CLOSED TYPE I ANCIENT SOLUTIONS TO RICCI FLOW

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Ancient solutions to Ricci flow arise as the blow up limit of the finite singularities. According to the speed of the singularity forming, it is divided into two types of ancient solutions. An ancient solution $(M, g(t))$ defined on $M \times (-\infty, 0)$ is called of type I if there exists a C such that

$$|Rm|(x,t) \le \frac{A}{|t|}. \tag{0.1}$$

Ancient solutions in dimension 2 and 3 have nonnegative curvature operator by Hamilton-Ivey's pinching estimate [**H**]. In this note we shall restrict to the type I ancient solutions with nonnegative curvature operators. In general, nonnegative curved ancient solutions (even κ-non-collapsed) may not be (locally) symmetric as shown by examples from [**H**] and [**P3**] (see also [**CLN**]). However in [**H**], Hamilton proves that any closed simply-connected ancient type I solution to Ricci flow must be the round sphere. In dimension 3, Perelman [**P1**] observed that any positively curved κ-non-collapsed type I ancient solution must also be the quotient of spheres. Using the recent result of [**BW**] we show the following high dimensional analogue.

Proposition 0.1. *Assume that $(M, g(t))$ is a closed type I, κ-non-collapsed (for some $\kappa > 0$) ancient solution to the Ricci flow with positive curvature operator. Then $(M, g(t))$ must be the quotients $\mathbb{S}^n$.*

It is an easy consequence of [**BW**] that any compact shrinking soliton with positive curvature operator (or 2-positive curvature operator) must be the quotient of spheres. We first observe the following generalizations to closed *gradient* shrinking solitons.

Lemma 0.2. *Assume that $(M, g(t))$ is a closed gradient shrinking soliton with 2-nonnegative curvature operator. Then $(M, g(t))$ must be the quotients of the product of compact symmetric spaces.*

Proof. By Corollary 2.4 of [**NW**], we know that the universal cover $(\tilde{M}, \tilde{g})$ must be the product of closed symmetric spaces and some Euclidean space $\mathbb{R}^k$. We only need to rule out the possibility of $\mathbb{R}^k$ in the factor. A simple way of showing that is by Theorem 1 of [**L**], which implies that the gradient shrinking soliton with nonnegative Ricci curvature must has finite fundamental group. This implies that the universal cover does not have the factor $\mathbb{R}^k$. q.e.d.

Lemma 0.3. *Assume that $(M, g(t))$ is an type I ancient solution of Ricci flow satisfying (0.1). Assume that the diameter of M $D_1 \doteqdot \mathrm{Diam}(M, g(-1)) < \infty$. Then there exists a $C = C(n, A)$ such that for any $t \le -1$, the diameter of $(M, g(t)$ satisfies*

$$\mathrm{Diam}(M, g(t)) \le (2C + \max\{D_1, 1\} + 1)\sqrt{-t}. \tag{0.2}$$

Proof. Let $\tau = -t$. We prove this through arguing by the contradiction. Assume that there exist x and y such that $d_\tau(x, y) \ge C_1\sqrt{\tau}$ for some C_1 to be chosen later. Here $d_\tau(\cdot, \cdot)$ is the distance function with respect to metric $g(t)$. By Theorem 17.1 of [**H**] (see also Lemma 8.3 of [**P2**]) we know that there exists $C_2(n, A)$ such that for any x_1, x_1 with $d_\tau(x_1, x_2) \ge \sqrt{\tau}$

$$\frac{d}{dt} d_\tau(x_1, x_2) \ge -\frac{C_2}{\sqrt{\tau}}. \tag{0.3}$$

Hence for any $1 \leq \tau_1 \leq \tau$ as far as (0.3) holds for all $\tau_1 \leq \eta \leq \tau$, (which follows from $d_\eta(x,y) \geq \sqrt{\eta}$, by Theorem 17.1 of [**H**]) we have that

$$\begin{aligned} d_{\tau_1}(x,y) &\geq d_\tau(x,y) - \int_{\tau_1}^{\tau} \frac{C_2}{\sqrt{\eta}} d\eta \\ &\geq C_1\sqrt{\tau} - 2C_2\sqrt{\tau} \\ &\geq \max\{\mathrm{Diam}(M, g(-1)), 1\}\sqrt{\tau_1} \end{aligned}$$

if we have chosen $C_1 = 2C_2 + \max\{\mathrm{Diam}(M, g(-1)), 1\} + 1$. This inductively shows that we can apply the above estimate for all $1 \leq \tau_1 \leq \tau$. In particular, we have that

$$d_1(x,y) \geq \max\{\mathrm{Diam}(M, g(-1)), 1\} + 1$$

which is clearly a contradiction. q.e.d.

Proof. (of the proposition 0.1.) Apply the the blow-down to $(M, g(t))$ as $t \to -\infty$ and using Proposition 11.2 of Perelman to show that the blow-down limit is a *closed gradient shrinking soliton*. For this we need the assumption that $(M, g(t))$ is κ-non-collapsing for some $\kappa > 0$. Here, since we are dealing with type I solution we can pick $o \in M$ and perform the blow down at (o, τ) by factor $\frac{1}{\tau}$. The κ non-collapsing and Proposition 11.2 of [**P2**] implies that $(M, o, g_\tau(s))$ with $g_\tau(t') = \frac{1}{\tau} g(t'\tau)$ converge to a *gradient shrinking soliton* $(M_\infty, o_\infty, g_\infty(t'))$. By (0.2) we know that for each $(M, (o,\tau), g_\tau(t'))$, at the slice $t' = -1$, its diameter is bounded by some uniform constant C_1. Hence (M_∞, g_∞) is closed and topologically same as $(M, g(-1))$, namely a quotient of $\mathbb{S}^n$. By Lemma 0.2 we conclude that it is metrically a quotient of $\mathbb{S}^n$.

Finally, by [**P2**], the entropy invariant $\nu(M, g(t))$ is monotone non-decreasing in t. $\nu(M, g(t))$ is defined to be $\inf_{\tau > 0} \mu(M, g(t), \tau)$ with

$$\mu(M, g(t), \tau) \doteqdot \inf_{\int_M u\, d\mu_{g(t)} = 1} \int_M \left(\tau(|\nabla f|^2 + R) + f - n\right) u d\mu_{g(t)}$$

with $u = \frac{e^{-f}}{(4\pi\tau)^{n/2}}$. By fact that $(M, g(t)) \to (M_\infty, g_\infty(t))$ after re-scaling, as $t\infty - \infty$, we conclude that $\nu(M, g(t)) \geq \nu(M_\infty, g_\infty(t))$, which is a constant. However, by [**BW**], we also have that $(M, g(t)) \to (M_\infty, g_\infty(t))$ as $t \to 0$. This implies that $\nu(M, g(t))$ is constant. Hence $(M, g(t))$ is a gradient shrinking soliton. q.e.d.

The argument employed in the above proof also yields the following corollary (using [**BS**] instead of [**BW**]).

Corollary 0.4. *1) Assume that $(M, g(t))$ is a closed type I, κ-non-collapsed (for some $\kappa > 0$) ancient solution to the Ricci flow with positive complex sectional curvature in the sense that $\langle R(Z,W)\overline{Z}, \overline{W}\rangle > 0$ for any nonzero complex tangent vectors Z and W. Then $(M, g(t))$ must be the quotients $\mathbb{S}^n$.*

2) Assume that $(M, g(t))$ is a closed type I, κ-non-collapsed (for some $\kappa > 0$) ancient solution to the Ricci flow with 2-nonnegative curvature operator. Then $(M, g(t))$ must be quotients of products of symmetric spaces.

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