t) = 0$ and initial conditions $u(x,0) = v(x,0) = \sin \pi x$.

   (a) Show $u(x,t) = v(x,2t)$.

   (b) If we use first order forward differencing in time and second order central differencing in space to solve for $u(x,t)$ using stepsize $\Delta x = 0.01$, what is the largest we can safely take Δt? What about for $v(x,t)$?

   (c) If we want to approximate $u(x,1)$, how many time steps will it take using the above Δt? If we want to approximate $v(x,2)$, how many time steps will it take using the above Δt?

2. Consider $u_t = u_{xx}$ with $x \in [-\pi, \pi]$ and Dirichlet boundary conditions $u(-\pi, t) = u(\pi, t) = 0$ and initial condition $u(x, 0) = \sin x$.

   (a) Verify the exact solution is $u(x,t) = e^{-t} \sin x$.

   (b) Use first order forward differencing in time and second order central differencing in space with $\Delta x = 2\pi/25$ and $\Delta t = \Delta x^2/2$ in Matlab to plot the solution after 20 time steps.

   (c) Compute the infinity norm error of your approximation (maximum of the absolute errors at all gridpoints). Call it $E(\Delta x)$.

   (d) Compute the infinity norm errors $E(\Delta x/2)$ and $E(\Delta x/4)$ for approximations run up to the same time (80 time steps and 320 time steps, respectively).

   (e) From these numbers, what do you guess to be the order of accuracy of the approximation scheme?

3. Consider $u_t = u_{xx}$ with $x \in [-\pi, \pi]$ and von Neumann boundary conditions $u_x(-\pi, t) = u_x(\pi, t) = 0$ and initial condition $u(x, 0) = \sin x$. Use first order forward differencing in time and second order central differencing in space with $\Delta x = 2\pi/100$ and $\Delta t = \Delta x^2/2$ and $U_{-1}^n = U_0^n$ and $U_{J+1}^n = U_J^n$ in Matlab to plot the solution after 400 time steps.

4. Consider $u_t = 4u_{xx} + f(x)$ with $x \in [-\pi, \pi]$ and $f(x) = x^2$ and Dirichlet boundary conditions $u(-\pi, t) = u(\pi, t) = 0$ and initial condition $u(x, 0) = \sin x$. Use the approximation scheme

$$U_j^{n+1} = U_j^n + 4\Delta t \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} + x_j^2 \Delta t$$

   with $\Delta x = 2\pi/100$ and $\Delta t = \Delta x^2/8$ in Matlab to plot the solution after 400 time steps.

5. Consider $u_t + 2u_x = 1$ with $x \in [-\pi, \pi]$ and Dirichlet boundary condition $u(-\pi, t) = 0$ and initial condition $u(x, 0) = \sin x$. Use the approximation scheme

$$U_j^{n+1} = U_j^n - 2\Delta t \frac{U_j^n - U_{j-1}^n}{\Delta x} + \Delta t$$

with $\Delta x = 2\pi/100$ and $\Delta t = \Delta x/4$ in Matlab to plot the solution after 200 time steps.