Lecture 19: Derivation of the flow integral. Let S be a closed surface and for each point (x, y, z) on the surface let $f(x, y, z)$ be the rate of flow of fluid out through the surface per unit surface area and unit time, i.e. the flow out of a small area $\triangle S$ during a small time $\triangle t$ is approximately $f \triangle S \triangle t$. The total flow of fluid out from the region enclosed by the surface per unit time is the **surface integral**

$$\iint_S f\, dS = \lim_{\triangle S_{ij} \to 0} \sum_{i,j} f(x_{ij}, y_{ij}, z_{ij}) \triangle S_{ij}$$

where the sum is over a partition of S into smaller surface areas $\triangle S_{ij}$, (x_{ij}, y_{ij}, z_{ij}) is any point in $\triangle S_{ij}$ and we take the limit as the partition becomes finer.

Let us now calculate the rate of flow of fluid f out per unit area and unit time, given the velocity vector field of the fluid $\mathbf{V}$ and the density μ. We define the **flow rate density** by

$$\mathbf{F} = \mu \mathbf{V},$$

If $\triangle S$ is a small area of a piece of a plane with outward unit normal $\mathbf{n}$ then we claim that the flow rate out of $\triangle S$ per unit time is given by

$$\mathbf{F} \cdot \mathbf{n} \triangle S$$

In fact, in a small time $\triangle t$, the fluid particles that will reach $\triangle S$ are at most $\mathbf{V} \triangle t$ away, and all particles within reach form a sloped cylinder with $\triangle S$ as its base and height $\mathbf{V} \cdot \mathbf{n} \triangle t$. Since the volume is the area of the base times the height the amount of fluid in the cylinder is the density times the volume: $\mu \mathbf{V} \cdot \mathbf{n} \triangle t \triangle S$. If we divide by $\triangle t$ we get the rate per unit time and if we divide this by $\triangle S$ we get the flow rate out of per unit surface area and unit time

$$f = \mathbf{F} \cdot \mathbf{n}$$

The flow rate of fluid out of the total surface S, or the **flux** of the velocity vector field $\mathbf{F}$ out of the surface S, with outward unit normal $\mathbf{n}$, is the **surface integral**

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

Ex. Find $\displaystyle\iint_S x\, dS$, where S is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.
Sol. The surface is a piece of a plane $ax+by+cz=d$ and putting in the 3 points we get $a=d$, $b=d$ and $c=d$, e.g. $a=b=c=d=1$. The surface is therefore given by $h(x, y, z) = x + y + z = 1$ and $(x, y) \in D = \{(x, y); x \geq 0, y \geq 0, x + y \leq 1\}$. The normal to the surface is therefore $\mathbf{n} = \nabla h/|\nabla h| = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$. We have

$$dS = \frac{dxdy}{|\cos \gamma|} = \frac{dxdy}{|\mathbf{n} \cdot \mathbf{k}|} = \sqrt{3} dxdy$$

If we rewrite $D = \{(x, y); 0 \leq x \leq 1,\ 0 \leq y \leq 1 - x\}$ we get

$$\iint_S x\, dS = \iint_D x\sqrt{3}\, dxdy = \int_0^1 \int_0^{1-x} x\sqrt{3}\, dy\, dx = \int_0^1 xy\sqrt{3}\Big|_{y=0}^{1-x} dx = \int_0^1 x(1-x)\sqrt{3}\, dx = \frac{\sqrt{3}}{6}$$

Ex. Let S be the part of the hyperboloid $x^2 + y^2 - z^2 = 1$ with $0 \leq z \leq 1$.
A parametrization of the surface is given by
$\mathbf{T}(u, v) = (\cos u - v \sin u)\mathbf{i} + (\sin u + v \cos u)\mathbf{j} + v\mathbf{k}, \qquad 0 \leq u \leq 2\pi, \quad 0 \leq v \leq 1.$
a) Find the area element dS expressed in terms of the parametrization $du\,dv$.
b) Find the surface integral $\iint_S z\,dS$.

Sol. a) $\mathbf{T}_u = (-\sin u - v \cos u)\mathbf{i} + (\cos u - v \sin u)\mathbf{j}$ and $\mathbf{T}_v = -\sin u\mathbf{i} + \cos u\mathbf{j} + \mathbf{k}$;

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u - v\cos u & \cos u - v \sin u & 0 \\ -\sin u & \cos u & 1 \end{bmatrix} = (\cos u - v \sin u)\mathbf{i} + (\sin u + v\cos u)\mathbf{j} - v\mathbf{k}$$

Hence $dS = |\mathbf{T}_u \times \mathbf{T}_v|\,dudv = \sqrt{1 + 2v^2}\,dudv$.

$$b) \iint_S z\,dS = \int_0^1 \int_0^{2\pi} v\sqrt{1 + 2v^2}\,dudv = 2\pi \int_0^1 v\sqrt{1 + 2v^2}\,dv = \pi \frac{(1 + 2v^2)^{3/2}}{3}\Big|_0^1 = \pi \frac{3^{3/2} - 1}{3}$$

Ex Let R be the 3-dimensional region $R = \{x^2/4 + y + z^2/4 \leq 1,\ y \geq 0\}$. Let S be the surface of R with the normal oriented outwards. Note that S has two parts $\{x^2/4 + y + z^2/4 = 1,\ y \geq 0\}$ and $\{y = 0,\ x^2/4 + z^2/4 \leq 1\}$.
a) Find the area of S.
b) Find the flux of $\mathbf{F} = x\mathbf{i} - y\mathbf{j} + \mathbf{k}$ through S; $\iint_S \mathbf{F} \cdot \mathbf{n}\,dS$.

2. Let $S_1 = \{(x, y, z);\ y = 0,\ x^2/4 + z^2/4 \leq 1\}$, $S_2 = \{(x, y, z);\ x^2/4 + y + z^2/4 = 1,\ y \geq 0\}$
The area of S_1 is 4π. S_2 can be viewed as a graph $y = g(x, z) = 1 - x^2/4 - z^2/4$ over the disc $D = \{(x, z);\ x^2/4 + z^2/4 \leq 1\}$ in the xz-plane. With $G(x, y, z) = y - g(x, z)$ the unit normal is $\mathbf{n} = \dfrac{\nabla G}{|\nabla G|} = \dfrac{-g_x\mathbf{i} - g_z\mathbf{k} + \mathbf{j}}{\sqrt{1 + g_x^2 + g_z^2}} = \dfrac{x\mathbf{i}/2 + z\mathbf{k}/2 + \mathbf{j}}{\sqrt{1 + x^2/4 + z^2/4}}$. Now $dS = dxdz/\mathbf{n}\cdot\mathbf{j} = \sqrt{1 + (x^2 + z^2)/4}\,dxdz$. Introducing polar coordinates in the xz-plane:

$$\iint_{S_2} dS = \iint_D \Big(1 + \frac{x^2 + z^2}{4}\Big)^{1/2} dxdz = \int_0^2 \int_0^{2\pi} \Big(1 + \frac{r^2}{4}\Big)^{1/2} d\theta r dr = 2\pi \frac{4}{3}\Big(1 + \frac{r^2}{4}\Big)^{3/2}\Big|_0^2 = 2\pi \frac{4}{3}(2^{3/2} - 1)$$

(b) The normal to S_1 is $\mathbf{n} = -\mathbf{j}$ and there $\mathbf{F} \cdot \mathbf{n} = y = 0$ so the integral over S_1 vanishes. Since $\mathbf{F} \cdot \mathbf{n} = (x^2/2 + z/2 - y)/\sqrt{1 + x^2/4 + z^2/4}$ we obtain

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\,dS = \iint_D (x^2/2 + z/2 - y)dxdz = \iint_D \big(x^2/2 + z/2 - (1 - (x^2 + z^2)/4)\big)\,dxdz$$

Introducing polar coordinates in the xz-plane; $x = r\cos\theta$, $z = r\sin\theta$, $dxdz = rdrd\theta$;

$$\int_0^2 \int_0^{2\pi} \Big(\frac{r^2}{4}(2\cos^2\theta + 1) + \frac{r}{2}\sin\theta - 1\Big) d\theta\,rdr = \int_0^2 \int_0^{2\pi} \Big(\frac{r^2}{4}(\cos 2\theta + 2) - \frac{r}{2}\sin\theta - 1\Big) d\theta\,rdr = \ldots = 0$$

Alternative solution: Let $\mathbf{T}(x, z) = x\mathbf{i} + g(x, z)\mathbf{j} + z\mathbf{k}$ and

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\,dS = \pm \iint_{x^2 + z^2 \leq 4} \mathbf{F} \cdot (\mathbf{T}_x \times \mathbf{T}_z)\,dxdz = \ldots$$