Homework #7

* In this assignment, given a matrix $A \in M_n$, $\lambda_1(A), \ldots, \lambda_n(A)$ denote the eigenvalues of $A$, including multiplicity, and ordered as

$$\lambda_1(A) \leq \ldots \leq \lambda_n(A).$$

1. Let $A \in M_n$ be Hermitian. Prove

$$\lambda_k(A) = \min_{S \subseteq \mathbb{C}^n \text{ subspace}, \dim(S) = k} \left( \max_{x \in S, x \neq 0} \frac{x^*Ax}{x^*x} \right).$$

2. (a) Suppose $A \in M_n$ is Hermitian, and

$$A = \begin{bmatrix} B & C \\ C^* & D \end{bmatrix},$$

where $B \in M_m$. Prove

$$\lambda_k(A) \leq \lambda_k(B),$$

for $k = 1, \ldots, m$. (So, together with the result from class, we have

$$\lambda_k(A) \leq \lambda_k(B) \leq \lambda_{n-m+k}(A),$$

for $k = 1, \ldots, m$)

(b) Let $A \in M_n$ be Hermitian. Conclude that if $U \in M_{n,m}$ has orthonormal columns, then

$$\lambda_k(A) \leq \lambda_k(U^*AU) \leq \lambda_{n-m+k}(A),$$

for $k = 1, \ldots, m$.

3. Let $A, B \in M_n$ be Hermitian, and let $C = A + B$.

(a) Suppose we know $\lambda_k(A) + \lambda_1(B) \leq \lambda_k(C)$, for $k = 1, \ldots, n$. Use $A = C + (-B)$ to prove

$$\lambda_k(C) \leq \lambda_k(A) + \lambda_n(B),$$

for $k = 1, \ldots, n$. (So, together with the result from class, we have

$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(C) \leq \lambda_k(A) + \lambda_n(B),$$

for $k = 1, \ldots, n$)

(b) Let $z \in \mathbb{C}^n$. Conclude that $C = A - zz^*$ implies

$$\lambda_k(A) - z^*z \leq \lambda_k(C) \leq \lambda_k(A),$$

for all $k = 1, \ldots, n$. What about if $C = A + zz^*$?
4. Let $V$ be a finite dimensional inner product space over the field $\mathcal{F}$, where $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$, and let $S \subseteq V$ be a subspace of dimension $k \geq 1$. We want to prove, given $z \in V$, $\dim(S \cap \{z\}^\perp) \geq k - 1$.

(a) If $z = 0$, conclude that the proof is complete. If $z \neq 0$, and $\{x_1, \ldots, x_k\}$ is a basis for $S$, conclude, if $<x_j, z> = 0$, for all $j = 1, \ldots, k$, that the proof is complete.

(b) For the case $z \neq 0$ and there exists an $i \in \{1, \ldots, k\}$ such that $<x_i, z> \neq 0$, prove, for each $j \in \{1, \ldots, k\} \setminus \{i\}$, there exists a scalar $\alpha_j \in \mathcal{F}$ such that $<x_j - \alpha_j x_i, z> = 0$.

(c) Continuing, prove $x_j - \alpha_j x_i$, for $j \in \{1, \ldots, k\} \setminus \{i\}$, are linearly independent. Then conclude that the proof is complete.

(d) What is $\dim(S \cap \{z\}^\perp)$ in the case $z \in S, z \neq 0$?

5. Let $A \in M_n$ be Hermitian, and let $z \in \mathbb{C}^n$.

(a) Prove if $C = A \pm zz^*$, then $\lambda_k(A) \leq \lambda_{k+1}(C)$, for $k = 1, \ldots, n - 1$. (So, together with the result from class, we have $\lambda_{k-1}(C) \leq \lambda_k(A) \leq \lambda_{k+1}(C)$, for $k = 2, \ldots, n - 1$)

(b) Using $A = C \mp zz^*$, prove $\lambda_{k-1}(A) \leq \lambda_k(C) \leq \lambda_{k+1}(A)$, for $k = 2, \ldots, n - 1$. 
