Lecture 9: Section 4.4. The material below only took half a lecture so I also did the Flow rate density and the Flux from the previous lecture.

We define the **curl** of a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ by

$$\mathbf{curl}\,\mathbf{F} = \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\mathbf{i} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\mathbf{j} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\mathbf{k}.$$

One way to remember this formula is that it looks like a cross product:

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix}\mathbf{k},$$

or with the del or nabla notation:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},$$

we symbolically write

$$\mathbf{curl}\,\mathbf{F} = \nabla \times \mathbf{F}.$$

In the same way we can symbolically write

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

and

$$\mathbf{grad}\ f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Ex. 1 Find $\nabla \times \mathbf{F}$ if $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$. **Sol.**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ x & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ x & y \end{vmatrix}\mathbf{k} = \mathbf{0}.$$

Ex. 2 Find $\nabla \times \mathbf{F}$ if $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. **Sol.**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ -y & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ -y & x \end{vmatrix}\mathbf{k} = 2\mathbf{k}.$$

The curl tells us how the vector field "swirls" around. From section 3.2 we know that the flow lines for Ex. 1 are lines going out from the origin where the flow lines for Ex. 2 are circles around the origin. Ex. 2 represents the velocity vector field of a body rotating around the z axis at angular velocity 1. In fact, $\mathbf{R}(t) = r\cos t\mathbf{i} + r\sin t\mathbf{j} + c\mathbf{k}$ represents the rotation of a particle at angular velocity 1 and $\mathbf{R}'(t) = \mathbf{F}(\mathbf{R}(t))$, if $\mathbf{F}$ is the vector field in Ex. 2. Curl is a vector; the magnitude tells us how much it curls and the direction tells us the axis around which it curls. From these examples we might suspect that curl is how much the flow lines curves around, but there is more to it as we shall see.

Ex. 3 Find $\nabla \times \mathbf{F}$ if $\mathbf{F} = (-y\mathbf{i} + x\mathbf{k})/(x^2+y^2)$. **Sol.**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{vmatrix} = \Big[\frac{\partial}{\partial x}\Big(\frac{x}{x^2+y^2}\Big) - \frac{\partial}{\partial y}\Big(\frac{-y}{x^2+y^2}\Big)\Big]\mathbf{k} = ... = \mathbf{0}$$

It is easy to check that for Ex. 3 the flow lines are still circles around the z-axis (in fact the same as Ex. 2.) It appears that our description of the curl as how much the flow lines curves around was not quite sufficient. A more accurate description is that curl is how much the fluid swirls around at a microscopic level at each point. Curl is how much a small paddle wheel "swirls" around its own axis. In Ex. 2 when the paddle wheel have made a complete rotation around the z-axis it has made a complete rotation around its own axis. However, in Ex. 3 when the paddle wheel has made a complete rotation around the z-axis it has in fact not rotated around its own axis. In the first case if you are sitting on the paddle wheel facing away from the origin (or z-axis) you are going to face away from the origin the complete rotation around the origin so you will have made a turn as well. However, in the second case you are during the complete rotation around the origin facing in a fixed direction. The explanation for this is that in the second example the velocity vector field gets stronger as we get closer to the origin so the side of the paddle wheel close to the origin will have larger angular velocity. Consider the following example.

Ex. 4 Find $\nabla \times \mathbf{F}$ if $\mathbf{F} = F_3(y)\mathbf{k}$. **Sol.**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & F_3 \end{vmatrix} = \frac{\partial F_3}{\partial y}\mathbf{i}$$

The curl can hence be non-vanishing even though in this case the flow lines are straight lines parallel to the z-axis. The magnitude of the curl then is simply the derivative of F_3 in the y direction and that curl is non-vanishing simply means that the velocity vector field is larger on one side of the paddle wheel then on the other. A vector field $\mathbf{F}$ is called **irrotational** if $\nabla\times\mathbf{F}=\mathbf{0}$. A gradient $\mathbf{F}=\nabla f$ is irrotational

$$\nabla \times \nabla f = 0 \tag{1}$$

i.e. the curl of the gradient of any scalar field is zero. f is called a **potential**. The converse is also true locally, i.e. an irrotational vector field is a gradient. A vector field $\mathbf{F}$ is called **divergence free** if $\nabla\cdot\mathbf{F}=0$. A curl $\mathbf{F}=\nabla\times\mathbf{G}$ is divergence free;

$$\nabla \cdot (\nabla \times \mathbf{G}) = 0 \tag{2}$$

i.e. the divergence of the curl of any vector field is zero. The converse is also true locally, i.e. a solenoidal vector field is a curl. The proofs of (1) and (2) use the equality of mixed partial derivatives. They should formally be compared to the identities for the cross product and dot products of vectors $\mathbf{v}\times\mathbf{v}=\mathbf{0}$ and $\mathbf{v}\cdot\big(\mathbf{v}\times\mathbf{w}\big)=0$.

Another operator that shows up a lot in the applications is the **Laplacian**

$$\triangle f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

If a vector field $\mathbf{F}$ is irrotational and divergence free then $\mathbf{F}=\nabla f$, where f satisfy

$$\triangle f = 0,$$

The water in the bath tub is typically both divergence free and irrotational except for in the center of the drain.