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Estimates on the modulus of expansion for vector fields solving nonlinear equations

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Abstract

In this article, by extending the method of Andrews and Clutterbuck (2011) [2] we prove a sharp estimate on the expansion modulus of the gradient of the logarithm of the parabolic kernel to the Schrödinger operator with convex potential on a bounded convex domain. The result improves an earlier work of Brascamp–Lieb which asserts the log-concavity of the parabolic kernel. We also give an alternate proof to a corresponding estimate on the first eigenfunction of the Schrödinger operator, obtained firstly by Andrews and Clutterbuck via the study of the asymptotics to a parabolic problem. Our proof is more direct via an elliptic maximum principle. An alternate proof of the fundamental gap theorem of Andrews and Clutterbuck (2011) [2], by considering the quotient of moduli of continuity, is also obtained. Moreover we derive a Neumann eigenvalue comparison result and some other lower estimates on the first Neumann eigenvalue for Laplace operator with a drifting term, including an explicit estimate on a conjecture of P. Li.

Résumé

Dans cet article, en étendant la méthode de Andrews et Clutterbuck (2011) [2] on obtient une inégalité fine sur le module d'expansion du gradient du logarithme du noyau parabolique de l'opérateur de Schrödinger avec potentiel convexe sur un domaine borné et convexe. Ceci améliore le résultat de Brascamp–Lieb qui établit la log-concavité du noyau parabolique. On donne également une démonstration différente de l'inégalité sur la première function propre de l'opérateur de Schrödinger avec potentiel convexe obtenue par Andrews and Clutterbuck via l'étude de l'asymptote parabolique. Notre demonstration, utilisant un principe de maximum elliptique, est plus directe. On donne aussi une autre démonstration du théorème de l'écart fondamental de Andrews et Clutterbuck (2011) [2], en considérant le quotient du module de contuinité. De plus, on déduit un résultat de comparaison de valeurs propres de Neumann, et d'autres inégalités sur la première valeur propre de Neumann du laplacien avec dérive, ce qui inclut une inégalité explicite d'une une conjecture de P. Li.

Keywords: Parabolic kernel; Eigenfunctions; Modulus of expansion; Energy of a path; Eigenvalue estimates

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1. Introduction

In [6], for any bounded convex domain $\Omega \subset \mathbb{R}^n$, it was proved that the first eigenfunction ϕ_0, of the operator $\mathcal{L}_q = \Delta - q(x)$ with the Dirichlet boundary value, has the property that $-\log\phi_0$ is a convex function on Ω, provided that the potential $q(x)$ is a convex function on Ω. See also [7,8,13,18] for alternate proofs, as well as generalizations/applications of this important result, via the maximum principle.

Recently, this log-concavity property of the first eigenfunction was sharpened into the following convexity estimate:

$$-\big((\nabla\log\phi_0)(y) - (\nabla\log\phi_0)(x)\big)\cdot\frac{y-x}{|y-x|} \geqslant 2\frac{\pi}{D}\tan\left(\frac{\pi|y-x|}{2D}\right) \tag{1.1}$$

for any pair of points (x, y) in Ω with $x \neq y$, by Andrews and Clutterbuck [2]. Here D is the diameter $\mathrm{Diam}(\Omega)$. This improved convexity (or log-concavity for ϕ_0) estimate, which is the novel crucial step in the proof to the fundamental gap conjecture, was proved in [2] by the study of the precise asymptotics (as $t \to \infty$) to a parabolic equation (cf. Theorem 4.1 and Corollary 4.4 in [2]) as well as some delicate constructions of barrier functions via the Prüfer transformation.

Motivated by this result, via studying a modified parabolic equation for the vector fields we prove a log-concavity estimate for the fundamental solution (with Dirichlet boundary data). The result sharpens a corresponding result by Brascamp and Lieb [5] (see also [6] Theorem 6.1) on the log-concavity of the fundamental solution. *Precisely, if $H(x, y, t)$ is the fundamental solution of $\frac{\partial}{\partial t} - \mathcal{L}_q$ and if $\bar{H}(s, t)$ is the fundamental solution of $\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}$ centered at* 0 *with Dirichlet boundary condition on* $[-\frac{D}{2}, \frac{D}{2}]$*. Then for any $t > 0$, $x \neq y$,*

$$-\big(\nabla_y \log H(z, y, t) - \nabla_x \log H(z, x, t)\big)\cdot\frac{y-x}{|y-x|} \geqslant -2(\log\bar{H})'\left(\frac{|y-x|}{2}, t\right), \tag{1.2}$$

provided that $q(x)$ is convex. This estimate yields (1.1) *by taking $t \to \infty$.* As in [2], even when $q(x)$ is not convex, the proof can still give a comparison result with the one-dimensional case. Since $\bar{H}' > 0$ on $(0, \frac{D}{2})$, the estimate (1.2) sharpens the log-concavity assertion of [5] on the fundamental solution. Our approach allows a more direct alternate argument to the estimate (1.1) via an elliptic maximum principle on vector fields satisfying nonlinear equations. Furthermore we interpret estimates (1.1) and (1.2) as comparisons of the energy of line intervals with respect to complete Riemannian metrics (in the case that Ω is a strictly convex C^2-domain) associated with ϕ_0 and H. See Corollary 3.3. This geometric interpretation may be of interest to a refined understanding of ϕ_0.

We also derive an eigenvalue (Neumann) comparison result for such manifolds and other lower estimates on the first Neumann eigenvalue for Laplace operator with a drifting term. This includes an explicit estimate addressing a conjecture of P. Li (see also [3,10] for earlier related works).

Here is how the paper is organized. In Section 2 we prove a maximum principle for vector fields satisfying a nonlinear PDE. This together with the precise boundary asymptotics in Section 3 provides an elliptic proof to (1.1). The estimate (1.2) was proved in Section 4 along with a parabolic maximum principle. Section 5 is devoted to an alternate perturbation in Andrews–Clutterbuck's parabolic approach to the fundamental gap conjecture. In Section 6 we give an elliptic proof to the fundamental gap theorem of [2]. Section 7 is on various applications including an explicit estimate on Li's conjecture.

2. Maximum principles

Recall that a function $\omega(s) : [0, +\infty) \to \mathbb{R}$, is called a modulus of the expansion for a vector field X if

$$\big(X(y) - X(x)\big)\cdot\frac{y-x}{|y-x|} \geqslant 2\omega\left(\frac{|y-x|}{2}\right). \tag{2.1}$$

Under this terminology (1.1) amounts to show that $\frac{\pi}{D}\tan(\frac{\pi}{D}s)$ is a modulus of expansion for $X = -\nabla\log\phi_0$. On a Riemannian manifold, (2.1) can be modified into a condition:

$$X\big(\gamma(d)\big)\cdot\gamma'(d) - X\big(\gamma(0)\big)\cdot\gamma'(0) \geqslant 2\omega\left(\frac{r(y,x)}{2}\right) \tag{2.2}$$

for a minimizing geodesic $\gamma(s) : [0, d] \to M$ with $\gamma(0) = x$, $\gamma(d) = y$, $d = r(x, y)$.

Let $X(x)$ be a C^2-vector field on Ω. Assume that X satisfies the differential equation:

$$\Delta X = 2\nabla_X X - V(x, X), \tag{2.3}$$

where $V(x,p)$ is a C^1-vector field defined on $\Omega \times \mathbb{R}^n$, which we assume that it is jointly convex in the sense that $\omega(s) \equiv 0$ is an expansion modulus of V, namely

$$\big(V\big(y, X(y)\big) - V\big(x, X(x)\big)\big) \cdot \frac{y-x}{|y-x|} \geqslant 0. \tag{2.4}$$

Let $\psi(s) : [0, \frac{D}{2}) \to \mathbb{R}$ be a C^2 function which satisfies that $\psi(0) = 0$,

$$\psi'' \leqslant -2\psi'\psi \tag{2.5}$$

and $\psi' < 0$.

Theorem 2.1. *Assume that $X(x)$ is a solution to (2.3) on Ω, a bounded domain in $\mathbb{R}^n$ with diameter D. Let ψ be a function defined above. Then*

$$\mathcal{C}(x,y) \doteqdot \big(X(y) - X(x)\big) \cdot \frac{y-x}{|y-x|} + 2\psi\left(\frac{|y-x|}{2}\right)$$

cannot attain a negative minimum in the interior, namely for some (x_0, y_0) with $x_0, y_0 \in \Omega$.

Proof. Argue by contradiction. Assume that at (x_0, y_0), $\mathcal{C}(x,y)$ attains a negative minimum. Clearly $x_0 \neq y_0$ since $\mathcal{C}(x,x) = 0$. Since for any $w_1 \in T_{x_0}\mathbb{R}^n$ and $w_2 \in T_{y_0}\mathbb{R}^n$, $\nabla_{w_1 \oplus w_2}\mathcal{C}(x,y)|_{(x_0,y_0)} = 0$, if we choose as in [2] a local orthonormal frame $\{e_i\}$ at x_0 such that $e_n = \frac{y_0 - x_0}{|y_0 - x_0|}$ and parallel translate them along the line interval joining x_0, y_0, it then implies that at (x_0, y_0), with $1 \leqslant i \leqslant n-1$,

$$\nabla_{e_i} X(y) \cdot \frac{y-x}{|y-x|} = -\frac{X(y) - X(x)}{|y-x|} \cdot e_i = \nabla_{e_i} X(x) \cdot \frac{y-x}{|y-x|}, \tag{2.6}$$

$$\nabla_{e_n} X(y) \cdot \frac{y-x}{|y-x|} = -\psi'\left(\frac{|y-x|}{2}\right) = \nabla_{e_n} X(x) \cdot \frac{y-x}{|y-x|}. \tag{2.7}$$

Let $E_i = e_i \oplus e_i \in T_{(x_0,y_0)}\mathbb{R}^n \times \mathbb{R}^n$ for $1 \leqslant i \leqslant n-1$, and $E_n = e_n \oplus (-e_n)$. Then the fact that $\mathcal{C}(x,y)$ attains its minimum at (x_0, y_0) implies that

$$\nabla^2_{E_j E_j}\mathcal{C}\big|_{(x_0,y_0)} \geqslant 0, \quad \text{for } 0 \leqslant j \leqslant n.$$

Direct calculation shows that at (x_0, y_0), with $1 \leqslant i \leqslant n-1$,

$$0 \leqslant \nabla^2_{E_i E_i}\mathcal{C} = \big(\nabla^2_{e_i e_i} X(y) - \nabla^2_{e_i e_i} X(x)\big) \cdot \frac{y-x}{|y-x|}, \tag{2.8}$$

$$0 \leqslant \nabla^2_{E_n E_n}\mathcal{C} = \big(\nabla^2_{e_n e_n} X(y) - \nabla^2_{e_n e_n} X(x)\big) \cdot \frac{y-x}{|y-x|} + 2\psi''. \tag{2.9}$$

On the other hand using Eq. (2.3), assumption (2.4) we have that at (x_0, y_0),

$$\begin{aligned}\sum_{j=1}^n \nabla^2_{E_j E_j}\mathcal{C} &= \big(\Delta X(y) - \Delta X(x)\big) \cdot \frac{y-x}{|y-x|} + 2\psi'' \\ &\leqslant 2\big(\nabla_{X(y)} X(y) - \nabla_{X(x)} X(x)\big) \cdot \frac{y-x}{|y-x|} - 4\psi'\psi.\end{aligned}$$

Now note that at (x_0, y_0), using (2.6) and (2.7),

$$\begin{aligned}\nabla_{X(y)}X(y)\cdot\frac{y-x}{|y-x|}&=\big\langle\nabla_{X(y)}X(y),e_n\big\rangle\\&=\sum_{j=1}^{n}\big\langle X(y),e_j\big\rangle\big\langle\nabla_{e_j}X(y),e_n\big\rangle\\&=-\frac{1}{|y-x|}\sum_{i=1}^{n-1}\big\langle X(y),e_i\big\rangle\big\langle X(y)-X(x),e_i\big\rangle-\psi'X(y)\cdot\frac{y-x}{|y-x|}.\end{aligned}$$

Combining the above two inequalities we conclude that at (x_0, y_0),

$$\begin{aligned}\sum_{j=1}^{n}\nabla^2_{E_jE_j}\mathcal{C}&\leqslant-\frac{2}{|y-x|}\sum_{i=1}^{n-1}\big\langle X(y)-X(x),e_i\big\rangle^2-2\psi'\big(X(y)-X(x)\big)\cdot\frac{y-x}{|y-x|}-4\psi'\psi\\&\leqslant-2\psi'\mathcal{C}.\end{aligned}\tag{2.10}$$

By assumption that $\mathcal{C}(x_0, y_0) < 0$ and $\psi' < 0$, estimate (2.10) is contradictory to the fact that

$$\sum_{j=1}^{n}\nabla^2_{E_jE_j}\mathcal{C}|_{(x_0,y_0)}\geqslant 0. \qquad \Box$$

With little modification, the proof gives the same result for V with non-vanishing expansion modulus $\omega(s)$. In this case ψ is assumed to satisfy:

$$\psi''\leqslant-2\psi\psi'+\omega.\tag{2.11}$$

Similar argument also proves the following result for a related parabolic equation, which is quite close to Theorem 4.1 of [2].

Theorem 2.2. *Assume that $X(x,t)$ on $\Omega\times[0,T]$ satisfies the equation*:

$$\left(\frac{\partial}{\partial t}-\Delta\right)X(x,t)=-2\nabla_{X(x,t)}X(x,t)+V\big(x,X(x,t)\big)\tag{2.12}$$

with the vector field $V(x,X(x))$ having a modulus of expansion $\omega(s)$. Let $\psi(s,t)$ be a function defined on $[0,\frac{D}{2})\times[0,T]$, with $\psi(0,t))=0$ and $\psi'(s,t)<0$, satisfies the parabolic inequality

$$\psi_t-\psi''\geqslant 2\psi'\psi-\omega.\tag{2.13}$$

Then $\mathcal{C}(x,y,t)=\langle X(y,t)-X(x,t),\frac{y-x}{|y-x|}\rangle+2\psi(\frac{|y-x|}{2},t)$ cannot attain the negative minimum in the (parabolic) interior.

3. Boundary asymptotics

First we show how to apply Theorem 2.1 to obtain the estimate (1.1) by establishing the boundary asymptotical estimates on $\mathcal{C}(x,y)$ for the case $X(x)=-\log\phi_0$. For this application we assume that Ω is C^2, and strictly convex, and we take $\psi(s)=-\frac{\pi}{D'}\tan(\frac{\pi}{D'}s)$ with $D'>D$, and $X=-\nabla\log\phi_0$. It is easy to see that X satisfies (2.3) with $V(x)=\nabla q$. One can also check that $\psi'(\frac{|y-x|}{2})<0$ and $\psi''=-2\psi\psi'$. The strategy is to prove that $\mathcal{C}(x,y)\geqslant 0$ for any $D'>D$ and then taking $D'\to D$ to obtain the estimate (1.1). Clearly for $D'>D$, $\psi(\frac{|y-x|}{2})$ is uniformly continuous on $\overline{\Omega}\times\overline{\Omega}$.

Recall that ϕ_0, the first eigenfunction (with the Dirichlet boundary) of $\mathcal{L}_q$, is a smooth function on $\overline{\Omega}$ such that $\phi_0(x)>0$ for any $x\in\Omega$, $\phi_0|_{\partial\Omega}=0$ and $\frac{\partial\phi_0}{\partial\nu}|_{\partial\Omega}<0$, where ν is the exterior unit normal. We assume that

$$\|\phi_0\|_{C^2(\overline{\Omega})}\leqslant\frac{A}{2}\tag{3.1}$$

for some $A>0$. For any given $\epsilon>0$ we shall prove that $\mathcal{C}(x,y)\geqslant -\epsilon$ on $\Omega\times\Omega$. Note that on the diagonal $\Delta=\{(x,x)\mid x\in\Omega\}$, $\psi(\frac{|y-x|}{2})\equiv 0$. Thus by the uniform continuity of $\psi(\frac{|y-x|}{2})$, there exists $\eta>0$ such that on $\{(x,y)\in\Omega\times\Omega\mid |y-x|\leqslant\eta\}$, a η-neighborhood of Δ, denoted by Δ_η, $2\psi\geqslant-\epsilon$.

Now let $\Omega_\delta=\{x\mid\phi_0(x)\geqslant\delta\}$. Also assume that $\delta\ll\eta$. We shall show that

$$\mathcal{C}(x,y)\geqslant 0\quad\text{on } \partial(\Omega_\delta\times\Omega_\delta)\setminus\Delta_\eta, \tag{3.2}$$

for δ sufficiently small. On $\partial\Delta_\eta$, by the log-concavity of Brascamp–Lieb (we can avoid appealing to this result, as explained in the remark below), $\mathcal{C}(x,y)\geqslant 2\psi\geqslant-\epsilon$. Hence Theorem 2.1 implies that $\mathcal{C}(x,y)\geqslant-\epsilon$. Below we shall show claim (3.2). This can be seen via the following considerations.

Since $|\nabla\phi_0|>0$ on $\partial\Omega$, we assume that there exist $\delta_0>0$ and $\theta_1>0$ such that

$$|\nabla\phi_0|\geqslant\theta_1 \tag{3.3}$$

for $x\in\Omega\setminus\Omega_{\delta_0}$. This in particular implies that for $\delta\leqslant\delta_0$, $\partial\Omega_\delta$ is a smooth hypersurface. Since Ω is convex, we may choose δ_0 small enough so that there exists $\theta_2>0$ such that for any $\delta\leqslant\delta_0$, the second fundamental form $\mathrm{II}(\cdot,\cdot)$ of $\partial\Omega_\delta$ satisfies

$$\mathrm{II}(\cdot,\cdot)\geqslant\theta_2\,\mathrm{I}(\cdot,\cdot), \tag{3.4}$$

where $\mathrm{I}(\cdot,\cdot)$ denotes the induced metric tensor on $\partial\Omega_\delta$. On the level hypersurface $\partial\Omega_\delta$, the following formula is also well known:

$$\mathrm{II}(\cdot,\cdot)=\frac{\nabla^2\phi_0(\cdot,\cdot)}{|\nabla\phi_0|} \tag{3.5}$$

as symmetric tensors on $T\partial\Omega_\delta$. We also make $\delta_0\leqslant\eta$.

Now let $C_1=\frac{1}{2}(\frac{A^2}{\theta_1\theta_2}+A)$ and $\delta_1=\min\{\delta_0,\frac{\theta_1^2}{4C_1},\frac{1}{\theta_2}\}$. Since Ω is strictly convex, if $x\in\partial\Omega$ and $y\in\overline{\Omega}$ such that $|y-x|\geqslant\delta_1/A>0$ there exists $\theta_3>0$, depending only on δ_1/A and Ω such that

$$\left\langle -\nu_x,\frac{y-x}{|y-x|}\right\rangle\geqslant\theta_3. \tag{3.6}$$

By the continuity we may also assume that the same estimate holds if Ω is replaced by Ω_δ for $\delta\leqslant\delta_0$.

For $x\in\partial\Omega_{\delta/2}$, $y\in\overline{\Omega_{\delta/2}}$, let $\gamma(s)$ be the line interval joining x to y parametrized by the arc-length. Denote $\gamma'(s)$ by W. Along $\gamma(s)$, W can be splitted into the tangential part W^{T} and the normal part $W^\perp$ with respect to $T_{\gamma(s)}\Omega_{\phi_0(\gamma(s))}$ and the inner-normal $-\nu_{\gamma(s)}$. The estimate (3.6) asserts that $|W^\perp|\geqslant\theta_3$. We also have the estimate

$$\begin{aligned}\nabla^2\phi_0(W,W)&=\nabla^2\phi_0\big(W^{\mathrm{T}},W^{\mathrm{T}}\big)+2\nabla^2\phi_0\big(W^{\mathrm{T}},W^\perp\big)+\nabla^2\phi_0\big(W^\perp,W^\perp\big)\\&\leqslant-|\nabla\phi_0|\,\mathrm{II}\big(W^{\mathrm{T}},W^{\mathrm{T}}\big)+A\big|W^\perp\big|\big|W^{\mathrm{T}}\big|+\frac{A}{2}\big|W^\perp\big|^2\\&\leqslant-\theta_1\theta_2\big|W^{\mathrm{T}}\big|^2+A\big|W^\perp\big|\big|W^{\mathrm{T}}\big|+\frac{A}{2}\big|W^\perp\big|^2\\&\leqslant-\frac{\theta_1\theta_2}{2}\big|W^{\mathrm{T}}\big|^2+C_1(\theta_1,\theta_2,A)\big|W^\perp\big|^2.\end{aligned} \tag{3.7}$$

Here in second line above we used (3.1), and in the third line we used (3.3) and (3.4).

Hence if for some integer $j\geqslant-1$, $\delta'=2^{j+1}\delta$ and $\delta'/2\leqslant\phi_0\leqslant\delta'$, we estimate

$$\begin{aligned}\nabla^2\log\phi_0(W,W)&=\frac{\nabla^2\phi_0(W,W)}{\phi_0}-\frac{|\nabla\phi_0|^2}{\phi_0^2}\big|W^\perp\big|^2\\&\leqslant-\frac{\theta_1\theta_2}{2\delta'}\big|W^{\mathrm{T}}\big|^2+\frac{2C_1}{\delta'}\big|W^\perp\big|^2-\frac{\theta_1^2}{\delta'^2}\big|W^\perp\big|^2\\&\leqslant-\frac{\theta_1\theta_2}{2\delta'}\big|W^{\mathrm{T}}\big|^2-\frac{\theta_1^2}{2(\delta')^2}\big|W^\perp\big|^2,\end{aligned} \tag{3.8}$$

for $\delta'\leqslant\delta_1$ and $\gamma(s)\in\Omega\setminus\Omega_{\delta_1}$. Here in the second line we used (3.7) and in the third line we used the definition of δ_1.

On the other hand, direct calculation shows

$$\begin{aligned}\big(X(y)-X(x)\big)\cdot\frac{y-x}{|y-x|}&=\big\langle X\big(\gamma(s)\big),\gamma'(s)\big\rangle\Big|_0^{|y-x|}\\&=\int_0^{|y-x|}\frac{d}{ds}\big(\big\langle X\big(\gamma(s)\big),\gamma'(s)\big\rangle\big)\,ds\\&=\int_0^{|y-x|}\nabla^2(-\log\phi_0)\big(\gamma'(s),\gamma'(s)\big)\,ds.\end{aligned}$$

Thus if $\gamma(s)\in\Omega\setminus\Omega_{\delta_1}$, (3.8) implies

$$\big(X(y)-X(x)\big)\cdot\frac{y-x}{|y-x|}\geqslant 0. \tag{3.9}$$

Otherwise, there exists s'', the first s such that $\phi_0(\gamma(s))=\delta_1$. Let k be the integer such that $2^k\delta\leqslant\delta_1<2^{k+1}\delta$. Since $|\nabla\phi_0|\leqslant\frac{A}{2}$, we deduce that if s_j is the first s with $\phi_0(\gamma(s))=\delta'/2$ and s'_j is the first s with $\phi_0(\gamma(s))=\delta'$, then $s'_j-s_j\geqslant\frac{\delta'}{A}$. Similarly $s''\geqslant\frac{\delta_1}{A}$, which particularly implies $|y-x|\geqslant\frac{\delta_1}{A}$. Clearly $s_{-1}=0$, $s'_j=s_{j+1}$. Now

$$\begin{aligned}\big(X(y)-X(x)\big)\cdot\frac{y-x}{|y-x|}&\geqslant\sum_{j=-1}^{k-1}\int_{s_j}^{s'_j}\nabla^2(-\log\phi_0)\big(\gamma',\gamma'\big)+\int_{s_k}^{s''}+\int_{\phi_0(\gamma(s))\geqslant\delta_1}\nabla^2(-\log\phi_0)\big(\gamma',\gamma'\big)\\&\geqslant\frac{\theta_1^2\theta_3^2}{4\delta A}\sum_{j=-1}^{k-1}\frac{1}{2^j}-\frac{4A^2}{\delta_1^2}D\\&\geqslant\frac{C_2}{\delta}-\frac{C_3}{\delta_1^2}.\end{aligned}\tag{3.10}$$

Here in the first line (3.9) is used, in the second line (3.8), (3.6) are used, and that $C_2=\frac{\theta_1^2\theta_3^2}{2A}$, $C_3=4A^2D$.

The estimates (3.9), (3.10) together with the fact that ψ is uniformly continuous on $\overline{\Omega}\times\overline{\Omega}$ implies the claim (3.2) hence that for δ small $\mathcal{C}(x,y)\geqslant-\epsilon$ on $\partial(\Omega_{\frac{\delta}{2}}\times\Omega_{\frac{\delta}{2}}\setminus\Delta_\eta)$. Taking $\delta\to0$ we have that

$$\mathcal{C}(x,y)\geqslant-\epsilon$$

for $x,y\in\Omega$. Taking $\epsilon\to0$ and then $D'\to D$ we have (1.1).

Remark 3.1. First observe that (3.10) implies that for δ sufficiently small, (3.9) holds for any $(x,y)\in\partial(\Omega_{\frac{\delta}{2}}\times\Omega_{\frac{\delta}{2}})$. By taking $\psi(s)=-\epsilon\tan(\epsilon s)$ with $\epsilon>0$ small, Theorem 2.1 and estimate (3.9) showed $\mathcal{C}(x,y)\geqslant0$ for this case, which implies Brascamp–Lieb's result on the log-concavity of ϕ_0 as $\epsilon\to0$.

Also note that in the above proof of the asymptotics of $\mathcal{C}(x,y)$ we did not use any other properties of ϕ_0 being the first eigenfunction besides that $\phi_0=0$, with non-vanishing gradient, on the boundary.

Replacing ψ by $\tilde{\psi}=c\psi(cs)$ with $0<c<1$ and letting $c\to1$, the same argument as above proves the following corollary which asserts that 'boundary convexity' implies the 'strong convexity' even the domain may not be convex.

Corollary 3.2. *Assume that Ω is a bounded domain such that there exists a smooth exhaustion Ω_δ with $\Omega_\delta\to\Omega$ as $\delta\to0$. Assume that X be a $C^2(\Omega)$ vector field satisfying (2.3), such that there exists $\delta_1>0$, for $(x,y)\in\partial(\Omega_\delta\times\Omega_\delta)$ with $|y-x|\leqslant\delta_1$ (3.9) holds, and for (x,y) with $|x-y|\geqslant\delta_1$, $(X(y)-X(x))\cdot\frac{y-x}{|y-x|}\to+\infty$ as $\delta\to0$. Let ψ be as in Theorem 2.1. Then $\mathcal{C}(x,y)\geqslant0$ for any $x,y\in\Omega$.*

Corollary 3.3. *Assume that Ω is a strictly convex bounded C^2-domain. Let $g_{ij}(x) = -\nabla^2_{ij}\log\phi_0$. Then $g_{ij}dx^i \otimes dx^j$ is a complete Riemannian metric on Ω. The estimate* (1.1) (*as well as the estimate* (1.2)) *can be interpreted as a comparison on the energy of the line segment in terms of this metric* (*in case of* (1.2)*, in terms of the metric $g_{ij}(x,t) = -\nabla^2_{ij}\log H(z,x,t)$*) *with the corresponding* (*complete*) *metric in one-dimensional model.*

Proof. We shall prove the result under the assumption Ω is C^3 since the argument is cleaner. The C^2 case can be derived by applying the similar argument to a neighborhood of points on $\partial\Omega$ with $\frac{\partial^2\phi_0}{\partial x^i\partial x^j}\frac{\partial\phi_0}{\partial x^i}\frac{\partial\phi_0}{\partial x^j} > 0$, $= 0$ and < 0 respectively.

Let $F(x) = -\log\phi_0(x)$. Then $g_{ij}(x) = F_{ij}$, the Hessian of F. Clearly $F(x)\to\infty$ as $x\to\partial\Omega$. We shall write ∇_g as the covariant derivative with respect to g and denote by $|\cdot|_g$ and $\langle\cdot,\cdot\rangle_g$ the norm and the inner product with respect to the metric g. Observe that $\frac{\phi_0}{\rho}$, with ρ being the distance to $\partial\Omega$, is a positive C^2-function on $\overline{\Omega}$, if Ω is C^3. Hence the completeness of g is equivalent to the completeness of $-\nabla^2_{ij}\log\rho$ near $\partial\Omega$. Thus we may assume that ϕ_0 is the distance function to $\partial\Omega$.

We shall prove that $|\nabla_g F|_g \leqslant 1$ near $\partial\Omega$. This implies that the metric g is complete, since for any curve $\gamma(t)$, $a\leqslant t\leqslant b$, with $\gamma(b)$ approaching to $\partial\Omega$ we have that

$$\int_a^b |\gamma'|_g\,dt \geqslant \int_a^b \langle\nabla_g F,\gamma'\rangle_g\,dt \geqslant F\big(\gamma(b)\big) - F\big(\gamma(a)\big)\to\infty.$$

Consider $g^{\epsilon}_{ij} = g_{ij} + \epsilon\delta_{ij} = -\frac{(\phi_0)_{ij}-\epsilon\phi_0\delta_{ij}}{\phi_0} + \frac{(\phi_0)_i(\phi_0)_j}{\phi_0^2}$, with δ_{ij} being the Euclidean metric. For simplicity we denote $(\phi_0)_{ij} - \epsilon\phi_0\delta_{ij}$ by A_{ij}.

Now we compute the inverse of (g^{ϵ}_{ij}). First let (A^{ij}) being the inverse of A_{ij}. Also let $\phi_0^i = A^{ij}\frac{\partial\phi_0}{\partial x^j}$. Then direct calculation shows that

$$\big(g^{ij}_{\epsilon}\big) = -\phi_0\left(A^{ij} + \frac{\phi_0^i\phi_0^j}{\phi_0 - \beta}\right)$$

with $\beta = A^{ij}\frac{\partial\phi_0}{\partial x^i}\frac{\partial\phi_0}{\partial x^j} < 0$. Using the above expression for (g^{ij}_{ϵ}), we have

$$\big|\nabla_{g^\epsilon}F\big|^2_{g^\epsilon} = g^{ij}_{\epsilon}F_iF_j = \frac{-\beta}{-\beta+\phi_0}\leqslant 1.$$

Taking $\epsilon\to 0$ we have that $|\nabla_g F|_g\leqslant 1$. Hence we prove the completeness.

Recall that for any path $\gamma(s)$ with $a\leqslant s\leqslant b$, the energy of the path (with respect to a Riemannian metric g) $\mathcal{E}_g(\gamma)$ is defined as $\int_a^b|\gamma(s)|^2_g\,ds$. If $\gamma(s) = x + s\frac{y-x}{|y-x|}$, with $0\leqslant s\leqslant |y-x|$, $X(x) = -\nabla\log\phi_0(x)$, direct calculation shows that

$$\big(X(y)-X(x)\big)\cdot\frac{y-x}{|y-x|} = \int_0^{|y-x|} -\nabla^2\log\phi_0\big(\gamma',\gamma'\big)\,ds = \int_0^{|y-x|}|\gamma'|^2_g\,ds.$$

It was Bruce Driver who suggested the possible path energy interpretation of (1.1). $\square$

Remark 3.4. The location where the maximum of ϕ_0 is attained has attracted much study. It is easy to show by direct calculation that the metric g in the above result has vanishing Ricci curvature at the maximum point of ϕ_0.

4. The modulus of expansion estimate for the fundamental solution

Here we improve the log-concavity of the fundamental solution proved in [6]. Let $H(z,x,t)$ be the fundamental solution to the heat operator $\frac{\partial}{\partial t} - \mathcal{L}_q$ with Dirichlet boundary value on a strictly convex domain D. We use $K(z,x,t)$ to denote the Euclidean heat kernel $\frac{1}{(4\pi t)^{n/2}}\exp(-\frac{|x-z|^2}{4t})$. It was proved in [6] that $\phi(x,t)\doteqdot(\frac{H}{K})(z,x,t)$ is a log-concave function of x. The improved estimate asserts that $-\nabla\log\phi$ has an expansion modulus given by the one-dimensional case. Before we state the improved version precisely, first let $\bar{H}(s,t)$ and $\bar{K}(s,t)$ be the corresponding fundamental solutions (concentrated at $s=0$) on $[-\frac{D}{2},\frac{D}{2}]$ and $\mathbb{R}$ for operator $\frac{\partial}{\partial t}-\frac{\partial^2}{\partial s^2}$. Let $\psi(s,t)=(\log(\frac{\bar{H}}{\bar{K}}))'$.

Theorem 4.1. *With the notation above, if q is convex, then for any $t\geqslant 0$*

$$-\big(\nabla_y\log\phi(y,t)-\nabla_x\log\phi(x,t)\big)\cdot\frac{y-x}{|y-x|}\geqslant -2\psi\left(\frac{|y-x|}{2},t\right). \tag{4.1}$$

The estimate (4.1) has the following equivalent form:

$$\big(\nabla_y\log H(z,y,t)-\nabla_x\log H(z,x,t)\big)\cdot\frac{y-x}{|y-x|}\leqslant 2(\log\bar{H})'\left(\frac{|y-x|}{2},t\right). \tag{4.2}$$

Since $(\log\bar{H})'<0$ and $(\log(\frac{\bar{H}}{\bar{K}}))'<0$ on $(0,\frac{D}{2})$, Theorem 4.1 and (4.2) improve the earlier result of Brascamp–Lieb. It is sharp since the equality holds for dimension one. For its proof we need the following variation of Theorem 2.2.

Theorem 4.2. *Let Ω be bounded domain of $\mathbb{R}^n$ with diameter D. Let $X(x,t)$ be a C^2-vector field defined on $\Omega\times(0,T]$ satisfying the equation*

$$\left(\frac{\partial}{\partial t}-\Delta\right)X(x,t)=V\big(x,X(x,t)\big)-2\nabla_X X(x,t)-\frac{1}{t}\big\langle\nabla_{(\cdot)}X(x,t),x-z\big\rangle-\frac{1}{t}X(x,t) \tag{4.3}$$

where $z\in\mathbb{R}^n$ is fixed, with $V(x,X)$ being jointly convex. Assume further that X is symmetric i.e. $\langle\nabla_{W_1}X,W_2\rangle=\langle\nabla_{W_2}X,W_1\rangle$ for any W_i. Let $\psi(s,t)$ be as in Theorem 4.1, or more generally a $C^{2,1}$-function on $[0,\frac{D}{2})\times\mathbb{R}_+\to\mathbb{R}$ with $\psi(0,t)=0$, $\psi'(s,t)<0$ for $s>0$, and satisfying

$$\psi_t-\psi''\geqslant 2\psi\psi'-\frac{\psi}{t}-\frac{\psi'}{t}s. \tag{4.4}$$

Then

$$\mathcal{C}(x,y,t)\doteqdot t\left(\left\langle X(y,t)-X(x,t),\frac{y-x}{|y-x|}\right\rangle+2\psi\left(\frac{|y-x|}{2},t\right)\right)$$

cannot attain a negative minimum in the parabolic interior.

Here $\langle\nabla_{(\cdot)}X,x-z\rangle$ is a vector whose inner product with any vector W is $\langle\nabla_W X,x-z\rangle$. Direct calculation shows that $X(x,t)\doteqdot-\nabla\log\phi(x,t)$ satisfies Eq. (4.3) with $V(x,X)=\nabla q(x)$. We first prove Theorem 4.2.

Proof. Argue by contradiction. Assume that at (x_0,y_0,t) with $t>0$, $x_0,y_0\in\Omega$, $\mathcal{C}$ attains a negative minimum on $\Omega\times\Omega\times(0,T]$. Following the notations from the proof to Theorem 2.1, the first variation consideration yields (2.6) and (2.7), which together imply that

$$\big\langle\nabla_{(\cdot)}X(y),e_n\big\rangle=\big\langle\nabla_{(\cdot)}X(x),e_n\big\rangle. \tag{4.5}$$

From now on, in the proof, when the meaning is clear we omit t variable dependence in $X(x,t)$. Now we compute

$$\begin{aligned}
0\geqslant{}&\left(\frac{\partial}{\partial t}-\sum_{j=1}^n\nabla^2_{E_jE_j}\right)\mathcal{C}(x,y,t)\Big|_{(x_0,y_0,t)}\\
={}&t\left\langle V\big(y,X(y)\big)-V\big(x,X(x)\big),\frac{y-x}{|y-x|}\right\rangle\\
&-2t\left\langle\nabla_{X(y)}X(y)-\nabla_{X(x)}X(x),\frac{y-x}{|y-x|}\right\rangle\\
&-\big\langle\nabla_{e_n}X(y),y-z\big\rangle+\big\langle\nabla_{e_n}X(x),x-z\big\rangle-\left\langle X(y)-X(x),\frac{y-x}{|y-x|}\right\rangle\\
&+2t\big(\psi_t-\psi''\big)+\left\langle X(y)-X(x),\frac{y-x}{|y-x|}\right\rangle+2\psi.
\end{aligned}$$

Here the right-hand side is evaluated at (x_0, y_0) and we have used (4.3). The first term is nonnegative by the convexity assumption on $V(x, X)$. As in the proof of Theorem 2.1, Eqs. (2.6) and (2.7) imply that the second term equals to

$$\frac{2t}{|y-x|}\sum_{i=1}^{n-1}\langle X(y)-X(x), e_i\rangle^2 + 2t\psi'\langle X(y)-X(x), e_n\rangle. \tag{4.6}$$

Applying (2.7) again, at (x_0, y_0, t), we have

$$-\langle \nabla_{e_n} X(y), y-z\rangle + \langle \nabla_{e_n} X(x), x-z\rangle = \psi'|y-x|. \tag{4.7}$$

Combining the previous computation with (4.6) and (4.7) we have that, at (x_0, y_0, t),

$$0 \geqslant 2t\psi'\langle X(y)-X(x), e_n\rangle + \psi'|y-x| + 2\psi + 2t(\psi_t - \psi''). \tag{4.8}$$

On the other hand ψ satisfies (4.4). Plugging (4.4) and $s = \frac{|y-x|}{2}$ into it, (4.8) then implies at (x_0, y_0, t)

$$0 \geqslant 2t\psi'\langle X(y)-X(x), e_n\rangle + 4t\psi'\psi = 2\psi'\mathcal{C}.$$

This is a contradiction to $\mathcal{C}(x_0, y_0, t) < 0$ and the fact that $\psi' < 0$ (which follows from the log-concavity of $\frac{\bar{H}}{\bar{K}}$ and the strong maximum principle, noticing $x_0 \neq y_0$). $\square$

Observe that $\phi(x,t) = \frac{H(z,x,t)}{K(z,x,t)}$ here also satisfies that $\phi(x,t) > 0$ on Ω, $\phi(x,t) = 0$ on $\partial\Omega$ and the partial differential equation:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi = -q\phi + 2\langle \nabla\phi, \nabla \log K\rangle.$$

Hence the parabolic Hopf's lemma implies that $\frac{\partial \phi}{\partial \nu} < 0$ on $\partial\Omega$. Therefore the same argument of Section 3 implies the estimates (3.9) and (3.10) for $X(x,t) = -\nabla \log \phi(x,t)$ on the points near the boundary. Now replacing ψ with $\widetilde{\psi}(s,t) = \epsilon\psi(\epsilon s, \epsilon^2 t)$ with $\epsilon \in (0,1)$, and observing that the heat kernel asymptotics (cf. [14,15]) imply that $\mathcal{C}(x,y,t) \geqslant 0$ holds at $t = 0$, Theorem 4.2 implies that $\mathcal{C}(x,y,t) \geqslant 0$. Letting $\epsilon \to 0$ we get a maximum principle proof for Brascamp–Lieb's log-concavity of $\frac{H}{K}$ and letting $\epsilon \to 1$ we get Theorem 4.1. The general ϵ serves a natural interpolation between the strong and the weak result.

Remark 4.3. By taking $t \to \infty$, since $e^{\lambda_0 t} H(z,x,t) \to \phi_0(z)\phi_0(x)$, the estimate (4.2) implies the improved log-concavity estimate (1.1). One can formulate a general maximum principle for the case that $V(x, X)$ has a convexity module as in Theorem 2.2. Similarly, Theorem 4.1 can be generalized to the case that ∇q has an expansion modulus $\omega(s)$ as in [2]. Without insisting $\psi' < 0$ in Theorem 4.2, the argument also proves that $\mathcal{C}(x,y,t) \geqslant 0$ is preserved by (4.3).

5. Alternate perturbation for the proof of fundamental gap theorem

In [2], by relating the fundamental gap to the exponential decay rate of the solution to a parabolic equation, the authors proved the following result.

Theorem 5.1 *(Andrews–Clutterbuck). Let Ω be a strictly convex bounded domain in $\mathbb{R}^n$ with diameter D. Then the gap between the second eigenvalue λ_1 and the first λ_0 (for the operator $\mathcal{L}_q$ with q being convex) satisfies*:

$$\lambda_1 - \lambda_0 \geqslant \frac{3\pi^2}{D^2}. \tag{5.1}$$

Recall that $\bar{w}(s)$ is called a modulus of continuity of a continuous function $w(x): \Omega \to \mathbb{R}$ if $w(y) - w(x) \leqslant 2\bar{w}(\frac{r(x,y)}{2})$ for any x, y. In [2], the result was proved by applying the following comparison result on the continuity modulus.

Theorem 5.2. *(See [2, Theorem 2.1].) Let $\Omega \subset \mathbb{R}^n$ be a strictly convex domain with diameter D and smooth boundary. Let $w(x,t)$ be a solution to*

$$\frac{\partial v}{\partial t} = \Delta v - 2\langle X, \nabla v\rangle \tag{5.2}$$

with Neumann boundary condition. Suppose that

(a) *$X(\cdot,t)$ has modulus of expansion $\omega(\cdot,t)$ for each $t>0$, where $\omega(s,t):[0,D/2]\times\mathbb{R}_+\to\mathbb{R}$ is smooth;*
(b) *$v(\cdot,0)$ has a modulus of continuity φ_0, where $\varphi_0(s):[0,D/2]\to\mathbb{R}$ is smooth with $\varphi_0(0)=0$ and $\varphi_0'(s)>0$ on $[0,\frac{D}{2}]$;*
(c) *$\varphi(s,t):[0,D/2]\times\mathbb{R}_+\to\mathbb{R}$ satisfies*
   (i) *$\varphi(s,0)=\varphi_0(s)$ on $[0,D/2]$;*
   (ii) *$\frac{\partial\varphi}{\partial t}\geqslant \varphi''-2\omega\varphi'$ on $[0,D/2]\times\mathbb{R}_+$;*
   (iii) *$\varphi'(s,t)>0$ on $[0,\frac{D}{2}]\times\mathbb{R}_+$;*
   (iv) *$\varphi(0,t)\geqslant 0$ for each $t\geqslant 0$.*

Then $\varphi(s,t)$ is a modulus of continuity of $v(x,t)$.

Theorem 5.1 can be derived from Theorem 5.2 by letting $v(x,t)=e^{-(\lambda_1-\lambda_0)t}\frac{\phi_1(x)}{\phi_0(x)}$ with ϕ_i being $(i+1)$-th eigenfunction and λ_i being the corresponding eigenvalue, and letting $\varphi(s,t)=Ce^{-(\mu_1-\mu_0)t}\frac{\bar\phi_1}{\bar\phi_0}$ with $\bar\phi_i$ being $(i+1)$-th eigenfunction and μ_i being the corresponding eigenvalue for the operator $\frac{d^2}{ds^2}$ on $[-\frac{D}{2},\frac{D}{2}]$. Here $X=-\nabla\log\phi_0$, $\omega(s)=-(\log\bar\phi_0)'$ and C is a constant sufficiently large. Estimate (1.1) asserts that $\omega(s)$ is an expansion modulus of X. This is how (1.1) plays the crucial role in the proof of Theorem 5.1.

The only problem here is that $\varphi'(\frac{D}{2},t)>0$ does not hold and the corresponding $\omega(s)=-(\log\bar\phi_0)'$ is not smooth on $[0,\frac{D}{2}]$. In [2] a perturbation via the Prüfer transformation was introduced to handle this.

On the other hand, the issue can be also dealt with in the following way. Let $D'>D$. Let $\bar\phi_i^{D'}$ be the corresponding eigenfunctions on $[-\frac{D'}{2},\frac{D'}{2}]$ (which are given by $\cos(\frac{\pi}{D'}s)$ and $\sin(\frac{2\pi}{D'}s)$ for $i=0,1$). Let $\omega^{D'}(s)=-(\log\bar\phi_0^{D'})'$. Since $\omega(s)\geqslant\omega^{D'}(s)$, $\omega^{D'}(s)$ is still an expansion modulus of X. And now the corresponding $\varphi^{D'}(s)=2Ce^{-(\mu_1^{D'}-\mu_0^{D'})t}\sin(\frac{\pi}{D'}s)$ does have $\varphi'^{D'}(s,t)>0$ on $[0,\frac{D}{2}]\times\mathbb{R}_+$. Hence Theorem 5.2 can be applied to conclude that

$$e^{-(\lambda_1-\lambda_0)t}\left(\frac{\phi_1(y)}{\phi_0(y)}-\frac{\phi_1(x)}{\phi_0(x)}\right)\leqslant Ce^{-(\bar\mu_1^{D'}-\bar\mu_0^{D'})t}\sin\left(\frac{\pi}{D'}\frac{|y-x|}{2}\right)$$

since there exists such a constant C at $t=0$. From this one has that $\lambda_1-\lambda_0\geqslant\bar\mu_1^{D'}-\bar\mu_0^{D'}$. The claimed result follows by letting $D'\to D$.

Theorem 5.2 effectively implies a type of Barta's theorem on Neumann eigenvalue comparison:

Let Ω be a strictly convex domain in $\mathbb{R}^n$ with diameter D. Suppose that $\omega(s)$ is a modulus of expansion of the vector field X and there exists $\varphi_0(s):[0,\frac{D}{2}]\to\mathbb{R}_+$ with $\varphi_0(0)=0$, $\varphi_0'(s)>0$ on $[0,\frac{D}{2})$ and $\varphi_0''-2\omega\varphi_0'\leqslant-\bar\lambda_1\varphi_0$. Then for any nonzero Neumann eigenvalue $\tilde\lambda_1$, of the operator $\Delta-2\langle X,\nabla(\cdot)\rangle$, the real part $\mathcal{R}e(\tilde\lambda_1)$ is bounded from below by $\bar\lambda_1$.

We refer the interested readers to [2] for the motivation, history and comprehensive literatures on previous works related to Theorem 5.1.

6. An elliptic proof

In this section we give an alternate proof of Theorem 5.1 without evoking the parabolic method. Here we apply the maximum principle to the quotient of the continuity moduli. The estimate (1.1) plays the crucial role again.

Let $w(x)=\frac{\phi_1(x)}{\phi_0(x)}$ and $\bar w(s)=\frac{1}{2}\frac{\bar\phi_1(s)}{\bar\phi_0(s)}$. Here ϕ_i and $\bar\phi_i$ are eigenfunctions of the Schrödinger operator as well as the 1-dimension model operator as in the last section. Direct calculation shows that w satisfies

$$\Delta w=-(\lambda_1-\lambda_0)w-2\langle\nabla\log\phi_0,\nabla w\rangle. \tag{6.1}$$

By [18], w can be extended to a smooth function on $\overline{\Omega}$ (still denoted by w) and $\frac{\partial w}{\partial\nu}=0$ on $\partial\Omega$.

Consider the quotient of the oscillations of w and $\bar{w}(s)$ and let

$$\mathcal{Q}(x,y) \doteqdot \frac{w(x)-w(y)}{\bar{w}(\frac{|x-y|}{2})}$$

on $\overline{\Omega}\times\overline{\Omega}\setminus\Delta$ with $\Delta=\{(x,x)\,|\,x\in\overline{\Omega}\}$ being the diagonal. The function $\mathcal{Q}$ can be extended to a set C, a natural 'compactification' of $\overline{\Omega}\times\overline{\Omega}\setminus\Delta$ (or a 'blow-up' of Δ), namely the disjoint union of $\overline{\Omega}\times\overline{\Omega}\setminus\Delta$ with the unit sphere bundle $U\Omega=\{(x,X)\,|\,x\in\overline{\Omega},\,|X|=1\}$, with the extension defined as

$$\mathcal{Q}(x,X)=\frac{2\langle\nabla w(x),X\rangle}{\bar{w}'(0)}.$$

Note here that $\bar{w}(s)=\sin(\frac{\pi}{D}s)$, hence $\bar{w}''(s)=-\mu_0\bar{w}(s)$, with $\mu_0=\frac{\pi^2}{D^2}$, and $\bar{w}'(0)=\frac{\pi}{D}$. By the scaling we can assume without the loss of the generality that $D=\pi$. The extended function $\mathcal{Q}$, which is continuous, attains its maximum somewhere on C. We consider two cases.

Case 1: the maximum of $\mathcal{Q}$, which is clearly nonzero and denoted by m, is attained at some (x_0,y_0) with $x_0\neq y_0$. The Neumann condition $\frac{\partial w}{\partial\nu}=0$, and the convexity of Ω force that both x_0 and y_0 must be in Ω. Now we pick a local orthonormal frame $\{e_i\}$, $1\leqslant i\leqslant n$ such that $e_n=\frac{y_0-x_0}{|y_0-x_0|}$. We then parallel translate the frame to y_0 and let $E_i=e_i\oplus e_i$ for $1\leqslant i\leqslant n-1$ and $E_n=e_n\oplus(-e_n)$. Now $\{E_i\}$ be tangent vectors of the product $\Omega\times\Omega$ as before. Since $\mathcal{Q}$ has an interior maximum, the first variation consideration implies that at (x_0,y_0):

$$\frac{\nabla^x_{e_i}w(x)}{\bar{w}}=\frac{\nabla^y_{e_i}w(y)}{\bar{w}}=0;$$
$$\nabla_{e_n}w(x)=\nabla_{e_n}w(y)=-\frac{m}{2}\bar{w}'.$$

Putting them together we have that at (x_0,y_0),

$$\nabla w(x)=\nabla w(y)=-\frac{m}{2}\bar{w}'\frac{y-x}{|y-x|}. \tag{6.2}$$

The second variation consideration yields for $1\leqslant i\leqslant n-1$, at (x_0,y_0),

$$0\geqslant\nabla^2_{E_i,E_i}\mathcal{Q}=\frac{\nabla^2_{e_ie_i}w(x)-\nabla^2_{e_ie_i}w(y)}{\bar{w}} \tag{6.3}$$

and

$$\begin{aligned}0&\geqslant\nabla^2_{E_nE_n}\mathcal{Q}\\&=\frac{\nabla^2_{e_ne_n}w(x)-\nabla^2_{e_ne_n}w(y)}{\bar{w}}+2\frac{\nabla_{e_n}w(x)+\nabla_{e_n}w(y)}{\bar{w}^2}\bar{w}'+2m\left(\frac{\bar{w}'}{\bar{w}}\right)^2-m\frac{\bar{w}''}{\bar{w}}.\end{aligned} \tag{6.4}$$

Using (6.2) and $\bar{w}''=-\mu_0\bar{w}$ we can write (6.4) as

$$0\geqslant\frac{\nabla^2_{e_ne_n}w(x)-\nabla^2_{e_ne_n}w(y)}{\bar{w}}+\mu_0 m. \tag{6.5}$$

Adding (6.3) (6.5), and using (6.1), (6.2) we have that at (x_0,y_0),

$$\begin{aligned}0&\geqslant\frac{\Delta w(x)-\Delta w(y)}{\bar{w}}+\mu_0 m\\&=-(\lambda_1-\lambda_0)m+2\frac{\langle-\nabla\log\phi_0(x),\nabla w(x)\rangle-\langle-\nabla\log\phi_0(y),\nabla w(y)\rangle}{\bar{w}}+\mu_0 m\\&=-(\lambda_1-\lambda_0)m+m\bar{w}'\frac{(-\nabla\log\phi_0(y)+\nabla\log\phi_0(x))\cdot\frac{y-x}{|y-x|}}{\bar{w}}+\mu_0 m.\end{aligned}$$

Applying (1.1), it then implies that

$$0 \geqslant -(\lambda_1 - \lambda_0)m + 2\frac{\pi^2}{D^2}m + \mu_0 m \tag{6.6}$$

from which one concludes that $\lambda_1 - \lambda_0 \geqslant 3\frac{\pi^2}{D^2}$.

Case 2: the maximum of $\mathcal{Q}$ is attained at some $(x_0, X_0) \in U\Omega$. It is easy to see that $X_0 = \frac{\nabla w}{|\nabla w|}$ at x_0 and $m = 2|\nabla w|(x_0)$. Moreover $|\nabla w|(x_0) \geqslant |\nabla w|(x)$ for any $x \in \overline{\Omega}$. Again the convexity of Ω and $\frac{\partial w}{\partial \nu} = 0$ rule out the possibility that x_0 is on $\partial\Omega$. Now pick an orthonormal frame $\{e_i\}$ at x_0 so that $e_n = \frac{\nabla w}{|\nabla w|}$. We also parallel translate it to a neighborhood of x_0. Under such a frame it is easy to see that $w_i = 0$ for $1 \leqslant i \leqslant n-1$ and $w_n = \nabla w$.

Since $|\nabla w|^2(x)$ attains its maximum at the interior point x_0, we have that

$$\nabla|\nabla w|^2 = 0, \quad \text{hence } w_{kn} = 0,\ 1 \leqslant k \leqslant n. \tag{6.7}$$

Moreover, the maximum principle concludes for any $1 \leqslant k \leqslant n-1$,

$$0 \geqslant \left(|\nabla w|^2\right)_{kk} = 2\sum_{j=1}^{n}\left(w_{jk}^2 + w_{kkj}w_j\right) \geqslant 2w_{kkn}w_n. \tag{6.8}$$

Above we have used (6.7) and $w_j = 0$ for $1 \leqslant j \leqslant n-1$.

Let $x(s) = x_0 + se_n$, $y(s) = x_0 - se_n$ and $g(s) = \mathcal{Q}(x(s), y(s))$. Since $\mathcal{Q}$ achieves its maximum at $(x_0, \frac{\nabla w}{|\nabla w|}(x_0))$ we have that $g(s) \leqslant g(0) = m$ for all $s \in (-\epsilon, \epsilon)$, which implies that $\lim_{s\to 0} g'(s) = 0$ and $\lim_{s\to 0} g''(s) \leqslant 0$. In the following we shall exploit these facts.

Direct calculation shows that

$$\begin{aligned} g'(s) &= \frac{\langle \nabla w(x(s)), e_n\rangle + \langle \nabla w(y(s)), e_n\rangle}{\bar{w}} - g(s)\frac{\bar{w}'}{\bar{w}}, \\ g''(s) &= \frac{\nabla^2_{e_n e_n} w(x(s)) - \nabla^2_{e_n e_n} w(y(s))}{\bar{w}(s)} - 2\frac{\langle \nabla w(x(s)), e_n\rangle + \langle \nabla w(y(s)), e_n\rangle}{\bar{w}}\frac{\bar{w}'}{\bar{w}} \\ &\quad + \mu_0 g(s) + 2g(s)\left(\frac{\bar{w}'}{\bar{w}}\right)^2. \end{aligned}$$

Observing that $\lim_{s\to 0}\frac{g'(s)}{\bar{w}(s)} = g''(0)$, and making use of the first equation above, the second equation implies that

$$\lim_{s\to 0} g''(s) = 2w_{nnn}(x_0) - 2\lim_{s\to 0} g''(s) + \mu_0 m.$$

Hence we have that at x_0,

$$0 \geqslant 2w_{nnn} + \mu_0 m. \tag{6.9}$$

Now multiply w_n on (6.9) and add the resulting inequality to (6.8). Then we have that at x_0,

$$0 \geqslant \left\langle \nabla(\Delta w), \nabla w\right\rangle + \mu_0|\nabla w|^2.$$

Apply (6.1) to the above estimate and note that $w_{kn} = 0$. Then we conclude that at x_0

$$0 \geqslant -(\lambda_1 - \lambda_0)|\nabla w|^2 - 2\nabla^2(\log\phi_0)(\nabla w, \nabla w) + \mu_0|\nabla w|^2.$$

The estimate $\lambda_1 - \lambda_0 \geqslant 3\frac{\pi^2}{D^2}$ follows from the estimate $-\nabla^2\log\phi_0 \geqslant \frac{\pi^2}{D^2}\,\mathrm{id}$, which is a direct consequence of (1.1).

Putting them together we have a proof of Theorem 5.1. In some sense, the above argument combines those of [2] and [18].

7. Further applications

Theorem 2.1 of [2] and the argument of the last section effectively give the following result on a lower bound of the second Neumann eigenvalue for the operator $\Delta_X \doteqdot \Delta - 2\langle\nabla(\cdot), X\rangle$ (with non-constant eigenfunction) on a strictly convex domain Ω with diameter D. In case that the eigenvaluese $\tilde{\lambda}_1$ of Δ_X are complex (with complex-valued eigenfunction w), the argument of Section 5/Section 6 applying to $\mathcal{R}e(e^{-\tilde{\lambda}_1 t}w)$, effectively provides the same lower bounds for $\mathcal{R}e(\tilde{\lambda}_1)$, the real part of any nonzero eigenvalues. For the simplicity below we only prove the results for the case $\tilde{\lambda}$ is real. Recall that $\mu_0 = (\frac{\pi}{D})^2$ and let $\bar{w} = \sin(\frac{\pi}{D}s)$.

Corollary 7.1.

(i) *If X has an expansion modulus given by $-(\log\bar{\phi}_0)'$, then for any nonzero Neumann eigenvalue $\tilde{\lambda}_1$ of the operator $\Delta - 2\langle\nabla(\cdot), X\rangle$, $\mathcal{R}e(\tilde{\lambda}_1)$ is bounded from below by $\mu_1 - \mu_0$;*

(ii) *If X is merely convex, or more generally X has $\epsilon'\frac{\bar{w}}{\bar{w}'}$ as its expansion modulus for $\epsilon' > -\frac{\mu_0}{2}$, then $\mathcal{R}e(\tilde{\lambda}_1) \geqslant 2\epsilon' + \mu_0$. The convexity of X amounts to $\epsilon = 0$.*

Both results still hold for Ω being a strictly convex domain in a Riemannian manifold with nonnegative Ricci curvature, or for any compact Riemannian manifold (*without boundary*) *with nonnegative Ricci curvature.*

Since the argument of previous section works equally well if $v(x,t) = e^{-\tilde{\lambda}_1 t}w(x)$ with $w(x)$ being the Neumann eigenfunction of Δ_X. This implies Part (i). For the proof of the second statement in Corollary 7.1, letting w be the first non-trivial eigenfunction in the argument of the last section, it suffices to let $\varphi(s,t) = Ce^{-(2\epsilon'+\mu_0)t}\bar{w}$ with and observe that $\bar{w}'' = -\mu_0\bar{w}$. The part (ii) generalizes an earlier result of Payne and Weinberger [16] which asserts the same statement for $X(x) \equiv 0$. Note that it even applies to the case that $\epsilon < 0$. One candidate of the vector field X satisfying the assumption of the part (i) is $-\nabla\log\tilde{\phi}_0$ with $\tilde{\phi}_0$ being the first eigenfunction of some domain Ω' containing Ω, but with the same diameter.

For the last statement, after some obvious modifications on the definition of the modulus of expansion and replacing $|y - x|$ by $r(x, y)$ (the distance function), it suffices to observe that the second variation (without fixing either end) of the distance function $r(x, y)$ is non-positive while $\varphi' > 0$. Hence the proof to Theorem 5.2 in [2] goes without any changes. Note that on a compact Riemannian manifold any convex vector field X (being convex is equivalent to that $\langle\nabla_W X, W\rangle \geqslant 0$) must be parallel. Hence statement here generalizes a corresponding result of Li and Yau [11] and Zhong and Yang [19] for strictly convex domains in a Riemannian manifold (or for a compact Riemannian manifold when X is non-convex).

Secondly we consider the case that Ω is a compact manifold with $\mathrm{Ric} \geqslant n-1$ or a bounded convex domain in such a Riemannian manifold. The argument in the last section can yield an interpolating estimate on $\mathcal{R}e(\tilde{\lambda}_1)$, where $\tilde{\lambda}_1$ is any nonzero eigenvalue. First recall the following lemma which may be well known for experts.

Lemma 7.1. *Assume that $x, y \in M$ with $\mathrm{Ric} \geqslant n-1$. Let $\gamma(s)$ be a minimizing geodesic joining x and y. Let $\{e_i\}$ be an orthonormal frame at x and parallel translate it along $\gamma(s)$ with $e_n = \gamma'(s)$. Then for $x, y \in M$, with distance $r(x, y) < \pi$,*

$$\sum_{i=1}^{n-1} \nabla^2_{E_i E_i} r(x,y) \leqslant -2(n-1)\frac{\sin(\frac{r(x,y)}{2})}{\cos(\frac{r(x,y)}{2})}. \tag{7.1}$$

Proof. Since the distance function $r(x, y)$ may not be smooth, the estimate is understood in the sense of support. Let $\gamma_i(s,\eta) = \exp_{\gamma(s)}(\eta V_i(s))$ for $i = 1, \ldots, n-1$ with $V_i = (\cos(s) + \frac{1-\cos d}{\sin d}\sin s)e_i(s)$. Here we denote $r(x, y)$ by d. Since $\frac{D\gamma}{\partial\eta}(0,\eta) = e_i(0)$, $\frac{D\gamma}{\partial\eta}(d,\eta) = e_i(d)$ and $r(\gamma(0,\eta), \gamma(d,\eta)) \leqslant L(\gamma(s,\eta))$, the arc-length of $\gamma(\cdot,\eta)$, the second variation formula implies the following differential inequality in the barrier sense,

$$\begin{aligned}
\nabla^2_{E_i E_i} r(x,y) &\leqslant \frac{d^2}{d\eta^2}\int_0^d \left|\frac{D\gamma}{\partial s}\right|(s,\eta)\,ds\,\bigg|_{\eta=0} \\
&= \int_0^d \left(\left|V_i'\right|^2 - \left\langle R\left(V_i, \gamma'\right)V_i, \gamma'\right\rangle\right) ds.
\end{aligned}$$

The lemma follows by summing the above for $i = 1$ to $n - 1$, plugging in the assumption $\mathrm{Ric} \geqslant n - 1$, and elementary identities. □

The argument in the last section then shows the following result regarding for any nonzero Neumann eigenvalue of $\Delta - 2\langle \nabla(\cdot), X\rangle$.

Corollary 7.2. *Let Ω be a compact manifold without boundary, or a strictly convex domain in a Riemannian manifold, with $\mathrm{Ric} \geqslant (n-1)K$. Assume that the diameter of Ω satisfies $D < \frac{\pi}{\sqrt{K}}$, and that X has a modulus of expansion $\epsilon' \frac{\bar{w}}{\bar{w}'}$. Then for any $D' \in (D, \frac{\pi}{\sqrt{K}}]$,*

$$\mathcal{R}e(\tilde{\lambda}_1) \geqslant 2\epsilon' + (n-1)\sqrt{K}\frac{\pi}{D'} \inf_{0 \leqslant r \leqslant D/2} \frac{\tan(\sqrt{K}r)}{\tan(\frac{\pi}{D'}r)} + \left(\frac{\pi}{D'}\right)^2. \tag{7.2}$$

We remark that a similar, probably more geometrically formulated result, but with $X = 0$, was obtained by Andrews and Clutterbuck as described in [1]. The formulation here is a bit simple-minded. Nevertheless it gives an explicit lower bound. Another explicit lower bound was also obtained in [4, Corollary 17]. By taking $D' = \pi/\sqrt{K}$, the result contains the Lichnerowicz's $\tilde{\lambda}_1 \geqslant nK$ (so does the formulation of Andrews–Clutterbuck, as well as Corollary 17 of [4]), and when $K = 0$ it recovers Li and Yau [11], Zhong and Yang's [19] estimate. Hence it addresses a conjecture of P. Li [12]. The above result also generalizes an earlier estimate of Reilly [17] for convex domains in a Riemannian manifold with $\mathrm{Ric} \geqslant (n-1)g$. Note that Reilly's method is quite different, and may not apply to the case $X \neq 0$ considered here.

Proof. It suffices to prove for $K = 1$. Now observe that the proof of Theorem 2.1 in [2] implies that Theorem 5.2 still holds on such a Riemannian manifold if the part (ii) in assumption (c) is replaced by

$$\frac{\partial \varphi}{\partial t} \geqslant \varphi'' - \big(2\omega + (n-1)\sqrt{K}\tan(\sqrt{K}s)\big)\varphi'. \tag{7.3}$$

Now let $v(x,t) = e^{-\tilde{\lambda}_1 t}w(x)$ with $w(x)$ being the Neumann eigenfunction of Δ_X and let $\varphi(s,t) = Ce^{-\bar{\lambda}t}\bar{\omega}$ with $\bar{\lambda}$ being the right-hand side of (7.2). It is now easy to check that (7.3) and other assumptions of Theorem 5.2 hold for any $D' > D$. Hence we conclude that $\tilde{\lambda}_1 \geqslant \bar{\lambda}$ as before. □

The following slightly more general result can be obtained by tracing the argument carefully.

Corollary 7.3. *Let Ω be a compact manifold without boundary, or a strictly convex domain in a Riemannian manifold, with $\mathrm{Ric} \geqslant (n-1)K$. Assume that the diameter $D < \frac{\pi}{\sqrt{K}}$, and that X has a modulus of expansion $\sigma(s)$. If for any $D' \in (D, \frac{\pi}{\sqrt{K}}]$, there exist a constant $\bar{\lambda}$ and a C^2-function $\bar{w} : [0, \frac{D'}{2}] \to [0, \infty)$ with $\bar{w}' > 0$ on $[0, \frac{D}{2}]$, and $-\bar{w}''(s) + (n-1)\sqrt{K}\tan(\sqrt{K}s)\bar{w}'(s) + 2\sigma(\frac{D}{D'}s)\bar{w}'(s) \geqslant \bar{\lambda}\bar{w}(s)$, on $[0, \frac{D}{2}]$, then $\mathcal{R}e(\tilde{\lambda}_1) \geqslant \bar{\lambda}$.*

The estimate (1.1) has another application on the lower estimate of λ_0, the first (Dirichlet) eigenvalue of the operator $\mathcal{L}_q$. Note that when $q(x) \equiv 0$, the estimate is only sharp for $n = 1$ (cf. [9]).

Corollary 7.4. *Assume that Ω is a bounded strictly convex domain in $\mathbb{R}^n$ with diameter D. Assume that $q(x)$ is convex. Then*

$$\lambda_0 \geqslant n\left(\frac{\pi}{D}\right)^2 + \inf_{x \in \Omega} q(x). \tag{7.4}$$

It then implies that the second eigenvalue has the lower bound estimate:

$$\lambda_1 \geqslant (n+3)\left(\frac{\pi}{D}\right)^2 + \inf_{x \in \Omega} q(x). \tag{7.5}$$

Proof. Since $\phi_0 = 0$ on $\partial\Omega$ and $\phi_0 > 0$, it must attain its maximum for some $x_0 \in \Omega$. For r small integrate the estimate (1.1) over the $\partial B_{x_0}(r)$:

$$\begin{aligned}\omega_{n-1} r^{n-1} 2\frac{\pi}{D}\tan\left(\frac{\pi r}{2D}\right) &\leqslant \int\limits_{\partial B_{x_0}(r)} \big(X(y) - X(x_0)\big)\cdot \nu \, dA(y)\\ &= \int\limits_{B_{x_0}(r)} \operatorname{div} X(y)\, d\mu(y)\\ &= \lambda_0 \frac{\omega_{n-1}}{n} r^n + \int\limits_{B_{x_0}(r)} \big(|\nabla \log\phi_0|^2 - q\big)\, d\mu(y).\end{aligned}$$

Here ω_{n-1} is the area of the $\partial B_0(1)$ and recall that $X = -\nabla \log\phi_0$. Hence we have that

$$\lambda_0 \geqslant \frac{2n\pi}{rD}\tan\left(\frac{\pi r}{2D}\right) + \fint_{B_{x_0}(r)} \big(q - |\nabla\log\phi_0|^2\big)\, d\mu(y).$$

Taking $r \to 0$ in the right-hand side above we get the desired result, since $\nabla\phi_0(x_0) = 0$ and

$$\lim_{r\to 0} \frac{2n\pi}{rD}\tan\left(\frac{\pi r}{2D}\right) = \frac{n\pi^2}{D^2}. \qquad \square$$

Using a similar argument as in the proof of Corollary 7.4, (1.2) implies the following sharp upper bound on the growth rate of $H(x, y, t)$.

Corollary 7.5. *Assume that Ω is strictly convex and $q(x)$ is convex. Let $H(z, x, t)$ be the Dirichlet heat kernel for $\frac{\partial}{\partial t} - \mathcal{L}_q$ with potential function q. For any fixed $z \in \Omega$, let $m(z,t) \doteqdot \max_{x\in\Omega} H(z, x, t)$. Then*

$$\frac{d}{dt}\log m(z,t) \leqslant n\frac{d}{dt}\log \bar{H}(0,t) - \inf q.$$

Proof. Since $m(y, t)$ may not be smooth in general, the derivative is understood as the Dini derivative from the left. Since $H(z, x, t)$ takes the 0 value on the boundary, it attains its maximum interior. Let $x(t)$ be such a point where $m(z, t)$ is attained. Then

$$\begin{aligned}\frac{d}{dt}\log m(z,t) &\leqslant \lim_{h\to 0} \frac{\log H(z, x(t), t) - \log H(z, x(t), t-h)}{h}\\ &\leqslant \frac{\Delta_x H(z, x(t), t)}{H(z, x(t), t)} - \inf q\\ &= \Delta \log H\big(z, x(t), t\big) - \inf q\\ &\leqslant n \lim_{s\to 0} \frac{(\log \bar{H}(s,t))'}{s} - \inf q\\ &\leqslant n\big(\log \bar{H}(0,t)\big)'' - \inf q\\ &= n(\log \bar{H})_t(0,t) - \inf q.\end{aligned}$$

Here in the third equation $\nabla H(z, x(t), t) = 0$ is used; in line 4 estimate (1.2) is used; in the last line the fact that $\bar{H}'(0, t) = 0$ is used. $\square$

Note that Corollary 7.5 implies Corollary 7.4 since the decay rate of $H(z, x, t)$ is $e^{-\lambda_0 t}$ and the decay rate of $\bar{H}(0, t)$ is $e^{-\mu_0 t}$.

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