1. Consider the Jordan block $J_k(\lambda)$.
   (a) Find $J_k(\lambda)^m$, for integers $m \geq 1$.
   (b) For $m \geq 1$ an integer, what is rank($J_k(\lambda)^m$) when $\lambda = 0$? What about when $\lambda \neq 0$?
   (c) A matrix $A = (a_{ij}) \in M_n$ is called convergent if $\lim_{m \to \infty} (A^m)_{ij} = 0$ for all $i = 1, \ldots, n$, $j = 1, \ldots, n$. Find necessary and sufficient conditions for $A = J_k(\lambda)$ to be convergent, for $k \geq 1$ an integer. What about when $A$ is a direct sum of Jordan blocks?

2. Consider square matrices that are direct sums of Jordan blocks, and only have one distinct eigenvalue of 2, with $am(2) = 5$ and $gm(2) = 3$.
   (a) Find all such matrices, up to similarity.
   (b) For each of your matrices, $A$, compute rank($(A - 2I)^m$), for $m = 1, 2, \ldots$.

3. A matrix $A$ is idempotent if $A^2 = A$.
   (a) Prove a Jordan block $J_k(\lambda)$ is idempotent if and only if $k = 1$ and $\lambda = 0$ or 1.
   (b) Prove a matrix $A \in M_n$ is idempotent if and only if it is similar to 
   $$
   \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.
   $$

4. Let $A \in M_n$. Prove $A$ can be written in the form $A = B + C$, where $B$ is diagonalizable, $C$ is nilpotent, and $BC = CB$.

5. Use Jordan canonical form to prove a matrix in $M_n$ whose eigenvalues are all real is similar to a matrix in $M_n(\mathbb{R})$.

6. Given $\alpha \neq 0$, use diagonal matrices in a similarity transform to show the Jordan block $J_k(\lambda)$ is similar to $B = (b_{ij}) \in M_k$, defined by
   $$
   b_{ij} = \begin{cases} 
   \lambda, & \text{if } i = j \\
   \alpha, & \text{if } i + 1 = j \\
   0, & \text{otherwise.}
   \end{cases}
   $$
   What if $\alpha = 0$?

7. Let $A \in M_n$ and suppose there exists nonsingular $X \in M_n$ such that $X^{-1}AX$ is a direct sum of matrices: $Y_1 \oplus \ldots \oplus Y_s$, for some $s$, where $Y_j \in M_{k_j}$. Also partition $X = [X_1 \ldots X_s]$, where $X_j \in M_{n,k_j}$.
   (a) Let $\mathcal{B} = \text{set of columns of } X_j$, and let $v \in \text{span}(\mathcal{B})$. Find $[Av]_\mathcal{B}$ in terms of $[v]_\mathcal{B}$. 

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(b) Suppose further that each $Y_j$ happens to be a Jordan block $J_{k_j}(\mu_j)$, for some $\mu_j \in \mathbb{C}$, and partition $X_j = [x^{(j)}_1 \ldots x^{(j)}_{k_j}]$, where $x^{(j)}_k \in \mathbb{C}^n$, for $k = 1, \ldots, k_j$. Prove $x^{(j)}_1$ is an eigenvector of $A$. Also find an expression for $Ax^{(j)}_{k+1}$ in terms of $x^{(j)}_k$ and $x^{(j)}_{k+1}$, for $k = 1, \ldots, k_j - 1$. 