J. reine angew. Math., Ahead of Print
DOI 10.1515/crelle-2023-0061

Journal für die reine und angewandte Mathematik

A Schur's theorem via a monotonicity and the expansion module

By *Lei Ni* at San Diego

Abstract. In this paper we present a monotonicity which extends a classical theorem of A. Schur comparing the chord length of a convex plane curve with that of a space curve of smaller curvature. The main result establishes a Schur's Theorem for spherical curves, which extends the well-known Cauchy's Arm Lemma. We also extend the result by allowing the curve with the smaller curvature being any curve in $\mathbb{S}^n$.

1. Introduction

For a convex curve $c(s) : [0, L] \to \mathbb{R}^2$ and a smooth curve in $\widetilde{c}(s) : [0, L] \to \mathbb{R}^3$ of the same length (both parametrized by the arc-length), A. Schur's theorem [10, Theorem A, p. 31] (see also [7]) asserts that: *If both curves are embedded, and the curvature of the space curve* $\widetilde{k}(s) := |\widetilde{T}'|(s)$, *where* $\widetilde{T}(s) = \widetilde{c}'(s)$ *is the tangent vector, is not greater than the curvature* $k(s)$ *of the convex curve, then* $d_{\mathbb{R}^3}(\widetilde{c}(0), \widetilde{c}(L)) \geq d_{\mathbb{R}^2}(c(0), c(L))$. Here $c'(s)$ denotes $\frac{d}{ds}c(s)$. From the proof of [10] it is easy to see $\mathbb{R}^3$ can be replaced by $\mathbb{R}^{n+1}$ with any $n \geq 1$.

The theorem can be proven for curves whose tangents have finite discontinuous jumps, and to the situation that the curvature of the smaller curve $\widetilde{c}(s)$ is a curve in $\mathbb{R}^{n+1}$ for $n \geq 1$. For each $1 \leq j \leq N-1$ at the point $c(s_j)$ the oriented turning angles (say counter-clock wisely) is measured by signed distance $\alpha_j := d_{\mathbb{S}^n}(c'(s_j-), c'(s_j+)) > 0$ (here note that $\{c'(s)\}$ is viewed as points in a great circle $\mathbb{S}^1$ inside $\mathbb{S}^n$ and an orientation is given based on the region enclosed by $c(s)$ and the chord $\overline{c(0)c(L)}$ is on the left side of $c'(s)$). The turning angle $\widetilde{\alpha}_j$ is measured simply by $\widetilde{\alpha}_j = d_{\mathbb{S}^n}(\widetilde{c}'(s_j-), \widetilde{c}'(s_j+))$. In terms of the generalization to curves with finite discontinuous points for the tangent, it assumes that there exists $\{s_j\}_{0\leq j\leq N}$ such that $0 = s_0 < s_1 < \cdots < s_k < \cdots < s_N = L$ such that both $c(s)$ and $\widetilde{c}(s)$ are regular embedded curves for $s \in (s_{j-1}, s_j)$ for all $1 \leq j \leq N$ satisfying $k(s) \geq \widetilde{k}(s)$ and above defined α_j and $\widetilde{\alpha}_j$ satisfy that $\alpha_j \geq \widetilde{\alpha}_j$ for all $1 \leq j \leq N-1$. The convexity of $c(s)$ and the simpleness assumption imply that $\alpha_j \in (0, \pi)$ and

$$\sum_{j=1}^{N} \int_{s_{j-1}}^{s_j} k(s)\, ds + \sum_{j=1}^{N-1} \alpha_j \leq 2\pi. \tag{1.1}$$

This extension, together with some ingenious applications of the hinge's theorem, allows one to prove the famous Cauchy's Arm Lemma for geodesic arms in the unit sphere (consisting of continuous broken great/geodesic arcs with finite jumps of the tangents) in [10, Lemma II, pp. 37–38]. The lemma became famous due to that it had an incomplete/false proof by Cauchy originally [6]. The corrected proof appeared in [1, 15] (cf. [1, p. 92 and Section 3.1]). This spherical Cauchy's Arm Lemma can also be proved by an induction argument [16], whose idea in fact in part resembles the proof of the smooth case to some degree. Note that this lemma of Cauchy plays a crucial role in the rigidity of convex polyhedra in $\mathbb{R}^3$, which finally was vastly generalized to convex surfaces (convex bodies enclosed) as the famous Pogorelov monotype theorem (cf. [5, Section 21]).

Schur's theorem also can be applied to prove the four-vertex theorem for convex plane curves, besides implying a theorem of H. A. Schwartz which asserts: *For any curve c of length L with curvature $k(s) \le \frac{1}{r}$, let C be the circle passing $c(0)$ and $c(L)$ of radius r, then L is either not greater than the length of the lesser circular arc, or not less than the length of the greater circular arc of C.* High-dimensional (intrinsic) analogues of A. Schur's theorem include the Rauch's comparison theorem and the Toponogov comparison theorem. The later however has the limit of requiring that the manifold with less curvature must be a space form of constant sectional curvature.

First we extend slightly the classical Schur's theorem in terms of a monotonicity.

Theorem 1.1. *Let $c : [0, L] \to \mathbb{R}^2$ be a simple piece wisely regular convex plane curve with curvature $k(s) \ge 0$ and finite many discontinuities for the tangent at $\{s_j\}_{j=1}^{N-1}$, and let $\widetilde{c} : [0, L] \to \mathbb{R}^{n+1}$ $(n \ge 1)$ be another simple curve such that $\widetilde{k}(s) = |\widetilde{T}'|(s) \le k(s)$. Moreover, we assume that the turning angles of $\{\alpha_j\}$ and $\widetilde{\alpha}_j$ satisfy $\alpha_j \ge \widetilde{\alpha}_j$. Then for any $0 \le s' < s'' \le L$ there exists a linear isometric map $\iota_{s',s''} : \mathbb{R}^2 \to \mathbb{R}^{n+1}$ with $\iota_{s',s''}(0) = 0$ such that*

$$I(s) := \langle \widetilde{c}(s) - \iota_{s',s''}(c(s)), \iota_{s',s''}(c(s'') - c(s')) \rangle$$

is monotone non-decreasing for $s \in [s', s'']$, or equivalently

$$\langle \widetilde{T}(s) - \iota_{s',s''}(T(s)), \iota_{s',s''}(c(s'') - c(s')) \rangle \ge 0, \quad s \in [s', s''] \setminus \bigcup_{k=1}^{N-1} \{s_j\}. \tag{1.2}$$

We note that there is a freedom of the linear isometry by a rotation fixing the vector $\iota_{s',s''}(c(s'') - c(s'))$. As $s' \to s''$, when s'' is a smooth point, the linear isometric embedding $\iota_{s',s''}$ converges to one identifying $T(s)$ with $\widetilde{T}(s)$ factoring this freedom.

Corollary 1.2. *Under the same assumption as in the theorem, for any $s' \le s'_* < s''_* \le s''$,*

$$\langle c(s''_*) - c(s'_*), c(s'') - c(s') \rangle \le \langle \widetilde{c}(s''_*) - \widetilde{c}(s'_*), \iota_{s',s''}(c(s'') - c(s')) \rangle. \tag{1.3}$$

When $s' = s'_*$ and $s'' = s''_*$, we have

$$|c(s'') - c(s')|^2 \le \langle \widetilde{c}(s'') - \widetilde{c}(s'), \iota_{s',s''}(c(s'') - c(s')) \rangle. \tag{1.4}$$

The equality holds if and only if $\iota(c(s))|_{[s',s'']}$ and $\widetilde{c}(s)|_{[s',s'']}$ are the same.

Estimate (1.4) implies Schur's theorem by the Cauchy–Schwarz inequality applied to the right-hand side of (1.4):

$$|c(s'') - c(s')| \le |\widetilde{c}(s'') - \widetilde{c}(s')|, \quad 0 \le s' < s'' \le L.$$

This extension allows one to rephrase the result in terms of the concept of the *expansion module* [2,12] of vector fields. If $X : \Omega \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a vector field defined on a convex domain, then the expansion module is a function of one variable $\psi(t)$ such that

$$\langle X(y) - X(x), \frac{y-x}{|y-x|}\rangle \geq 2\psi\left(\frac{|x-y|}{2}\right).$$

Since $\tilde{c}(s)$ and $c(s)$ are related via the parameter s, one may view $\tilde{c}$ as a related vector field defined over $\iota_{s',s''}(c(s)) \in \mathbb{R}^{n+1}$. Now the estimate in Theorem 1.1 simply asserts that the related vector fields $\tilde{c}(s)$ has an expansion module function $\psi(t) = t$ with respect to the associated vector $\iota_{s',s''}(c(s))$.

From the above connection between the concept of *curvature* and the *expansion module* it is our hope that a high-dimensional Schur's theorem could be discovered through the consideration involving the expansion module.

Given that Schur's theorem implies the Cauchy's Arm Lemma for the arms of great arcs in the unit sphere $\mathbb{S}^2$, a natural question is that if the spherical analogue of Schur's theorem still holds. Namely, given two embedded spherical curves $c(s)$ and $\tilde{c}(s)$ in the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ parametrized by the arc-length $s \in [0, L]$ with $L \leq \pi$. Assume that $c(s)$ is convex (in the sense that $c(s)$ together with a minimizing arc joining $c(0)$ and $c(L)$ bound a convex region in $\mathbb{S}^2$) with geodesic curvature $k(s) \geq 0$, and that the geodesic curvature of $\tilde{c}$ satisfies $|\tilde{k}|(s) \leq k(s)$. Does it still hold that $d_{\mathbb{S}^2}(c(0), c(L)) \leq d_{\mathbb{S}^2}(\tilde{c}(0), \tilde{c}(L))$? One could also allow the tangent of curves to have same amount of finite many jumps at $\{s_j\}$. In that case, at each s_j, the oriented angle $\alpha_j \in (0, \pi)$ and the turning angle $\tilde{\alpha}_j = d_{\mathbb{S}^2}(\tilde{T}(s_j-), \tilde{T}(s_j+))$ are assumed to satisfy that $\alpha_j \geq \tilde{\alpha}_j$ as in the previous case. The Cauchy's Arm Lemma in the sphere provides a positive answer in the special case where both curves have zero geodesic curvature for the smooth parts. Given that the classical Schur's theorem implies the four vertex theorem, and an open problem in [3] is a spherical analogue of the four vertex theorem, a positive answer to this question, beside a natural extension of Cauchy's Arm Lemma, could be possibly useful in the study of mechanics. Here we confirm this conjecture by proving

Theorem 1.3. *The Schur's theorem holds for two curves in $\mathbb{S}^2 \subset \mathbb{R}^3$ under the above configurations similar to that of Theorem* 1.1. *The equality holds if and only if the two curves are congruent by an isometry of* $\mathbb{S}^2$.

In fact, what can be proved in Theorem 3.1 is more general. The proof is by a construction of auxiliary curves via the cones over $c(s)$ and $\tilde{c}(s)$, with one of them being a convex plane curve, and then reduce it to the above generalized Schur's theorem, namely Theorem 1.1. A similar construction was employed in [14] to reduce the interior regularity of a strictly convex Alexandrov solution to that of a Minkowski problem solution. For the sake of completeness we present the proof of Theorem 1.1 with details. Note that Theorem 1.3 generalizes the spherical Cauchy's Arm Lemma. It would be interesting to see if it plays any role in the proof of Pogorelov's monotype theorem. There were extensions of A. Schur's theorem in hyperbolic spaces [8] and in the Minkowski plane [11] earlier. It is also interesting to see if the method of this paper can be used to provide a unified/alternate proof of the former work via Theorem 2.1.

The generalization in Theorem 1.1 allows a truly spherical version of Theorem 1.1 in Theorem 3.1 providing a comparison between a convex curve in $\mathbb{S}^2$ and a curve in $\mathbb{S}^n$, for any $n \geq 2$, with its geodesic curvature $\tilde{k}(s) \leq k(s)$.

The exposition [3] contains other interesting topics and constructions on spherical curves and their relevance in the study of mechanics. In particular, there exists a conjecture concerning the pseudo-functions in [3, Section 30] which is the analogue of four-vertex theorem for spherical curves. It would be interesting if the above theorem can be applied to this problem. Another possible application is towards the study of the flow of space curves.

2. Proof of Theorem 1.1

We prove the theorem and its corollary together. After a linear isometric embedding $\iota : \mathbb{R}^2 \to \mathbb{R}^{n+1}$, which shall be specified later, we may consider the tangent $T(s)$ and $\widetilde{T}(s)$ as two curves in $\mathbb{S}^n$. For the situation when the tangent $T(s)$ has a jump at s_j, the minimizing arc jointing $T(s_j-)$ and $T(s_j+)$ is also considered to be part of the image. They together form a part of a great circle which is denoted by Image$(T(s))$. Namely, we may view $T(s)$ as a set valued map into $\mathbb{S}^1 \subset \mathbb{R}^2$. First we choose a point $\mathcal{N} \in$ image$(T(s))$ as a *normalized tangent*. Without the loss of generality we assume s' and s'' are two smooth points.

The convexity of $c(s)$ implies that image$(T(s))$ is part of a great arc of $\mathbb{S}^1 \subset \mathbb{R}^2$. If we parametrize it with angle $\theta(s)$ from the positive x-axis, then $\theta(s'') \geq \theta(s')$ if $s'' \geq s'$. We first find an $s_* \in [s', s'']$ and a vector $\mathcal{N} \in T(s_*)$ such that it equals $(c(s'') - c(s'))/|c(s'') - c(s')|$. When s_* is a smooth point, $T(s_*)$ is single valued. If $s_* = s_j$ for some j, we pick one value in the arc spanning from $T(s_*-)$ to $T(s_*+)$. To make the argument easy, we put the plane curve $c(s)|_{[s',s'']}$ into a xy-plane so that the vector $c(s'') - c(s')$ is in the positive x-axis direction, and $c(s')$ and $c(s'')$ are on the x-axis. The curve $c(s)|_{(s',s'')}$ is inside the lower half plane $y \leq 0$. The curve $c(s)$ can also be expressed as $(x(s), y(s))$ with $y(s) \leq 0$. Note that $y(s') = y(s'') = 0$ (and $x(s') < x(s'')$). The continuous piece-wisely smooth function $y(s)$ attains a minimum somewhere at $s_* \in (s', s'')$. Without loss of generality we may assume that $y(s_*) < 0$, otherwise $c(s)$ is a line segment with $k(s) \equiv 0$ (which implies $\widetilde{k}(s) \equiv 0$, namely $\widetilde{c}(s)$ is also a line segment). If $s_* \neq s_j$, namely it is a smooth point, then $y'(s_*) = 0$, which implies that $T(s_*) = (x'(s_*), 0)$. Namely $T(s_*)$ is parallel to the x-axis. By the convexity, the angle between $T(s)$ and the x-axis starts with $\theta(s') \geq -\pi$, and monotonically increases into $\theta(s'') \leq \pi$. Hence the angle between $T(s_*)$ and the x-axis can only be $-\pi, 0, \pi$. We claim that at s_* the angle between $T(s_*)$ and the x-axis can only be zero, namely $x'(s_*) > 0$, otherwise $c(s_*)$ must be on x-axis, namely $y(s_*) = 0$, which contradicts to $y(s_*) < 0$. If $s_* = s_j$ for some s_j, namely the tangent has a turn at s_*, then $y'(s_*-) \leq 0$ and $y'(s_*+) \geq 0$ as the consequence of that $y(s_*)$ is the negative minimum of $y(s)|_{[s',s'']}$. Since the angle between $T(s_*)$ and the x-axis must be in $(-\pi, \pi)$ (otherwise $y(s_*) = 0$ as the above), and the angle between $T(s_*-)$ and the x-axis is in $(-\pi, 0]$, while the angle between $T(s_*+)$ and the x-axis is in $[0, \pi)$, we can find a value between $T(s_*-)$ and $T(s_*+)$ such that it is in the positive x-axis direction. Putting these together we have found $s_* \in [s', s'']$ and a vector $\mathcal{N} \in T(s_*)$, which is a positive multiple of $c(s'') - c(s')$.

After a possible rotation, when s_* is a smooth point, we find a linear isometric embedding $\iota : \mathbb{R}^2 \to \mathbb{R}^{n+1}$ such that

$$\iota(\mathcal{N}) = \widetilde{T}(s_*).$$

We will specify the linear isometric embedding/identification later when s_* is a point at which the tangent of $c(s)$ has a jump. Below we shall omit ι and denote the curve $\iota(c(s)$ $(\iota(T(s)))$ as $c(s)$ $(T(s)$ respectively).

Now consider the two products P_i defined as

$$P_1 := \langle c(s'') - c(s'), \mathcal{N} \rangle = \int_{s'}^{s''} \langle T(s), \mathcal{N} \rangle \, ds,$$
$$P_2 := \langle \widetilde{c}(s'') - \widetilde{c}(s'), \mathcal{N} \rangle = \int_{s'}^{s''} \langle \widetilde{T}(s), \mathcal{N} \rangle \, ds.$$

From the choice of s_* and $\mathcal{N}$,

$$\langle c(s'') - c(s'), \mathcal{N} \rangle = |c(s'') - c(s')|.$$

Now

$$P_1 = |c(s'') - c(s')| \quad \text{and} \quad P_2 = \frac{\langle \widetilde{c}(s'') - \widetilde{c}(s'), c(s'') - c(s') \rangle}{|c(s'') - c(s')|}.$$

The claimed estimate (1.4) amounts to showing that the second product is bounded from below by the first (after a linear isometric embedding). Let j_1 be the smallest j such that $s_j \geq s'$, j_2 be the biggest j such that $s_j \leq s''$ and j_3 be the biggest j with $s_j \leq s_*$. Observe the convexity of $c(s)$ implies that, for the case s_* is a smooth point,

$$\alpha_{j_0} + \sum_{j_1 \leq j \leq j_3} \alpha_j + \int_{s'}^{s_{j_1}} k(s)\, ds + \sum_{j_1 \leq j \leq j_3 - 1} \int_{s_j}^{s_{j+1}} k(s)\, ds + \int_{s_{j_3}}^{s_*} k(s)\, ds = \pi$$

and

$$\sum_{j_3+1 \leq j \leq j_2} \alpha_j + \alpha_{j_4} + \int_{s_*}^{s_{j_3+1}} k(s)\, ds + \sum_{j_3+1 \leq j \leq j_2 - 1} \int_{s_j}^{s_{j+1}} k(s)\, ds + \int_{s_{j_2}}^{s''} k(s)\, ds = \pi,$$

with α_{j_0} being the angle from $-\mathcal{N}$ to $T(s')$ and α_{j_4} being the angle from $T(s'')$ to $-\mathcal{N}$. This implies that the image of $T([s', s_*])$ (a piece-wisely smooth spherical curve denoted by Γ_1) and the image of $T([s_*, s''])$ (another a piece-wisely smooth spherical curve denoted as Γ_2) are two minimizing arcs of the great circle. Geometrically, Γ_1 is the piecing together of $T|_{[s', s_{j_1})}$, the connecting great arc jointing $T(s_{j_1}-)$ to $T(s_{j_1}+)$, $T|_{(s_j, s_{j+1})}$, for $j_1 \leq j \leq j_3 - 1$, and connecting arcs joining $T(s_j-)$ to $T(s_j+)$, for $j_1 + 1 \leq j \leq j_3$, and $T|_{(s_{j_3}, s_*]}$. One may express it in terms of the angle $\beta(\eta)$ from $\mathcal{N}$ and parametrize it using $\eta \in [s', s_* + \sum_{k=j_1}^{j_3} \alpha_k]$ as follows. Let $\theta(s)$ denote the angle between $\mathcal{N}$ and $T(s)$ when s is smooth (which is in $[-\pi, 0]$ for $s \in [s', s_*]$ and in $[0, \pi]$ for $s \in [s_*, s'']$). Here an orientation is picked for $c(s) \in \mathbb{R}^2$, as specified in the introduction. For s_j, $\theta(s_j-)$ denotes the angle between $\mathcal{N}$ to $T(s_j-)$. Define $\theta(s_j+)$ similarly. Then $\theta(s_j+) - \theta(s_j-) = \alpha_j$. Explicitly, $\beta(\eta)$, for $\eta \in [s', s_* + \sum_{j_1}^{j_3} \alpha_k]$, which takes value in $[-\pi, 0]$, can be given by

$$\begin{aligned}
\beta(\eta) &= \theta(\eta) && \text{for } s' \leq \eta \leq s_{j_1},\\
\beta(s_{j_1}) &= \theta(s_{j_1}-), &&\\
\beta(\eta) &= \theta(s_{j_1}-) + \eta - s_{j_1} && \text{for } s_{j_1} < \eta < s_{j_1} + \alpha_{j_1},\\
\beta(s_{j_1} + \alpha_{j_1}) &= \theta(s_{j_1}+), &&\\
\beta(\eta) &= \theta(\eta - \alpha_{j_1}) && \text{for } s_{j_1} + \alpha_{j_1} < \eta < s_{j_1+1} + \alpha_{j_1},\\
\beta(s_{j_1+1} + \alpha_{j_1}) &= \theta(s_{j_1+1}-). &&
\end{aligned}$$

For $j_3 - 1 \geq j \geq j_1 + 1$,

$$\beta\left(s_j + \sum_{j_1}^{j-1} \alpha_k\right) = \theta(s_j-), \quad \beta\left(s_j + \sum_{j_1}^{j} \alpha_k\right) = \theta(s_j+),$$

and

$$\beta(\eta) = \theta(s_j-) + \eta - \left(s_j + \sum_{j_1}^{j-1} \alpha_k\right) \quad \text{for } \eta \in \left(s_j + \sum_{j_1}^{j-1} \alpha_k, s_j + \sum_{j_1}^{j} \alpha_k\right),$$

$$\beta(\eta) = \theta\left(\eta - \sum_{j_1}^{j} \alpha_k\right) \quad \text{for } \eta \in \left(s_j + \sum_{j_1}^{j} \alpha_k, s_{j+1} + \sum_{j_1}^{j} \alpha_k\right).$$

Finally,

$$\beta\left(s_{j_3} + \sum_{j_1}^{j_3-1} \alpha_k\right) = \theta(s_{j_3}-), \quad \beta\left(s_{j_3} + \sum_{j_1}^{j_3} \alpha_k\right) = \theta(s_{j_3}+),$$

and

$$\beta(\eta) = \theta(s_{j_3}-) + \eta - \left(s_{j_3} + \sum_{j_1}^{j_3-1} \alpha_k\right) \quad \text{for } \eta \in \left(s_{j_3} + \sum_{j_1}^{j_3-1} \alpha_k, s_{j_3} + \sum_{j_1}^{j_3} \alpha_k\right),$$

$$\beta(\eta) = \theta\left(\eta - \sum_{j_1}^{j_3} \alpha_k\right) \quad \text{for } \eta \in \left(s_{j_3} + \sum_{j_1}^{j_3} \alpha_k, s_* + \sum_{j_1}^{j_3} \alpha_k\right].$$

One can check that $\Gamma_1(\eta)$ is piece-wisely smooth. Moreover, the convexity of $c(s)$ implies that $\beta(\eta'') \geq \beta(\eta')$ if $\eta'' \geq \eta'$. Similarly, a parametrization can be given for Γ_2 with the corresponding $\beta(\eta)$ valuing in $[0, \pi]$. The choice of $\mathcal{N}$, and that there is no 'folding' for Γ_i, imply that

$$\max\{\text{Length}(\Gamma_1), \text{Length}(\Gamma_2)\} \leq \pi. \tag{2.1}$$

Denote the spherical curves corresponding to $\widetilde{T}$ by $\widetilde{\Gamma}_i$. For example, $\widetilde{\Gamma}_1$ is obtained by piecing together of $\widetilde{T}|_{[s', s_{j_1})}$, the minimizing great arc jointing $\widetilde{T}(s_{j_1}-)$ to $\widetilde{T}(s_{j_1}+)$, $\widetilde{T}|_{(s_j, s_{j+1})}$ for $j_1 \leq j \leq j_3 - 1$, and minimizing arcs joining $\widetilde{T}(s_j-)$ to $\widetilde{T}(s_j+)$ for $j_1 + 1 \leq j \leq j_3$, and $\widetilde{T}|_{(s_{j_3}, s_*]}$. However, since $\tilde{c}(s)$ is not necessarily a convex plane curve, it follows that $\widetilde{\Gamma}_i$ over (s_j, s_{j+1}), namely $\widetilde{T}|_{(s_j, s_{j+1})}$ is not necessarily in a plane. Noting that $\mathcal{N} = T(s_*) = \widetilde{T}(s_*)$ we estimate

$$\begin{aligned}
\pi &\geq d_{\mathbb{S}^n}(T(s'), \mathcal{N}) = d_{\mathbb{S}^n}(T(s'), T(s_*)) = \text{Length}(\Gamma_1) \\
&= \sum_{j_1 \leq j \leq j_3} \alpha_j + \int_{s'}^{s_{j_1}} k(s)\, ds + \sum_{j_1 \leq j \leq j_3 - 1} \int_{s_j}^{s_{j+1}} k(s)\, ds + \int_{s_{j_3}}^{s_*} k(s)\, ds \\
&\geq \sum_{j_1 \leq j \leq j_3} \tilde{\alpha}_j + \int_{s'}^{s_{j_1}} \tilde{k}(s)\, ds + \sum_{j_1 \leq j \leq j_3 - 1} \int_{s_j}^{s_{j+1}} \tilde{k}(s)\, ds + \int_{s_{j_3}}^{s_*} \tilde{k}(s)\, ds \\
&= \sum_{j_1 \leq j \leq j_3} \tilde{\alpha}_j + \int_{s'}^{s_{j_1}} |\widetilde{T}'|(s)\, ds + \sum_{j_1 \leq j \leq j_3 - 1} \int_{s_j}^{s_{j+1}} |\widetilde{T}'|(s)\, ds + \int_{s_{j_3}}^{s''} |\widetilde{T}'|(s)\, ds \\
&= \text{Length}(\widetilde{\Gamma}_1) \geq d_{\mathbb{S}^n}(\widetilde{T}(s'), \widetilde{T}(s_*)) = d_{\mathbb{S}^n}(\widetilde{T}(s'), \mathcal{N}).
\end{aligned}$$

The second line above follows from the definition of the curvature (for the space curve) as the derivative of the angle $\theta(s)$ between the tangent $T(s)$ and $T(s+h)$ for a curve [13, p. 49] and that for a convex curve $k(s) \geq 0$. The third line uses the assumption of the theorem, and the last line follows from the definition of the (spherical) distance between two points being the infimum of the length of all possible piece-wisely smooth pathes (in $\mathbb{S}^n$) connecting them. By the monotonicity of the parametrization of $\beta(\eta)$ the same argument also implies that for any smooth $s \in [s', s_*]$,

$$\pi \geq d_{\mathbb{S}^n}(T(s), \mathcal{N}) = \text{Length}(\Gamma_1|_{[s,s_*]}) \geq \text{Length}(\widetilde{\Gamma}_1|_{[s,s_*]}) \geq d_{\mathbb{S}^n}(\widetilde{T}(s), \mathcal{N}).$$

Here $\Gamma_1|_{[s,s_*]}$ means the curve obtained by piecing together $T|_{[s,s_*]}$ as explained above. The same applies to $\widetilde{\Gamma}_1|_{[s,s_*]}$. The above estimate implies that

$$\begin{aligned} \langle T(s), \mathcal{N}\rangle &= \cos(d_{\mathbb{S}^n}(T(s), T(s_*)) \leq \cos(d_{\mathbb{S}^n}(\widetilde{T}(s), \widetilde{T}(s_*)) \\ &= \cos(d_{\mathbb{S}^n}(\widetilde{T}(s), \mathcal{N})) = \langle \widetilde{T}(s), \mathcal{N}\rangle. \end{aligned} \tag{2.2}$$

Rewriting the above estimate, we have

$$\langle \widetilde{T}(s) - T(s), \mathcal{N}\rangle \geq 0 \quad \text{for } s \in [s', s_*],$$

which implies (1.2). A similar argument proves that inequality (2.2) holds also for $s \in [s_*, s'']$. Putting them together, we have (1.2).

Now we explain how to handle the case when $s_* = s_{j_3}$ is not a smooth point. By the previous construction $\mathcal{N} \in T(s_{j_3})$. Recall that the image of $T(s_{j_3})$ is the minimizing arc joining $T(s_{j_3}-)$ to $T(s_{j_3}+)$. Assume that $\gamma(t) : [0, \alpha_{j_3}] \to \mathbb{S}^1$ is this arc parametrized by the arc-length t with $\gamma(0) = T(s_{j_3}-)$ and $\gamma(\alpha_{j_3}) = T(s_{j_3}+)$. Assume that $\gamma(t_1) = \mathcal{N}$ for some $t_1 \in [0, \alpha_{j_3}]$. There exists a similar minimizing geodesic $\widetilde{\gamma}(t) : [0, \widetilde{\alpha}_{j_3}] \to \mathbb{S}^n$ corresponding to $\widetilde{T}(s_{j_3})$. If $t_1 \leq \widetilde{\alpha}_{j_3}$, we choose a linear isometric embedding which identifies $\widetilde{\gamma}(t_1)$ with $\mathcal{N}$. If $\alpha_{j_3} - t_1 \leq \widetilde{\alpha}_{j_3}$, we identify $\mathcal{N}$ with $\widetilde{\gamma}(\widetilde{\alpha}_{j_3} - \alpha_{j_3} + t_1)$. If none of the previous two cases holds, namely $t_1 > \widetilde{\alpha}_{j_3}$ and $\alpha_{j_3} - t_1 > \widetilde{\alpha}_{j_3}$, we identify $\mathcal{N}$ with any arbitrarily chosen $\widetilde{\gamma}(t)$ with $t \in [0, \widetilde{\alpha}_{j_3}]$. For example, one choice of the linear isometric embedding is to identify $\mathcal{N}$ with $\widetilde{\gamma}(0) = \widetilde{T}(s_{j_3}-)$. The above identification ensures the comparison since

$$\begin{aligned} d_{\mathbb{S}^n}(T(s'), \mathcal{N}) &= \sum_{j_1}^{j_3-1} \alpha_j + \int_{s'}^{s_{j_1}} k(s)\,ds + \sum_{j_1}^{j_3-1} \int_{s_j}^{s_{j+1}} k(s)\,ds + t_1 \\ &\geq \sum_{j_1}^{j_3-1} \widetilde{\alpha}_j + \int_{s'}^{s_{j_1}} \widetilde{k}(s)\,ds + \sum_{j_1}^{j_3-1} \int_{s_j}^{s_{j+1}} \widetilde{k}(s)\,ds + d_{\mathbb{S}^n}(\widetilde{T}(s_{j_3}-), \mathcal{N}) \\ &\geq d_{\mathbb{S}^n}(\widetilde{T}(s'), \mathcal{N}) \end{aligned}$$

by $t_1 \geq d_{\mathbb{S}^n}(\widetilde{T}(s_{j_3}-), \mathcal{N})$, which equals either t_1, $\widetilde{\alpha}_{j_3} - \alpha_{j_3} + t_1$ or 0 according to the three possible identifications above. (If in the last case of the above three situations we choose a linear isometric embedding to identify $\mathcal{N}$ with $\widetilde{\gamma}(t_2)$ the arc jointing $\widetilde{T}(s_{j_3}-)$ with $\widetilde{T}(s_{j_3}+)$, the estimate remains true due to that $t_1 > \widetilde{\alpha}_{j_3} \geq t_2$ since this inequality holds when the first two situations do not arise.) The comparisons for $s \in [s', s_*]$ and $s \in [s_*, s'']$ are the same.

Now we compare the two products P_i by writing

$$P_1 = \int_{s'}^{s''} \langle T(s), \mathcal{N}\rangle\,ds = \left(\int_{s'}^{s_*} + \int_{s_*}^{s''}\right) \cos(d_{\mathbb{S}^n}(T(s), T(s_*))\,ds.$$

We express P_2 accordingly. The above estimate (2.2) implies that

$$\int_{\eta}^{s_*} \cos(d_{\mathbb{S}^n}(T(s), T(s_*))\, ds \le \int_{\eta}^{s_*} \cos(d_{\mathbb{S}^n}(\widetilde{T}(s), \widetilde{T}(s_*))\, ds, \quad \eta \in [s', s_*]. \tag{2.3}$$

Similarly, we have

$$\int_{s_*}^{\eta} \cos(d_{\mathbb{S}^n}(T(s), T(s_*))\, ds \le \int_{s_*}^{\eta} \cos(d_{\mathbb{S}^n}(\widetilde{T}(s), \widetilde{T}(s_*))\, ds, \quad \eta \in [s_*, s'']. \tag{2.4}$$

From (2.3) and (2.4), applied to $\eta = s'$ and $\eta = s''$ we have $P_1 \le P_2$, namely the desired claim (1.4). The equality case can be seen by tracing estimates. Applying them to $\eta = s'_*$ and s''_* respectively, we have (1.3). This completes the proof of corollary.

From the proof we have the following more general monotonicity since the proof for the monotonicity works as along as $\mathcal{N}$ satisfies (2.1), while (1.1) implies that one can always choose an $s_* \in [0, L]$ and $\mathcal{N} \in T(s_*)$ independent of s' and s''.

Proposition 2.1. *Let $c : [0, L] \to \mathbb{R}^2$ be an embedded convex plane curve with curvature $k(s) \ge 0$. Let $\widetilde{c} : [0, L] \to \mathbb{R}^{n+1}$ ($n \ge 1$) be a curve such that $\widetilde{k}(s) \le k(s)$. Then there exist $s_* \in [0, L]$, $\mathcal{N} \in T(s_*)$ and a linear isometric embedding $\iota : \mathbb{R}^2 \to \mathbb{R}^{n+1}$ with $\iota(0) = 0$ and $\iota(\mathcal{N}) \in \widetilde{T}(s_*)$ such that for any smooth s,*

$$I_1'(s) = \langle \widetilde{T}(s) - \iota(T(s)), \iota(\mathcal{N})\rangle \ge 0, \quad s \in [0, L],$$

where $I_1(s) = \langle \widetilde{c}(s) - \iota(c(s)), \iota(\mathcal{N})\rangle$.

The proof can easily be adopted to show a comparison between a time-like curves in a Minkowski plane L^2_1 and another time-like curve in the three-dimensional Minkowski space L^3_1 with signature $(+, -, -)$. In fact, in terms of the monotonicity one may choose s_* freely.

Following the convention of the physics, we call a vector u time-like if $\langle u, u\rangle > 0$. For a time-like curve $c(s)$, parametrized by the arc-length, $|T(s)|^2 = |c'(s)|^2 = 1$. Hence $T(s)$ can be viewed as a point in the hyperbolic line H^1 (hyperbolic plane H^2) defined as $x_1^2 - x_2^2 = 1$ ($x_1^2 - x_2^2 - x_3^2 = 1$, respectively). It can be checked easily that -1 multiple of the restricted metric on the line/surface is the standard hyperbolic metric. Using the arc-length s, $T(s)$ can be expressed as $(\cosh\theta(s), \sinh\theta(s))$. Hence $\theta(s)$ is the analogous angle function of $T(s)$ in H^1. A simple computation shows that, for a convex curve, the angle difference $\varphi(s_2, s_1) = \theta(s_2) - \theta(s_1)$ equals the hyperbolic distance between $T(s_1)$ and $T(s_2)$. In fact, $T'(s) = (\sinh\theta(s), \cosh\theta(s))\theta'(s)$. Note that

$$d_{H^1}(T(s_2), T(s_1)) = \int_{s_1}^{s_2} \sqrt{-\langle T'(s), T'(s)\rangle} = \int_{s_1}^{s_2} \theta'(s)\, ds = \theta(s_2) - \theta(s_1),$$

where we have used the fact that $\theta'(s) = k(s) \ge 0$. Moreover, for any two vectors $u, v \in H^1$ (or H^n), namely $|u| = |v| = 1$, without the loss of generality we may express them as $(1, 0)$ and $(\cosh\theta(s_1), \sinh\theta(s_1))$ for some $s_1 \ne 0$. For $s_1 > 0$, consider the curve $u(s) : [0, s_1] \to H^1$ given by $u(s) := (\cosh\theta(s), \sinh\theta(s))$ joining u and v,

$$\langle u, v\rangle - 1 = \left\langle u(0), \int_0^{s_1} u'(s)\, ds \right\rangle = \int_0^{s_1} \sinh\theta(s)\, \theta'(s)\, ds = \cosh\theta(s_1) - 1.$$

Hence the Minkowski inner product $\langle u, v\rangle$ of two unit time like vectors is given by $\cosh\theta$, with θ being the hyperbolic angle function. These observations are the hyperbolic analogues of the facts used in the above proof (concerning the angle difference being the distance between two tangents in $\mathbb{S}^1$), which allow a similar consideration as the above to yield the following result.

Theorem 2.1. *Let $c(s) : [0, L]$ be a time-like convex curve in L^2_1 parametrized by the arc-length, and let $\widetilde{c}(s) : [0, L]$ be a similarly parametrized regular time-like curve in L^3_1. Assume that $k(s) \geq |\widetilde{k}|(s)$. Then for any $s_* \in [0, L]$ and a linear isometric embedding of $\iota : L^2_1 \to L^3_1$, which identifies $T(s_*)$ with $\widetilde{T}(s_*)$, we have*

$$I_2'(s) = \langle \iota(T(s)) - \widetilde{T}(s), \widetilde{T}(s_*)\rangle \geq 0, \tag{2.5}$$

where $I_2(s) = \langle \iota(c(s)) - \widetilde{c}(s), \widetilde{T}(s_*)\rangle$. In particular, $|c(L) - c(0)| \geq |\widetilde{c}(L) - \widetilde{c}(0)|$. The equality holds if and only if $\iota(c(s)) = \widetilde{c}(s)$.

The last statement above generalizes the result of [11] by allowing the second curve $\widetilde{c}(s)$ to be a space curve in L^3_1. To derive this, by the first part of the argument in the proof of Theorem 1.1 and [11, Lemma 3], we can find s_* so that $T(s_*)$ (or a vector in $T(s_*)$ if s_* is not a smooth point) is a positive multiple of $c(L) - c(0)$. Now integrate (2.5) with s_* so chosen and then apply the reversed Cauchy–Schwarz inequality (which holds for two time-like vectors in L^3_1). Note also that for the curves in two Minkowski planes, the result for space-like curves is the same as that for the time-like curves.

3. Proof of Theorem 1.3

First note that if $k(s) \equiv 0$, then $\widetilde{c}(s)$ is a geodesic. Then claimed result holds. Hence we assume that $k(s) > 0$ somewhere.

We start with some basics on spherical (smooth) curves. Let $c(s)$ be a curve in $\mathbb{S}^2$ parametrized by the arc-length. Let $T(s)$ be its tangent, which is orthogonal to $c(s)$. Let $V(s) = c(s) \times T(s)$ be the cross product of $c(s)$ and $T(s)$ in $\mathbb{R}^3$, which is a normal of $c(s)$ in $T_{c(s)}\mathbb{S}^2$. The triple $\{c(s), T(s), V(s)\}$ forms an orthonormal moving frame (of $\mathbb{R}^3$) along $c(s)$. Since the geodesic curvature of a curve in the sphere (in a surface) is the changing rate of the tangential great circles (tangential geodesics in general, by [17, (8-3) of p. 157]), and that $V(s)$ provides a natural parametrization of the tangential great circles, the derivative of $V(s)$ yields the geodesic curvature of $c(s)$. This can also be formulated in terms of the following result, which resembles the Frenet formulas. (But they are not the same. It is closer to the derivation of Kepler's law in [9].)

Proposition 3.1. *Let $k(s)$ be the geodesic curvature of $c(s)$ (with respect to $\mathbb{S}^2$). Then the following holds for $\{c(s), T(s), V(s)\}$:*

$$\begin{aligned} c'(s) &= T(s), \\ T'(s) &= k(s)V(s) - c(s), \\ V'(s) &= -k(s)T(s). \end{aligned}$$

Proof. The first equation is definition. Also, by definition, $k(s) = \langle T'(s), V(s)\rangle$. Hence from

$$0 = \frac{d^2}{ds^2}(|c|^2(s)) = 2\langle T(s), T(s)\rangle + 2\langle c(s), T'(s)\rangle = 2 + 2\langle c(s), T'(s)\rangle$$

we deduce the second equation. Now by the second equation

$$V'(s) = T(s) \times T(s) + c(s) \times T'(s) = k(s)\, c(s) \times V(s) = -k(s)T(s).$$

This prove the third one, hence completes the proof of the proposition. □

The local convexity of $c(s)$ is equivalent to $k(s) \geq 0$ (cf. [4, Proposition 2.1]). The construction starts with the cone $\mathcal{C}(c(s))$ over the spherical curve $c(s)$ centered at the origin, and then obtain a plane curve $\mathcal{P}_c(s)$ by taking the intersection of $\mathcal{C}(c(s))$ with a plane P not passing the origin. The convexity of $c(s)$ implies that if we join $c(0)$ and $c(L)$ by a minimizing geodesic (namely part of a great circle), it bounds a convex region D in $\mathbb{S}^2$. By approximating $c(s)|_{[0,L]}$ with $c(s)|_{[0,L-\epsilon]}$ for positive $\epsilon \to 0$, we may assume that the total length of the closed simple curve ∂D is less than 2π. By [18, Problem 1.10.4], D is contained inside the interior of a semisphere. Hence the cone over this convex region D bounded by $c(s)$ and the minimizing part of the great arc is convex, and the cone minus the origin is contained inside the interior of a half space of $\mathbb{R}^3$. Hence there exists such a plane P which does not pass the origin such that the intersection with the cone over $c(s)$, $\mathcal{P}_c(s)$ together with the chord $\overline{\mathcal{P}_c(0)\mathcal{P}_c(L)}$ enclose a convex domain in P. (Such a plane often is not unique. We simply choose one such plane.)[1)] The curve $\mathcal{P}_c(s)$ can be expressed as $R(s)c(s)$ with $R(s)$ being the distance of $\mathcal{P}_c(s)$ to the origin. We need the following formula for the curvature of the space curve in $\mathbb{R}^3$ applied to $\mathcal{P}_c(s)$.

Proposition 3.2. *If $c(s)$ is a convex curve in $\mathbb{S}^2$, then $\mathcal{P}_c(s)$ is a convex curve in P. The curvature $\mathbf{k}(s)$ of $\mathcal{P}_c(s)$ (as a space curve of $\mathbb{R}^3$) is given by*

$$\mathbf{k}^2(s) = \frac{|\mathcal{P}_c'(s) \times \mathcal{P}_c''(s)|^2}{|\mathcal{P}_c'(s)|^6}. \tag{3.1}$$

Proof. From the geometric definition of the convexity we know that $c(s)$ lies in a signed semisphere cut out by any tangent great circle obtained by a plane P_0 passing the origin. Then it is clear that $\mathcal{P}_c(s)$ lies on the corresponding half plane cut out by the corresponding tangent line of $\mathcal{P}_c(s)$ in P which is the intersection of P_0 and P. This proves the convexity of $\mathcal{P}_c(s)$. The formula for the curvature of a space curve is well known and computational. See for example [13, p. 51]. Of course, the formula does apply to the case that the curve happens to be a plane curve. □

Now let τ be the arc-length parameter of $\mathcal{P}_c(s)$. Direct calculation shows that

$$\tau(s) = \int_0^s \sqrt{(R'(s))^2 + R^2(s)}\, ds. \tag{3.2}$$

We construct a space curve $\widetilde{\mathcal{P}}_{\tilde{c}}(s)$ corresponding to $\tilde{c}(s)$ by defining it as $R(s)\tilde{c}(s)$. In general, this is not a plane curve. The key observation is that the arc-length parameter for $\widetilde{\mathcal{P}}_{\tilde{c}}(s)$ is the same as that of $\mathcal{P}_c(s)$, namely it is given by (3.2) as well, since $|\tilde{c}|(s) = 1 = |c(s)|$ and $|\tilde{c}'|(s) = 1 = |c'(s)|$. Moreover, its curvature $\tilde{\mathbf{k}}(s)$ (as a curve in $\mathbb{R}^3$) can be expressed

1) This is only place where we used $L \leq \pi$. By [18, Problem 1.10.1], which asserts that the great circle and the lune biangle are the only two cases that the total length of the closed simple convex curves equals 2π (it is less than 2π otherwise), and [18, Problem 1.10.4], which asserts that if the length is less than 2π the bounded convex region can be contained inside an open hemisphere, the convexity of $c(s)$ together with the assumption that $c(s)$ is not part of a great circle implies the existence of such a plane without assuming $L \leq \pi$.

similarly as

$$\widetilde{\mathbf{k}}^2(s) = \frac{|\widetilde{\mathcal{P}}'_{\widetilde{c}}(s) \times \widetilde{\mathcal{P}}''_{\widetilde{c}}(s)|^2}{|\widetilde{\mathcal{P}}'_{\widetilde{c}}(s)|^6}.$$

Namely, the second part of Proposition 3.2 applies to $\widetilde{\mathcal{P}}_{\widetilde{c}}(s)$ as well since it holds for any space curve in $\mathbb{R}^3$. The key step is the following comparison.

Proposition 3.3. *Under the assumption that the geodesic curvature $k(s)$ of $c(s)$ and the geodesic curvature $\widetilde{k}(s)$ of $\widetilde{c}(s)$ satisfy $k(s) \geq |\widetilde{k}(s)| \geq 0$, the curvatures of $\mathcal{P}_c(s)$ and $\widetilde{\mathcal{P}}_{\widetilde{c}}(s)$ satisfy $\mathbf{k}(s) \geq 0$ and $\mathbf{k}(s) \geq |\widetilde{\mathbf{k}}|(s)$.*

Proof. Since $\mathcal{P}_c(s)$ is convex, we have $\mathbf{k}(s) \geq 0$. It suffices to show that $\mathbf{k}^2(s) \geq \widetilde{\mathbf{k}}^2(s)$. First we observe that

$$|\mathcal{P}'_c(s)|^2 = R^2(s) + (R'(s))^2 = |\widetilde{\mathcal{P}}'_{\widetilde{c}}(s)|^2.$$

This reduces the desired estimate to

$$|\mathcal{P}'_c(s) \times \mathcal{P}''_c(s)|^2 \geq |\widetilde{\mathcal{P}}'_{\widetilde{c}}(s) \times \widetilde{\mathcal{P}}''_{\widetilde{c}}(s)|^2. \tag{3.3}$$

Using the fact that $\{c(s), T(s), V(s)\}$ forms an oriented orthonormal moving frame, a direct calculation, using Proposition 3.1, shows that

$$\begin{aligned}
\mathcal{P}'_c(s) \times \mathcal{P}''_c(s) &= (R'(s)c(s) + R(s)T(s)) \times (R''(s)c(s) + 2R'(s)T(s) + R(s)T'(s)) \\
&= (2(R'(s))^2 - R(s)R''(s))V(s) - R'(s)R(s)k(s)\,T(s) \\
&\quad\quad + R^2(s)k(s)c(s) + R^2(s)V(s) \\
&= R^2(s)k(s)c(s) - R'(s)R(s)k(s)\,T(s) \\
&\quad\quad + (2(R'(s))^2 - R(s)R''(s) + R^2(s))V(s).
\end{aligned}$$

Hence we have

$$\begin{aligned}
|\mathcal{P}'_c(s) \times \mathcal{P}''_c(s)|^2 = (R^4(s) &+ (R'(s)R(s))^2)k^2(s) \\
&+ (2(R'(s))^2 - R(s)R''(s) + R^2(s))^2.
\end{aligned} \tag{3.4}$$

A similar calculation shows that

$$\begin{aligned}
|\widetilde{\mathcal{P}}'_{\widetilde{c}}(s) \times \widetilde{\mathcal{P}}''_{\widetilde{c}}(s)|^2 = (R^4(s) &+ (R'(s)R(s))^2)\widetilde{k}^2(s) \\
&+ (2(R'(s))^2 - R(s)R''(s) + R^2(s))^2.
\end{aligned} \tag{3.5}$$

From (3.4) and (3.5), the assumption $k(s) \geq |\widetilde{k}|(s)$ implies (3.3), hence the desired estimate of the proposition. □

Now Proposition 3.3 and (3.2) imply that $\mathcal{P}_c(\tau)$ and $\widetilde{\mathcal{P}}_{\widetilde{c}}(\tau)$ are two curves satisfying the assumption of Theorem 1.1. Hence we have

$$d_{\mathbb{R}^3}(\mathcal{P}_c(0), \mathcal{P}_c(\tau(L))) \leq d_{\mathbb{R}^3}(\widetilde{\mathcal{P}}_{\widetilde{c}}(0), \widetilde{\mathcal{P}}_{\widetilde{c}}(\tau(L))).$$

Theorem 1.3 for the smooth curves now follows from the hinge theorem of Euclidean geometry.

For the general case when the tangents of $c(s)$ and $\widetilde{c}(s)$ have finite jumps at $\{s_j\}$, if we denote the turning angles at $\mathcal{P}_c(s_j)$ and $\widetilde{\mathcal{P}}_{\widetilde{c}}(s_j)$ by θ_j and $\widetilde{\theta}_j$, then

$$\cos\theta_j = \frac{R'(s_j-)R'(s_j+) + R^2(s_j)\cos\alpha_j}{\sqrt{((R'(s_j-))^2 + R^2(s_j))((R'(s_j+))^2 + R^2(s_j))}}. \tag{3.6}$$

By a similar formula for $\cos\widetilde{\theta}_j$ we deduce that $\theta_j \geq \widetilde{\theta}_j$ if $\alpha_j \geq \widetilde{\alpha}_j$. Hence Theorem 1.3 follows from the general case of Theorem 1.1.

The argument can be modified to prove the following more general result.

Theorem 3.1. *Let $c(s) : [0, L] \to \mathbb{S}^2$ be a simple piece wisely smooth curve in $\mathbb{S}^2$ parametrized by the arc length such that its tangent has finite many discontinuities at $\{s_j\}_{j=1}^{N-1}$. Assume that the curve is convex. Namely together with a minimizing arc joining $c(0)$ and $c(L)$, it bounds a convex region in $\mathbb{S}^2$. Assume that $\widetilde{c}(s) : [0, L] \to \mathbb{S}^n$ is another simple piece wisely smooth curve satisfying that the only possible discontinuities of its tangent $\widetilde{T}(s)$ are at $\{s_j\}$, and the angle $\widetilde{\alpha}_j$ between $T(s_j-)$ and $T(s_j+)$ satisfies $\widetilde{\alpha}_j \leq \alpha_j$. Moreover, assume that the geodesic curvature $\widetilde{k}(s) = |\frac{D\widetilde{T}}{ds}|$ satisfies that $0 \leq \widetilde{k}(s) \leq k(s)$. Then we have that $d_{\mathbb{S}^2}(c(0), c(L)) \leq d_{\mathbb{S}^n}(\widetilde{c}(0), \widetilde{c}(L))$. The equality holds if and only if $\widetilde{c}(s)$ is congruent to $c(s)$ after a proper linear isometric embedding of $\mathbb{S}^2$ into $\mathbb{S}^n$.*

To prove this general result, we need modified versions of Propositions 3.1–3.3 for the curves $\widetilde{c}(s)$ in $\mathbb{S}^n$ and the corresponding curve $\widetilde{\mathcal{P}}_{\widetilde{c}}(s) = R(s) \cdot \widetilde{c}(s)$ in $\mathbb{R}^{n+1}$.

The modified Proposition 3.1 for $\widetilde{c}(s) \in \mathbb{S}^n$ amounts to properly defining $\widetilde{k}(s)$. Let $\frac{D\widetilde{T}}{ds}$ denote the covariant derivative of the tangent $\widetilde{T}$ as a vector in $T_{\widetilde{c}(s)}\mathbb{S}^n$. Then we have

$$\frac{D\widetilde{T}}{ds} = \widetilde{T}'(s) - \langle \widetilde{T}'(s), \widetilde{c}(s)\rangle = \widetilde{T}'(s) + \widetilde{c}(s).$$

Here we also used

$$0 = \frac{1}{2}\frac{d^2}{ds^2}|\widetilde{c}(s)|^2 = \langle \widetilde{T}'(s), \widetilde{c}(s)\rangle + |\widetilde{T}(s)|^2.$$

We define $\widetilde{k}(s) = |\frac{D\widetilde{T}}{ds}|$. When $\frac{D\widetilde{T}}{ds} \neq 0$, we may write it as $\widetilde{k}(s)\widetilde{V}(s)$ with $\widetilde{V}(s)$ being a unit vector which is normal to both $\widetilde{c}(s)$, and $\widetilde{T}(s)$. With this convention we do have

$$\widetilde{c}'(s) = \widetilde{T}(s), \quad \widetilde{T}'(s) = \widetilde{k}(s)\widetilde{V}(s) - \widetilde{c}(s), \tag{3.7}$$

which are sufficient for the comparison of the curvatures of the constructed curves $\mathcal{P}_c(s)$ and $\widetilde{\mathcal{P}}_{\widetilde{c}}(s)$ in $\mathbb{R}^3$ and $\mathbb{R}^{n+1}$. Note that $\widetilde{V}(s)$ is only defined locally when $|\frac{D\widetilde{T}}{ds}|(s) \neq 0$, and $\{\widetilde{c}(s), \widetilde{T}(s), \widetilde{V}(s)\}$ are orthonormal when $\widetilde{V}$ is defined. However, the second equation in (3.7) still holds even when $\widetilde{k}(s) = |\frac{D\widetilde{T}}{ds}| = 0$.

The key is a generalization of Proposition 3.2 since for this general case we can not use a formula involving the cross product to compute the curvature of a curve in $\mathbb{R}^{n+1}$.

Proposition 3.4. *Let $r(s) : (a, b) \to \mathbb{R}^{n+1}$ be a regular curve and τ the arc-length parameter. Let $\mathbf{k}(s)$ be its curvature (defined as $(\langle \frac{d^2r}{d\tau^2}(\tau), \frac{d^2r}{d\tau^2}(\tau)\rangle)^{\frac{1}{2}}$). Then*

$$\mathbf{k}^2(s) = \frac{\langle r''(s), r''(s)\rangle\langle r'(s), r'(s)\rangle - \langle r''(s), r'(s)\rangle^2}{\langle r'(s), r'(s)\rangle^3}. \tag{3.8}$$

Proof. Note that

$$\frac{d\tau}{ds} = |r'|, \quad \frac{dr}{d\tau} = r'\frac{ds}{d\tau} = \frac{r'}{|r'|}.$$

Hence we have

$$\frac{d^2 r}{d\tau^2} = \frac{d}{ds}\left(\frac{dr}{d\tau}\right)\frac{ds}{d\tau} = \frac{d}{ds}\left(\frac{r'}{|r'|}\right)|r'|^{-1} = \frac{r''|r'|^2 - \langle r'', r'\rangle r'}{\langle r', r'\rangle^2}.$$

Using the definition of $\mathbf{k}(s)$, we have

$$\mathbf{k}^2(s) = \left\langle \frac{d^2 r}{d\tau^2}, \frac{d^2 r}{d\tau^2}\right\rangle = \frac{|r''|^2|r'|^4 - \langle r'', r'\rangle^2 |r'|^2}{|r'|^8} = \frac{|r''|^2|r'|^2 - \langle r'', r'\rangle^2}{\langle r', r'\rangle^3}.$$

This proves the claimed formula. □

Now we prove a similar comparison result as Proposition 3.3.

Proposition 3.5. *Under the assumption that the geodesic curvature $k(s)$ of $c(s)$ and the geodesic curvature $\widetilde{k}(s)$ of $\widetilde{c}(s)$ satisfy $k(s) \geq \widetilde{k}(s) \geq 0$, the curvatures of $\mathcal{P}_c(s)$ and $\widetilde{\mathcal{P}}_{\widetilde{c}}(s)$ (as curves in $\mathbb{R}^2$ and in $\mathbb{R}^{n+1}$) satisfy $\mathbf{k}(s) \geq 0$ and $\mathbf{k}(s) \geq |\widetilde{\mathbf{k}}|(s)$.*

Proof. The only difference is we use equation (3.8) to compute the curvature $|\widetilde{\mathbf{k}}|^2(s)$ of $\widetilde{\mathcal{P}}_{\widetilde{c}}(s)$. To simplify the notation, we denote $\widetilde{\mathcal{P}}_{\widetilde{c}}(s)$ by $r(s)$, which is given by $R(s)\cdot\widetilde{c}(s)$. Clearly $r'(s) = R'(s)\widetilde{c}(s) + R(s)\widetilde{c}'(s) = R'(s)\widetilde{c}(s) + R(s)\widetilde{T}(s)$, and by (3.7),

$$r'' = (R'' - R)\widetilde{c} + 2R'\widetilde{T} + R\widetilde{k}\widetilde{V}.$$

Note that the above also holds when $\widetilde{k} = 0$. Now direct calculation shows

$$\begin{aligned}
\langle r'', r''\rangle &= (R'' - R)^2 + 4(R')^2 + R^2\widetilde{k}^2,\\
\langle r', r'\rangle &= (R')^2 + R^2,\\
\langle r'', r'\rangle^2 &= ((R'' - R)R' + 2R'R)^2.
\end{aligned}$$

Putting the above together, Proposition 3.4 implies that

$$((R')^2 + R^2)^3|\widetilde{\mathbf{k}}|^2(s) = ((R'' - R)R - 2(R')^2)^2 + R^2\widetilde{k}^2((R')^2 + R^2).$$

This together with (3.1), the fact that $|\mathcal{P}_c'(s)|^2 = R^2(s) + (R'(s))^2$, and (3.4) proves the proposition. □

Theorem 3.1 follows from Theorem 1.1 (precisely (1.4) applied to $\mathcal{P}_c(s)$ and $\widetilde{\mathcal{P}}_{\widetilde{c}}(s)$), Proposition 3.5, (3.6) and the argument of proving Theorem 1.3.

Acknowledgement. The author would like to thank Burkhard Wilking for helpful discussions, Paul Bryan, Jon Wolfson, H. Wu and Fangyang Zheng for their interests to the problem considered. He is grateful to a referee who spotted a discrepancy in an earlier version of the paper, and another referee for careful reading and suggestions which improve the exposition. Finally, the author thanks Pak-Yeung Chan who read a later version and made comments which also helped the presentation.

References

[1] *A. D. Alexandrov*, Convex polyhedra, Springer Monogr. Math., Springer, Berlin 2005.
[2] *B. Andrews* and *J. Clutterbuck*, Proof of the fundamental gap conjecture, J. Amer. Math. Soc. **24** (2011), no. 3, 899–916.
[3] *V. I. Arnol'd*, The geometry of spherical curves and quaternion algebra, Uspekhi Mat. Nauk **50** (1995), no. 1(301), 3–68.
[4] *P. Bryan* and *J. Louie*, Classification of convex ancient solutions to curve shortening flow on the sphere, J. Geom. Anal. **26** (2016), no. 2, 858–872.
[5] *H. Busemann*, Convex surfaces, Interscience Tracts Pure Appl. Math. **6**, Interscience, New York 1958.
[6] *A. Cauchy*, Sur les polygones et les polyèdres. Second Mémoire, in: Oeuvres Complètes, II^e Série, Gauthier-Villars, Paris (1905), 26–38.
[7] *S. S. Chern*, Curves and surfaces in Euclidean space, in: Studies in global geometry and analysis Mathematical Association of America, Buffalo (1967), 16–56.
[8] *C. L. Epstein*, The theorem of A Schur in hyperbolic spaces, preprint 1985.
[9] *M. W. Hirsch* and *S. Smale*, Differential equations, dynamical systems, and linear algebra, Pure Appl. Math. **60**, Academic Press, New York 1974.
[10] *H. Hopf*, Differential geometry in the large, 2nd ed., Lecture Notes in Math. **1000**, Springer, Berlin 1989.
[11] *R. López*, The theorem of Schur in the Minkowski plane, J. Geom. Phys. **61** (2011), no. 1, 342–346.
[12] *L. Ni*, Estimates on the modulus of expansion for vector fields solving nonlinear equations, J. Math. Pures Appl. (9) **99** (2013), no. 1, 1–16.
[13] *A. V. Pogorelov*, Differential geometry, P. Noordhoff, Groningen 1960.
[14] *A. V. Pogorelov*, The Minkowski multidimensional problem, V. H. Winston & Sons, Washington 1978.
[15] *E. Steinitz* and *H. Rademacher*, Vorlesungen über die Théorie der Polyeder, Springer, Berlin 1934.
[16] *I. J. Schoenberg* and *S. C. Zaremba*, On Cauchy's lemma concerning convex polygons, Canad. J. Math. **19** (1967), 1062–1071.
[17] *D. J. Struik*, Lectures on classical differential geometry, 2nd ed., Dover, New York 1988.
[18] *V. A. Toponogov*, Differential geometry of curves and surfaces, Birkhäuser, Boston 2006.

Lei Ni, Department of Mathematics, University of California, San Diego, La Jolla, CA 92093, USA
e-mail: leni@ucsd.edu

Eingegangen 8. Mai 2023, in revidierter Fassung 22. August 2023