The Fourier transform

The Fourier transform $\mathcal{F}:\ f \to \hat{f}$ is defined to be

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-ix\cdot\xi}\, dx \tag{1}$$

The Fourier transform is invertible, in fact we will prove Fourier's inversion formula:

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \hat{f}(\xi) e^{ix\cdot\xi}\, dx \tag{2}$$

The Fourier transform makes sense for a very general class of functions and even distributions. However, it is natural to first define it for a more restrictive class and afterwards extend the definition by continuity. This is the Schwartz class $\mathcal{S}$ consisting of all infinitely differentiable functions that are rapidly decreasing:

$$\sup_x |x^\beta \partial^\alpha \phi(x)| < \infty$$

for all multi-indices α and β. The importance of this class is that $\mathcal{F} : \mathcal{S} \to \mathcal{S}$. which follows from the following identities for the Fourier transform:

$$\mathcal{F} : \partial_j f(x) \to i\xi_j \hat{f}(\xi), \quad \mathcal{F} : x_j f(x) \to i\partial_j \hat{f}(\xi) \tag{3}$$

which follows from integrating by parts in (1) respectively differentiating below the integral sign. Note also that by changing variables we get a another simple property

$$\mathcal{F} : f(ax) \to a^{-n} \hat{f}(\xi/a) \tag{4}$$

The proof of (2) uses:

Lemma 1.

$$\mathcal{F} : e^{-|ax|^2/2} \to (2\pi)^{n/2} a^{-n} e^{-|\xi/a|^2/2} \tag{5}$$

Proof. Let $\phi(x) = e^{-|x|^2/2}$. Since $(\partial_j + x_j)\phi(x) = 0$ it follows from (4) that $(i\xi_j + i\partial_j)\hat{\phi}(\xi) = 0$. This differential equation has the solution $\hat{\phi}(\xi) = c_n e^{-|\xi|^2/2}$ so it only remains to calculate c_n. However, by (1) $c_n = \hat{\phi}(0) = \int e^{-|x|^2/2}\, dx$. If $n = 2$ this integral can easily be calculated by introducing polar coordinates $\int_{\mathbf{R}^2} e^{-|x|^2/2}\, dx = 2\pi \int_0^\infty e^{-r^2/2}\, r\, dr = 2\pi$. In general we can write $\int_{\mathbf{R}^n} e^{-|x|^2/2}\, dx = (\int_{\mathbf{R}} e^{-x_1^2/2}\, dx_1)^n$ and $\int_{\mathbf{R}} e^{-x_1^2/2}\, dx_1 = (\int_{\mathbf{R}^2} e^{-|x|^2/2}\, dx)^{1/2}$ so $c_n = (2\pi)^{n/2}$. $\square$

Lemma 2. *If $\phi \in \mathcal{S}$ set $\phi_\varepsilon(x) = \phi(x/\varepsilon)/\varepsilon^n$, then*

$$\int f(x)\, \phi_\varepsilon(x)\, dx = \int f(\varepsilon x)\, \phi(x)\, dx \to f(0) \int \phi(x)\, dx \quad \varepsilon \to 0, \quad \textit{for} \quad f \in \mathcal{S}$$

Proof. Since $|f(\varepsilon x)\, \phi(x)| \le \sup_y |f(y)|\, |\phi(x)|$ the lemma follows from the theorem of Dominated converge. It is also easy to prove directely; since

$|f(\varepsilon x)\,\phi(x) - f(0)\phi(x)| \le \varepsilon \sup_y ||y|f'(y)|\,|\phi(x)|$ the difference of the two integrals is bounded by $C\varepsilon$. $\square$

We also have

$$\int \hat{\phi}\psi\, d\xi = \int \phi\hat{\psi} dx, \quad \phi, \psi \in \mathcal{S} \tag{6}$$

In fact, both sides of (6) are equal to the double integral

$$\iint \phi(x)\psi(\xi)\, e^{-ix\cdot\xi}\, dx d\xi$$

Note now that it sufficies to prove (2) for $x = 0$ since its translation invariant. Using (7) and (5) gives

$$\int \hat{\phi}(x)\;\; f(\varepsilon x)\, dx = \int \phi(\varepsilon\xi)\, \hat{f}(\xi)\, d\xi$$

By Lemma 2 we get as $\varepsilon \to 0$

$$\int \hat{\phi}(x)\, dx\, f(0) = \phi(0) \int \hat{f}(\xi)\, d\xi$$

Picking $\phi(x) = e^{-|x|^2/2}$ we get from Lemma 1 and its proof that $\int \hat{\phi}(x)\, dx = (2\pi)^n$ and Fourier's inversion formula (2) follows. Using Fourier's inversion formula and (6) we get Parseval's formula

$$\int \phi(x)\overline{\psi(x)}\, dx = \int \hat{\phi}(\xi)\overline{\hat{\psi}(\xi)}\, d\xi \tag{7}$$

In particular;

$$\int |\phi(x)|^2\, dx = \frac{1}{(2\pi)^n} \int |\hat{\phi}(\xi)|^2\, d\xi$$

which shows that $\mathcal{F}:\, L^2 \to L^2$. It also follows from (3) that

$$\sum_{i=1}^{n} \int |\partial_i \phi(x)|^2\, dx = \frac{1}{(2\pi)^n} \sum_{i=1}^{n} \int |\xi_i \hat{\phi}(\xi)|^2\, d\xi = \frac{1}{(2\pi)^n} \int |\xi|^2 |\hat{\phi}(\xi)|^2\, d\xi$$

It is now natural to define the Sobolev norms

$$\|\phi\|_{H^s} = \sqrt{\int (1 + |\xi|^2)^s |\hat{\phi}(\xi)|^2\, d\xi} \tag{8}$$

For integer values of s this corresponds to L^2 norms of derivatives of ϕ, but the norm makes sense and is usefull also for real s. This shows that there is a reltion between decay of the Fourier transform and regularity of the function.

Let the convolution be defined by

$$K * g(x) = \int K(y) g(x-y)\, dy = \int K(x-y) g(y)\, dy$$

We have

$$\mathcal{F}:\; f * g \to \hat{f}\hat{g} \tag{9}$$

Some Distribution theory

Let $C_0^\infty(\mathbf{R}^n)$ (or $\mathcal{D}$) denote the set of infinitely differentiable functions that have compact support. The seminorms

$$\rho_{\alpha,K}(\phi) = \sup_{x\in K} |\partial^\alpha \phi(x)|,$$

where K is any compact subset, makes C_0^∞ into a topological space, a Freche' space.

Definition. Let $\mathcal{D}'$ denote the dual space of C_0^∞, i.e. the space of all continuous linear functionals : $C_0^\infty \to \mathbf{C}$. $\mathcal{D}'$ is called the space of distributions.

Recall that a linear map $L : C_0^\infty \to \mathbf{C}$ is continuous means that $L(\phi_n) \to L(\phi)$, if $\phi_n \to \phi$ in C_0^∞. The later statement means exactly that $\rho_{\alpha,K}(\phi_n - \phi) \to 0$ as $n \to \infty$ for every fixed α and K. However, since L is linear it is equivalent to only assume continuity at $\phi = 0$. Moreover, using the principle of uniform boundedness, this is equivalent to that for every compact set K there exists C and N such that

$$|L(\phi)| \le C \sum_{|\alpha|\le N} \sup_{x\in K} |x^\beta \partial^\alpha \phi(x)|, \qquad \operatorname{supp}\phi \subset K$$

If f is a distribution we will write $\langle f, \phi\rangle$ for what we just called $L(\phi)$. A bounded function can be viewed as a distribution given by $\langle f, \phi\rangle = \int f\phi\, dx$. In fact $|\int f\phi\, dx| \le \int |f|\, dx\, \sup_x |\phi(x)|$. Even if f is a distribution which is not a function we will sometimes use $\int f\phi\, dx$ to denote $\langle f, \phi\rangle$, keeping in mind that it is to be interpreted as a continuous linear functional and not an integral.

We have just seen that functions are distributions. Moreover, any derivative of a function is a distribution even if the function is not differentiable in the usual sense. In fact one of the main motivations to introduce distributions is to generalize the concept of derivative to all functions. We simply define the derivative of a distribution f by

$$\int (\partial^\alpha f)\, \phi\, dx = (-1)^{|\alpha|} \int f\, \partial^\alpha \phi\, dx$$

This obviously defines a distribution and it agrees with the usual derivative if f is smooth by integrating by parts.

Moreover, any weak limit of a distribution is a distribution. We say that $f_n \to f$ weakly if

$$\int f_n \phi\, dx \to \int f\phi\, dx, \quad \phi \in \mathcal{S}$$

In fact, any distribution f is the weak limit of a sequence of $f_n \in C_0^\infty$. If f has compact support then the convolution $f_n(x) = \phi_{1/n} * f(x) = \int f(y)\phi_{1/n}(x-y)\, dy$ is well defined if $\phi_\varepsilon(x) = \phi(x/\varepsilon)/\varepsilon^n$ and $\phi \in \mathcal{S}$. Its easy to show that $f_n \in C^\infty$ and that $f_n \to f$ weakly. Moreover, it follows directely from the definitions that $\partial^\alpha f_n \to \partial^\alpha f$ if $f_n \to f$.

The simplest example of a distribution which is not a function is the "delta function" at a $\delta_a(x) = \delta(x-a)$ defined by

$$\int \phi(x)\, \delta_a(x)\, dx = \phi(a), \quad \phi \in \mathcal{S}$$

It is a distribution since it satisfies $|\langle \delta_a, \phi \rangle| \leq \sup_x |\phi(x)|$. Physically one should think of the delta function as a point charge; $\delta_a(x) = \infty$ when $x = a$ and $\delta_a(x) = 0$ when $x \neq a$ in such a way that the total change is $\int \delta_a(x)\,dx = 1$. Another way to think of the delta function is as a limit of a the sequence $\phi_\varepsilon(x) = \phi(x/\varepsilon)/\varepsilon^n$, with $\int \phi\,dx = 1$ as in Lemma 2. A third interpretation of the delta function in one variable is as the derivative of the step function: For $x \in \mathbf{R}$ define the Heavyside function H by $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$. Then in the sense of distributions $H'(x) = \delta(x)$. In fact

$$\int H'(x)\phi(x)\,dx = -\int H(x)\phi'(x)\,dx = -\int_0^\infty \phi'(x)\,dx = \phi(0), \quad \phi \in \mathcal{S}$$

Note that

$$\delta * \phi = \phi$$

Note that multiplication of distributions is not always defined. E.g. we can't multiply $\delta(x)$ with itself. In fact, if this was possible then since $\delta(x)$ is the limit of $\phi_\varepsilon(x) = \phi(x/\varepsilon)/\varepsilon^n$ $\delta(x)^2$ would be the limit of $\phi_\varepsilon(x)^2 = \phi(x/\varepsilon)^2/\varepsilon^{2n}$. However $\int \phi_\varepsilon(x)^2\,dx = \int \phi(x)^2\,dx/\varepsilon^n \to \infty$ as $\varepsilon \to 0$. However, if f is a distribution and u is a smooth function then the product uf makes sense and is defined by $\langle uf,\, \phi\rangle = \langle f,\, u\phi\rangle$.
Problem 1:

$$u(t)\delta(t) = u(0)\delta(t), \quad u(t)\delta'(t) = u(0)\delta'(t) - u'(0)\delta(t)$$

Problem 2: If $t \in \mathbf{R}$ and $f \in C^\infty(\mathbf{R})$, $f(0) = 0$, $f(t) \neq 0$ when $t \neq 0$ and $f'(0) \neq 0$ then

$$\delta(f(t)) = \frac{1}{|f'(0)|}\delta(t)$$

and if $x \in \mathbf{R}^n$

$$\delta(ax) = \frac{1}{|a|^n}\delta(x)$$

The Fourier transform of a distribution f is defined through duality using (7) and the fact that $\mathcal{F} : \mathcal{S} \to \mathcal{S}$:

$$< \hat{f}, \phi > = < f, \hat{\phi} >$$

In order for the right hand side to be well defined we must assume that f is a tempered distribution, i.e. a continuous linear functional on $\mathcal{S}$ with respect to the seminorms

$$\rho_{\alpha,\beta}(\phi) = \sup_x |x^\beta \partial^\alpha \phi(x)|$$

Then

$$\mathcal{F} :\ \delta_a(x) \to e^{-ia\cdot\xi} \tag{10}$$

Some fundamentasl solutions using fourier transform

Let us first consider the transport equation

$$\partial_t u(t,x)+\partial_x u(t,x) = 0, \tag{11}$$
$$u(0,x) = g(x) \tag{12}$$

Let $\hat{u}(t,\xi) = \int u(t,x)e^{-ix\xi}\, dx$ be the Fourier transform of $u(t,x)$ with respect to x for t fixed. Then

$$\partial_t \hat{u}(t,\xi)+i\xi\hat{u}(t,\xi) = 0, \tag{13}$$
$$\hat{u}(0,\xi) = \hat{g}(\xi) \tag{14}$$

Solving the PDE (11)-(12) now reduces to solving the ODE (13)-(14) for fixed ξ:

$$\hat{u}(t,\xi) = e^{-i\xi t}\hat{g}(\xi) \tag{15}$$

Taking the inverse Fourier transform gives

$$u(t,x) = \frac{1}{2\pi}\int e^{ix\xi}e^{-i\xi t}\hat{g}(\xi)d\xi = \frac{1}{2\pi}\int e^{i(x-t)\xi}\hat{g}(\xi)d\xi = g(x-t) \tag{16}$$

One can also obtain the same result directely from (15) using (9) and (10). We have

$$K(t,x) = K_t(x) = \mathcal{F}^{-1}(e^{-it\xi}) = \delta_t(x)$$

and

$$u(t,x) = \mathcal{F}^{-1}(e^{-i\xi t})*\mathcal{F}^{-1}(\hat{g}(\xi)) = K_t * g(x) = \int \delta(y-t)g(x-y)\, dy = g(x-t) \tag{17}$$

Note also that by (15) $|\hat{u}(t,\xi)| = |\hat{g}(\xi)|$ so by Parseval's formula (7) we get the energy identity

$$\int |u(t,x)|^2\, dx = \int |g(x)|^2\, dx$$

and more generally $\|u(t,\cdot)\|_{H^s} = \|g\|_{H^s}$. One can now use K_t also to solve the inhomogeneous problem

$$\partial_t u(t,x)+\partial_x u(t,x) = F(t,x),$$
$$u(0,x) = 0$$

We claim that $u(t,x) = \int_0^t u_s(t,x)\, ds$ where u_s is the solution of

$$\partial_t u_s(t,x)+\partial_x u_s(t,x) = 0,$$
$$u_s(s,x) = g_s(x) = F(s,x)$$

In fact

$$(\partial_t+\partial_x)\int_0^t u_s(t,x)\, ds = u_t(t,x) + \int_0^t (\partial_t+\partial_x)u_s(t,x)\, ds = F(t,x)$$

It follows that is given by a translation of the solution of (11)-(12) so

$$u_s(t,x) = K_{t-s} * g_s(x) = \int_0^t \int K(t-s, x-y)\, F(s,y)\, dy\, ds$$

Let us now look on the heat equation

$$\partial_t u(t,x) - \triangle u(t,x) = 0 \tag{18}$$

$$u(0,x) = g(x) \tag{19}$$

taking the Fourier transform $\hat{u}(t,\xi) = \int u(t,x)\, e^{-ix\cdot\xi}\, dx$ gives

$$\partial_t \hat{u}(t,\xi) + |\xi|^2 \hat{u}(t,\xi) = 0 \tag{20}$$

$$\hat{u}(0,\xi) = \hat{g}(\xi) \tag{21}$$

Hence

$$\hat{u}(t,\xi) = e^{-t|\xi|^2} \hat{g}(\xi) \tag{22}$$

By Lemma 1 with $a = 1/\sqrt{2t}$

$$K_t(x) = \mathcal{F}^{-1}(e^{-t|\xi|^2})(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, \quad t > 0 \tag{23}$$

and by (9)

$$u(t,x) = K_t * g(x) = \int \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t} g(y)\, dy \tag{24}$$

Problem 3: Prove that $u(t,x) \to g(x)$, when $t \to 0$.

Let us now consider the Schrödinger equation

$$i\partial_t u(t,x) + \triangle u(t,x) = 0 \tag{25}$$

$$u(0,x) = g(x) \tag{26}$$

taking the Fourier transform $\hat{u}(t,\xi) = \int u(t,x)\, e^{-ix\cdot\xi}\, dx$ gives

$$i\partial_t \hat{u}(t,\xi) - |\xi|^2 \hat{u}(t,\xi) = 0 \tag{27}$$

$$\hat{u}(0,\xi) = \hat{g}(\xi) \tag{28}$$

Hence

$$\hat{u}(t,\xi) = e^{-it|\xi|^2} \hat{g}(\xi) \tag{29}$$

By Formally replacing t by it in (23) we get

$$K_t(x) = \mathcal{F}^{-1}(e^{it|\xi|^2})(x) = \frac{1}{(4\pi i t)^{n/2}} e^{i|x|^2/4t}, \quad t > 0 \tag{30}$$

where we interpret $i^{1/2}$ as $e^{i\pi/4}$, and if we can justify (30) we get

$$u(t,x) = K_t * g(x) = \int \frac{1}{(4\pi i t)^{n/2}} e^{i|x-y|^2/4t} g(y)\,dy \tag{31}$$

Now for $t > 0$ it is easy to see by direct calculation that

$$(i\partial_t + \triangle)K_t = 0 \tag{32}$$

But

$$(i\partial_t + \triangle)\int K_t(x-y)\,g(y)\,dy = \int (i\partial_t + \triangle)K_t(x-y)\,g(y)\,dy = 0 \tag{33}$$

so (31) is a solution of (25). However, it still remains to prove that $K_t * g(x) \to g(x)$, when $t \to 0$, which requires stationary phase which we will deal with later on. However, one can also prove that (30) follows from (23) by analytic continuation.

Let us now look on the wave equation

$$\partial_t^2 u(t,x) - \triangle u(t,x) = 0 \tag{34}$$

$$u(0,x) = f(x), \quad u_t(0,x) = g(x) \tag{35}$$

taking the Fourier transform $\hat{u}(t,\xi) = \int u(t,x)\, e^{-ix\cdot\xi}\,dx$ gives

$$\partial_t^2 \hat{u}(t,\xi) + |\xi|^2 \hat{u}(t,\xi) = 0 \tag{36}$$

$$\hat{u}(0,\xi) = \hat{f}(\xi), \quad \partial_t \hat{u}(0,\xi) = \hat{g}(\xi) \tag{37}$$

It is easy to see that this second order ODE has the solution

$$\hat{u}(t,\xi) = \cos(t|\xi|)\,\hat{f}(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\,\hat{g}(\xi) \tag{38}$$

The inverse Fourier transform of $\cos(t|\xi|)$ and $\sin(t|\xi|)/|\xi|$ are not functions but distributions.

Problem 4 If $\xi \in \mathbf{R}$ find the inverse Fourier transform of $\cos(t|\xi|) = \cos(t\xi) = \big(e^{it\xi} + e^{-it\xi}\big)/2$ and $\sin(t|\xi|)/|\xi| = \sin(t\xi)/\xi$ and use it to obtain the following integral representation of the solution of (34)-(35):

$$u(t,x) == \frac{1}{2}\big(f(x+t) + f(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(y)\,dy$$

Problem 5: Show that

$$|\partial_t \hat{u}(t,\xi)|^2 + |\,|\xi|\hat{u}(t,\xi)|^2 = |\hat{g}(\xi)|^2 + |\,|\xi|\hat{f}(\xi)|^2$$

and use it to prove the energy identity

$$\int |\partial_t u(t,x)|^2 + \sum_{i=1}^n |\partial_i u(t,x)|^2\,dx = \int |g|^2 + \sum_{i=1}^n |\partial_i f|^2\,dx$$

Some fundamental solutions using distribution theory

The fundamental solution E of a partial differential operator $P(D)$ is defined to be

$$P(D)E = \delta$$

Using the fundamental solution one can solve the equation

$$P(D)u = F,$$

In fact $u = E * F$ satisfies

$$P(D)(E * F) = (P(D)E) * F = \delta * F = F$$

Let us first derive the fundamental solution of $\triangle$. Since $\triangle$ is invariant under rotations we expect $E(x) = f(|x|)$. Since δ is homogeneous of degree $-n$ in $\mathbf{R}^n$ we expect E to be homogeneous of degree $-n+2$. Since the only distribution that is homogeneous of degree $-n+2$ is $|x|^{-n+2}$ we expect that $E(x) = c_n|x|^{-n+2}$. The constant c_n can be calculated rather easily, using that

$$\phi(0) = \langle \delta, \phi\rangle = \langle \triangle E, \phi\rangle = \langle E, \triangle\phi\rangle$$

either by using Green's theorem or just by assuming that $\phi(x)$ also is invariant under rotations and introducing polar coordinates.
Problem 6: Prove that $E(x) = c_n|x|^{-n+2}$ is a fundamental solution of $\triangle$ and find c_n.

The fundamental solution for the wave equation

$$\Box E = \delta(t,x), \quad \Box = \partial_t^2 - \triangle, \qquad (t,x)\in \mathbf{R}^{1+n}$$

is not hard to derive from the symmetries as well. Since $\Box$ is invariant under Lorentz transformations we expect the fundamental solution $E(t,x)$ to be invariant under Lorentz transformations as well, which means that it should be of the form $E(t,x) = f(t^2 - |x|^2)$, where f is a distribution. Plugging this into the equation gives after some calculation

$$4\rho f''(\rho) + 2(1+n)f'(\rho) = 0, \quad \rho = t^2 - |x|^2 \tag{40}$$

when $(t,x) \neq (0,0)$. This has the solution

$$\begin{cases} f(\rho) = & c_1 H(\rho), \quad \text{if} \quad n=1 \\ f(\rho) = & c_2 H(\rho)\rho^{-1/2}, \quad \text{if} \quad n=2 \\ f(\rho) = & c_3\delta(\rho), \quad \text{if} \quad n=3 \end{cases} \tag{41}$$

The constants can be calculated in the same way as we did for the fundamental solution of $\triangle$.

$$\begin{cases} E(t,x) = & c_1 H(t-|x|), \quad \text{if} \quad n=1 \\ E(t,x) = & c_2 H(t-|x|)(t^2-|x|^2)^{-1/2}, \quad \text{if} \quad n=2 \\ E(t,x) = & c_3\delta(t^2-|x|^2)H(t), \quad \text{if} \quad n=3 \end{cases} \tag{42}$$

Problem 7: Show in each case that (41) is a solution of (40).
Problem 8: If $n = 3$ prove that

$$\delta(t^2 - |x|^2)H(t) = \delta(t - |x|)/2|x| \tag{43}$$

Problem 9: Prove that $E(t, x)$ given above are fundamental solutions of $\Box$ and find the constants c_n.

Using the fundamental solution E for $\Box$ we can now solve the Cauchy problem

$$\Box u(t, x) = F$$
$$u(0, x) = f(x), \quad u_t(0, x) = g(x)$$

In fact let $u_0(t, x) = u(t, x)H(t)$ and $F_0(t, x) = F(t, x) = H(t)$ then

$$\Box u_0(t, x) = \Box u(t, x)H(t) = (\Box u(t, x))H(t) + 2u_t(t, x)\delta(t) + u(t, x)\delta'(t)$$
$$= F(t, x)H(t) + 2u_t(0, x)\delta(t) + u(0, x)\delta'(t) - u_t(0, x)\delta(t) = F_0(t, x) + g(x)\delta(t) + f(x)\delta'(t)$$

and hence

$$u_0(t, x) = E * \big(F_0(t, x) + g(x)\delta(t) + f(x)\delta'(t)\big) = E * F_0 + E * (g(x)\delta(t)) + \partial_t E * (f(x)\delta(t))$$

Let us now derive the solution formula if $n = 3$ in which case $E(t, x) = \delta(t - |x|)/4\pi|x|$ and hence
Problem 10 Show that
(44)

$$E * F_0(t, x) = \int\int F_0(t - s, x - y)\, \delta(s - |y|) \frac{1}{4\pi|y|} dy ds = \int_{|y| \le t} \frac{F_0(t - |y|, x - y)}{4\pi|y|} dy$$

and that

$$E * (g(x)\delta(t)) = t \int_{\omega \in S^2} \frac{g(x - t\omega)}{4\pi} dS(\omega)$$

where $dS(\omega)$ is the surface measure on the sphere S^2.