International Journal of Mathematics
(2021) 2140007 (13 pages)

DOI: 10.1142/S0129167X21400073

An application of a C^2-estimate for a complex Monge–Ampère equation

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Received 25 November 2020
Accepted 19 February 2021
Published

Dedicated to the 110th anniversary of S. S. Chern with a great honor.

By studying a complex Monge–Ampère equation, we present an alternate proof to a recent result of Chu-Lee-Tam concerning the projectivity of a compact Kähler manifold N^n with $\mathrm{Ric}_k < 0$ for some integer k with $1 < k < n$, and the ampleness of the canonical line bundle K_N.

Keywords: Kähler metrics; holomorphic sectional curvature; Ric_k; complex Monge–Ampère equation; Schwarz lemma; nef and ample line bundle; canonical line bundle.

Mathematics Subject Classification 2020: 32Q10, 32Q15, 32W20, 53C55

AQ: Please check the Keywords.

1. Introduction

In a recent preprint [4], the authors proved the following result.

Theorem 1.1 (Chu-Lee-Tam). *Assume that (N^n, ω_g) is a compact Kähler manifold ($n = \dim_{\mathbb{C}}(N)$) with $\mathrm{Ric}_k(X, \overline{X}) \leq -(k+1)\sigma|X|^2$ for some $\sigma \geq 0$. Then K_N is nef and is ample if $\sigma > 0$.*

Please indicate corresponding author.

The curvature notion Ric_k is defined as the Ricci curvature of the k-dimensional subspaces of the holomorphic tangent bundle $T'N$. Hence it coincides with the holomorphic sectional curvature $H(X)$ when $k = 1$, and with the Ricci curvature when $k = n = \dim_{\mathbb{C}}(N)$. The condition $\mathrm{Ric}_k > 0$ is significantly different from its Riemannian analogue, i.e. the so-called q-Ricci of Bishop-Wu [2], since it examines only the Ricci curvature of the subspaces in the holomorphic tangent space $T'N$, thus unlike its Riemannian analogue, $\mathrm{Ric}_k > 0$ (< 0) does not imply $\mathrm{Ric}_{k+1} > 0$ (< 0). The study of the condition of $\mathrm{Ric}_k < 0$ was initiated by the second author [10] to generalize the hyperbolicity of Kobayashi to the k-hyperbolicity of a compact Kähler manifold. It is closely related to the degeneracy of holomorphic mappings from $\mathbb{C}^k$ into concerned manifolds (cf. [10, Theorem 1.3]). In fact a generalization to Royden's result was proved there. Moreover it was proved recently by the second author that $\mathrm{Ric}_k > 0$ implies that M is projective and rational connected. The above result of Chu-Lee-Tam answers a question raised by the second author in [10], namely the projectivity of a compact Kähler manifold with $\mathrm{Ric}_k < 0$ affirmatively. In view of the fact that $\mathrm{Ric}_1 < 0$ is the same as the holomorphic section curvature $H < 0$, Theorem 1.1 generalizes the earlier work of [15, 16].

The proof of [4] is via the study of a twisted Kähler–Ricci flow. In this note we provide a direct alternate proof via the Aubin–Yau solution [1, 19] to a complex Monge–Ampère equation (which is similar to the equation for the Kähler–Einstein metric in the negative first Chern class). This was the method utilized in [16]. Comparing with [16] here a modification on the Monge–Ampère equation (cf. (3.2)) is necessary to adapt the method to the curvature condition $\mathrm{Ric}_k < 0$ when $n > k > 1$. This makes the derivation of the estimates more involved. The method here also extends to the more general setting considered in [4].

It was proved in [11] that any compact Këhler manifold with the second scalar curvature $S_2 > 0$ (simply put S_k is the average of Ric_k) must be projective. It remains an interesting question if $S_2 < 0$, or $\mathrm{Ric}_k^{\perp} < 0$, also implies the projectivity. For more backgrounds and references related to the theorem please refer to [4, 9, 10]. One can also find the definitions and motivations of several other curvature notions, including $S_2, \mathrm{Ric}_k^{\perp}$, and problems related to them in [9]. The geometry of $\mathrm{Ric}_k < 0$ compares sharply with the case of $\mathrm{Ric}_k > 0$. This is discussed at the end of Sec. 3 and in Appendix A via a construction of McKernan.

2. Preliminaries

Since the result is known for $k = 1, n$ below we assume that $1 < k < n$ in this section and the next. Here we collect some algebraic estimates as consequences of the assumption $\mathrm{Ric}_k(X, \overline{X}) \leq -(k+1)\sigma|X|^2, \forall\,(1,0)$-type tangent vector X. They are useful in obtaining key estimates for a Monge–Ampère equation (cf. (3.2) below). The first is [4, Lemma 2.1].

Lemma 2.1. *Under the assumption that* $\mathrm{Ric}_k(X,\overline{X}) \le -(k+1)\sigma|X|^2$ *the following estimate holds*

$$(k-1)|X|^2\,\mathrm{Ric}(X,\overline{X}) + (n-k)R(X,\overline{X},X,\overline{X}) \le -(n-1)(k+1)\sigma|X|^4. \quad (2.1)$$

The result follows by summing $\mathrm{Ric}_k(X,\overline{X}) \le -(k+1)\sigma|X|^2$ (the assumption) for a suitable chosen unitary basis. By using a Royden's trick [12] the following result was derived out of Lemma 2.1 (cf. [4, Lemma 2.2]).

Lemma 2.2. *Let* (N,ω) *be a compact Kähler manifold with*

$$\mathrm{Ric}_k(X,\overline{X}) \le -(k+1)\sigma|X|^2 \quad (2.2)$$

for some $\sigma \ge 0$. *Let* $\tilde{\omega} = \omega_{\tilde{g}}$ *be another Kähler metric on* N. *Set*

$$G = \mathrm{tr}_{\tilde{\omega}}\omega.$$

Then the following estimate holds

$$2\tilde{g}^{i\bar{j}}\tilde{g}^{k\bar{l}}\mathrm{R}_{i\bar{j}k\bar{l}} \le \frac{-(n-1)(k+1)\sigma}{n-k}(G^2+|g|^2_{\tilde{g}}) - \frac{k-1}{n-k}G\cdot\mathrm{tr}_{\tilde{g}}\mathrm{Ric} - \frac{k-1}{n-k}\langle\omega,\mathrm{Ric}\rangle_{\tilde{g}}. \quad (2.3)$$

Proof. Here we provide a proof using the averaging technique (cf. [10, Appendix]) instead of Royden's trick since the argument is more transparent. Pick a normal frame $\{\frac{\partial}{\partial z^i}\}$ so that $\tilde{g}_{i\bar{j}} = \delta_{ij}$ and $g_{i\bar{j}} = |\lambda_i|^2\delta_{ij}$. Then $\{\frac{\partial}{\partial w^i}\}$, with $\frac{\partial}{\partial w^i} := \frac{1}{\lambda_i}\frac{\partial}{\partial z^i}$, is a unitary frame for ω. Lemma 2.1 implies that (Einstein convention applied)

$$\begin{aligned}
2(n-k)&R^{\omega}_{i\bar{i}j\bar{j}}|\lambda_i|^2|\lambda_j|^2 + (k-1)G\,\mathrm{Ric}^{\omega}_{s\bar{s}}\,|\lambda_s|^2 + (k-1)\,\mathrm{Ric}^{\omega}_{i\bar{i}}\,|\lambda_i|^4 \\
&= n(n+1)\fint_{\mathbb{S}^{2n-1}}(k-1)|Y|^2\,\mathrm{Ric}^{\omega}(Y,\overline{Y}) + (n-k)R^{\omega}(Y,\overline{Y},Y,\overline{Y}) \\
&\le -(n-1)(k+1)n(n+1)\sigma\fint_{\mathbb{S}^{2n-1}}|Y|^4 \\
&= -(n-1)(k+1)\sigma(G^2+|\lambda_\gamma|^4).
\end{aligned}$$

Here $Y = \lambda_i w^i\frac{\partial}{\partial w^i}$ with respect to a normal frame $\{\frac{\partial}{\partial w^i}\}$, Ric^{ω} and $R^{\omega}_{i\bar{j}k\bar{l}}$ are the Ricci and curvature tensor expressed with respect to the metric ω (namely the unitary frame $\{\frac{\partial}{\partial w^i}\}$. The result then follows by identifying the terms invariantly. □

For the application it is useful to write (2.3) as

$$2\tilde{g}^{i\bar{j}}\tilde{g}^{k\bar{l}}\mathrm{R}_{i\bar{j}k\bar{l}} \le \frac{-(n-1)(k+1)\sigma}{n-k}(G^2+|g|^2_{\tilde{g}}) - 2\frac{k-1}{n-k}G\cdot\mathrm{tr}_{\tilde{g}}\mathrm{Ric} + \frac{k-1}{n-k}(\langle\mathrm{Ric},\tilde{\omega}\rangle_{\tilde{g}}\langle\omega,\tilde{\omega}\rangle_{\tilde{g}} - \langle\omega,\mathrm{Ric}\rangle_{\tilde{g}}). \quad (2.4)$$

3. Proof of Theorem 1.1

Assume that the canonical line bundle K_N of (N,ω) is not nef. Then there exists $\epsilon_0 > 0$ such that $\epsilon_0[\omega]-C_1(N)$ is nef but not Kähler. Thus, $\forall\epsilon > 0$, $(\epsilon+\epsilon_0)[\omega]-C_1(N)$ is Kähler. This means that there exists a smooth function ϕ_ϵ such that

$$\omega_\epsilon := (\epsilon_0+\epsilon)\omega - \mathrm{Ric}(\omega) + \sqrt{-1}\partial\bar{\partial}\phi_\epsilon > 0. \tag{3.1}$$

By Aubin–Yau's existence theorem and a prior estimate for a complex Monge–Ampère equation, we first prove the following theorem.

Theorem 3.1. *Let* (N,ω) *be a compact Kähler manifold which satisfies* (2.2) *for some* $\sigma \ge 0$. *Then* K_N *is nef.*

Proof. For any $\epsilon > 0$, we consider the complex Monge–Ampère equation for ψ_ϵ,

$$((\epsilon+\epsilon_0)\omega - \mathrm{Ric}(\omega) + \sqrt{-1}\partial\bar{\partial}(\phi_\epsilon+\psi_\epsilon))^n = e^{\phi_\epsilon+\psi_\epsilon+\frac{k-1}{2(n-k)}(\phi_\epsilon+\psi_\epsilon)}\omega^n \tag{3.2}$$

and

$$(\epsilon+\epsilon_0)\omega - \mathrm{Ric}(\omega) + \sqrt{-1}\partial\bar{\partial}(\phi_\epsilon+\psi_\epsilon) > 0. \tag{3.3}$$

By the Aubin–Yau theorem [1, 19], there is a unique solution ψ_ϵ of (3.2). For simplicity, let

$$\begin{aligned}\tilde{\omega}_\epsilon &:= (\epsilon+\epsilon_0)\omega - \mathrm{Ric}(\omega) + \sqrt{-1}\partial\bar{\partial}(\phi_\epsilon+\psi_\epsilon)\\ &= \omega_\epsilon + \sqrt{-1}\partial\bar{\partial}\psi_\epsilon, \qquad (3.4)\\ u_\epsilon &:= \phi_\epsilon + \psi_\epsilon.\end{aligned}$$

Then taking $\partial\bar{\partial}\log(\cdot)$ on both sides of (3.2), we see that (3.2) is equivalent to

$$\begin{aligned}\widetilde{\mathrm{Ric}}_\epsilon &:= \mathrm{Ric}(\tilde{\omega}_\epsilon) = \mathrm{Ric}(\omega) - \sqrt{-1}\partial\bar{\partial}\left(u_\epsilon + \frac{k-1}{2(n-k)}u_\epsilon\right)\\ &= -\tilde{\omega}_\epsilon + (\epsilon+\epsilon_0)\omega - \sqrt{-1}\frac{k-1}{2(n-k)}\partial\bar{\partial}u_\epsilon. \qquad (3.5)\end{aligned}$$

Let $G = G_\epsilon = \mathrm{tr}_{\tilde{\omega}_\epsilon}\omega$. Then by the calculation in the proof of the Schwarz lemma [18], and in particular (2.3) of [10] (also see computations in the earlier work of [3, 8]), as well as the C^2-estimate computations in [1, 13] (a slight different calculation was done in [14, 19]), we have that

$$\tilde{\Delta}_\epsilon(\log G) \ge \frac{1}{G}(\widetilde{\mathrm{Ric}}_{\epsilon_{p\bar{q}}}\tilde{g}_\epsilon^{p\bar{j}}\tilde{g}_\epsilon^{i\bar{q}}g_{i\bar{j}} - \tilde{g}_\epsilon^{i\bar{j}}\tilde{g}_\epsilon^{k\bar{l}}\mathrm{R}_{i\bar{j}k\bar{l}}). \tag{3.6}$$

Applying Lemma 2.2 (namely (2.4)) to $G = G_\epsilon$ we have the estimate

$$\begin{aligned}\frac{1}{G}\tilde{g}_\epsilon^{i\bar{j}}\tilde{g}_\epsilon^{k\bar{l}}\mathrm{R}_{i\bar{j}k\bar{l}} \le \frac{-(n-1)(k+1)\sigma}{2(n-k)}\left(G + \frac{1}{G}|g|^2_{\tilde{g}_\epsilon}\right) - \frac{k-1}{n-k}\mathrm{tr}_{\tilde{g}_\epsilon}\mathrm{Ric}\\ + \frac{k-1}{2(n-k)}\frac{1}{G}(G\cdot\mathrm{tr}_{\tilde{g}_\epsilon}\mathrm{Ric} - \langle\omega,\mathrm{Ric}\rangle_{\tilde{g}_\epsilon}). \qquad (3.7)\end{aligned}$$

Choosing local coordinates such that $(\tilde{g}_\epsilon)_{i\bar{j}} = \delta_{ij}$, $g_{i\bar{j}} = g_{i\bar{i}}\delta_{ij}$, then we also have

$$\begin{aligned}
&G \cdot \mathrm{tr}_{\tilde{g}_\epsilon}\mathrm{Ric} - \langle \omega, \mathrm{Ric}\rangle_{\tilde{g}_\epsilon} \\
&\quad = \sum_i \mathrm{Ric}_{i\bar{i}}\left(\sum_k g_{k\bar{k}} - g_{i\bar{i}}\right) \\
&\quad = \sum_i \left(\mathrm{Ric}_{i\bar{i}}\left(\sum_{k\neq i} g_{k\bar{k}}\right)\right) \\
&\quad \leq \sum_i ((\epsilon+\epsilon_0)g_{i\bar{i}} + u_{\epsilon_{i\bar{i}}})\left(\sum_k g_{k\bar{k}} - g_{i\bar{i}}\right) \\
&\quad = (\epsilon+\epsilon_0)G^2 - (\epsilon+\epsilon_0)|g|^2_{\tilde{g}_\epsilon} + G\cdot\tilde{\Delta}_\epsilon u_\epsilon - \langle\sqrt{-1}\partial\bar{\partial}u, \omega\rangle_{\tilde{g}_\epsilon}.
\end{aligned} \tag{3.8}$$

Here we used (3.3) in the third line. Plugging this into (3.7), we have

$$\begin{aligned}
-\frac{1}{G}\tilde{g}_\epsilon^{k\bar{l}}\tilde{g}_\epsilon^{i\bar{j}}R_{i\bar{j}k\bar{l}} \geq{}& \frac{\sigma(n-1)(k+1)-(k-1)(\epsilon+\epsilon_0)}{2(n-k)}G \\
&+ \frac{\sigma(n-1)(k+1)+(k-1)(\epsilon+\epsilon_0)}{2(n-k)}\frac{|g|^2_{\tilde{g}_\epsilon}}{G} + \frac{(k-1)}{(n-k)}\mathrm{tr}_{\tilde{g}_\epsilon}\mathrm{Ric} \\
&- \frac{k-1}{2(n-k)}\tilde{\Delta}_\epsilon u_\epsilon + \frac{k-1}{2(n-k)}\frac{1}{G}\langle\sqrt{-1}\partial\bar{\partial}u_\epsilon, \omega\rangle_{\tilde{g}_\epsilon}.
\end{aligned} \tag{3.9}$$

On the other hand, a direct calculation using (3.5) can express the first term in (3.6) as

$$\begin{aligned}
\frac{1}{G}\widetilde{\mathrm{Ric}}_{\epsilon_{p\bar{q}}}\tilde{g}_\epsilon^{p\bar{j}}\tilde{g}_\epsilon^{i\bar{q}}g_{i\bar{j}} &= \frac{1}{G}\langle\widetilde{\mathrm{Ric}}_\epsilon, \omega\rangle_{\tilde{g}_\epsilon} \\
&= \frac{1}{G}\left\langle -\tilde{w}_\epsilon + (\epsilon+\epsilon_0)\omega - \sqrt{-1}\partial\bar{\partial}\left(\frac{(k-1)}{2(n-k)}u_\epsilon\right), \omega\right\rangle_{\tilde{g}_\epsilon} \\
&= \frac{1}{G}\langle -\tilde{w}_\epsilon + (\epsilon+\epsilon_0)\omega, \omega\rangle_{\tilde{g}_\epsilon} - \frac{1}{G}\frac{(k-1)}{2(n-k)}\langle\sqrt{-1}\partial\bar{\partial}u_\epsilon, \omega\rangle_{\tilde{g}_\epsilon}.
\end{aligned} \tag{3.10}$$

Note that we used (3.5) in the third line above. Combining (3.6), (3.9) and (3.10), we have

$$\begin{aligned}
\tilde{\Delta}_\epsilon(\log G) \geq{}& \frac{\sigma(n-1)(k+1)-(k-1)(\epsilon+\epsilon_0)}{2(n-k)}G \\
&+ \frac{\sigma(n-1)(k+1)+(k-1)(\epsilon+\epsilon_0)}{2(n-k)}\frac{|g|^2_{\tilde{g}_\epsilon}}{G} + \frac{(k-1)}{(n-k)}\mathrm{tr}_{\tilde{g}_\epsilon}\mathrm{Ric} \\
&- \frac{(k-1)}{2(n-k)}\tilde{\Delta}_\epsilon u_\epsilon + \frac{1}{G}\langle -\tilde{\omega}_\epsilon + (\epsilon+\epsilon_0)\omega, \omega\rangle_{\tilde{g}_\epsilon}.
\end{aligned} \tag{3.11}$$

Hence

$$\tilde{\Delta}_\epsilon\left(\log G - \frac{k-1}{2(n-k)}u_\epsilon\right) \geq \frac{\sigma(n-1)(k+1)-(k-1)(\epsilon+\epsilon_0)}{2(n-k)}G + \frac{\sigma(n-1)(k+1)+(k-1)(\epsilon+\epsilon_0)}{2(n-k)}\frac{|g|^2_{\tilde{g}_\epsilon}}{G} + \frac{(k-1)}{(n-k)}(\mathrm{tr}_{\tilde{g}_\epsilon}\mathrm{Ric} - \tilde{\Delta}_\epsilon u_\epsilon) + \frac{1}{G}\langle -\tilde{\omega}_\epsilon + (\epsilon+\epsilon_0)\omega, \omega\rangle_{\tilde{g}_\epsilon}. \tag{3.12}$$

Next, we observe that

$$|g|^2_{\tilde{g}_\epsilon} \geq \frac{G^2}{n},$$
$$-\tilde{\Delta}_\epsilon u_\epsilon = -\tilde{g}_\epsilon^{i\bar{j}}(\tilde{g}_{\epsilon_{i\bar{j}}} + \mathrm{Ric}_{i\bar{j}} - (\epsilon+\epsilon_0)g_{i\bar{j}}) = -n - \mathrm{tr}_{\tilde{g}_\epsilon}\mathrm{Ric} + (\epsilon+\epsilon_0)G \quad \text{and}$$
$$\frac{1}{G}\langle(\epsilon+\epsilon_0)\omega - \tilde{w}_\epsilon, \omega\rangle_{\tilde{g}_\epsilon} = \frac{1}{G}(\epsilon+\epsilon_0)|g|^2_{\tilde{g}_\epsilon} - \frac{1}{G}G \geq -1.$$

Plugging these three inequalities/equation above into (3.12), we see that

$$\begin{aligned}
&\tilde{\Delta}_\epsilon\left(\log G - \frac{(k-1)}{2(n-k)}u_\epsilon\right)\\
&\geq \left(\frac{n}{n}\cdot\frac{\sigma(n-1)(k+1)-(k-1)(\epsilon+\epsilon_0)}{2(n-k)}\right)\cdot G + \left(\frac{(\epsilon+\epsilon_0)(k-1)2n}{(n-k)2n}\right)\cdot G\\
&\quad + \left(\frac{\sigma(n-1)(k+1)+(k-1)(\epsilon+\epsilon_0)}{2(n-k)n}\right)\cdot G - 1 - \frac{n(k-1)}{n-k}\\
&= \left(\frac{(n+1)\sigma(n-1)(k+1)}{2(n-k)n} + \frac{(\epsilon+\epsilon_0)(k-1)(n+1)}{2(n-k)n}\right)\cdot G - 1 - \frac{n(k-1)}{n-k}\\
&\geq \max\left\{\frac{(n+1)\sigma(n-1)(k+1)}{2(n-k)n}, \frac{(\epsilon+\epsilon_0)(k-1)(n+1)}{2(n-k)n}\right\}\cdot G - \frac{k(n-1)}{n-k}.
\end{aligned} \tag{3.13}$$

Now we apply the maximum principle to get a lower estimate of $\tilde{\omega}_\epsilon$. At the maximum of u_ϵ, say x_0, since $\sqrt{-1}\partial\bar{\partial}u_\epsilon \leq 0$ we have that $((\epsilon+\epsilon_0)\omega - \mathrm{Ric}(\omega))(x_0) \geq \tilde{\omega}_\epsilon > 0$ and

$$e^{\frac{2n-k-1}{2(n-k)}\sup_N u_\epsilon} = e^{\frac{2n-k-1}{2(n-k)}u_\epsilon(x_0)} \leq \left.\frac{((\epsilon+\epsilon_0)\omega - \mathrm{Ric}(\omega))^n}{\omega^n}\right|_{x=x_0} \leq C,$$

for some C independent of ϵ. This proves a uniform upper bound for u_ϵ, and hence that

$$\sup_N \frac{\tilde{\omega}_\epsilon^n}{\omega^n} \leq C, \text{ equivalently } W_n \geq C^{-1}, \text{ with } W_n := \frac{\omega^n}{\tilde{\omega}_\epsilon^n}. \tag{3.14}$$

Again we apply the maximum principle to $\log G - \frac{(k-1)}{2(n-k)} u_\epsilon$. By (3.13), at the point x_0', where the maximum of $\log G - \frac{(k-1)}{2(n-k)} u_\epsilon$ is attained, we have that

$$G(x_0') \leq C \text{ for some C independent of } \epsilon. \tag{3.15}$$

Since $GW_n^{\frac{k-1}{2n-k-1}} = Ge^{-\frac{k-1}{2(n-k)}u_\epsilon}$, we infer that $\sup_N(GW_n^{\frac{k-1}{2n-k-1}})$ is also attained at x_0'. By GM-AM inequality $G \cdot W_n^{\frac{k-1}{2n-k-1}} \leq G(\frac{G}{n})^{\frac{k-1}{(2n-k-1)n}}$ at x_0'. This, together with (3.15), implies $\sup_N(GW_n^{\frac{k-1}{2n-k-1}}) \leq C$ for some $C > 0$ independent of ϵ. Combining this with (3.14) we have that

$$G \leq C \text{ hence } \tilde{\omega}_\epsilon \geq A\omega, \tag{3.16}$$

for a constant $A > 0$ independent of ϵ. This is a contradiction to that $\epsilon_0[\omega] - C_1(N)$ is not Kähler by taking $\epsilon \to 0$. This completes the proof of Theorem 3.1. □

A remark is appropriate to compare the above proof with that of [16]. The idea of using an Aubin–Yau solution is the same. The difference lies in the details. First we came up with a modified Monge–Ampère equation to accommodate the new curvature condition. Secondly Wu-Yau's proof [16] of the C^2-estimate can be obtained by a direct application of Royden's version of Yau's Schwarz lemma (precisely, [12, Theorem 1, p. 554]). Namely no additional proof is necessary for bounding G under the assumption of [16] (namely the holomorphic sectional curvature $H < 0$), in view of an obvious lower bound on $\mathrm{Ric}(\tilde{\omega})$ from (3.5). By comparison, some nontrivial manipulations are needed above (at the least to the best knowledge of the authors) to get the C^2-estimate since one cannot infer any useful information from (3.6) directly under $\mathrm{Ric}_k < 0$ for some $k > 1$.

Once the nefness of K_N is established, the ampleness of K_N follows as [16, Theorem 7] provided that $\sigma > 0$. In this case we take $\epsilon_0 = 0$. By considering the Monge–Ampère equation (3.2), repeating the argument above, since $\sigma > 0$ is assumed now, we still can have the uniform estimates (3.14), (3.16) and the upper bound of u_ϵ from the key estimate (3.13) independent of ϵ. Moreover the elementary inequality $\mathrm{tr}_\omega \tilde{\omega}_\epsilon \leq \frac{1}{(n-1)!}(\mathrm{tr}_{\tilde{\omega}_\epsilon} \omega)^{n-1} \frac{\tilde{\omega}_\epsilon^n}{\omega^n}$ implies that $\mathrm{tr}_\omega \tilde{\omega}_\epsilon \leq C$. Hence we have that $\tilde{\omega}_\epsilon$ and ω are equivalent. Namely for some $C > 0$ independent of ϵ

$$C^{-1}\omega \leq \tilde{\omega}_\epsilon \leq C\omega. \tag{3.17}$$

This also gives the C^0-estimate (namely the lower bound of u_ϵ) by Eq. (3.2). The C^3-estimate of Calabi [1, 14, 19] also applies here (cf. [15] for an adapted calculation to a settings similar to (3.2)). Alternatively one can also use the $C^{2,\alpha}$-estimate of Evans as in [13]. Uniform estimates for up to the third-order derivatives of u_ϵ allow one to apply the Arzela–Ascoli compactness to get a convergent subsequence out of u_ϵ as $\epsilon \to 0$.

Taking $\epsilon \to 0$, and letting

$$u_\infty := \lim_{\epsilon\to 0} u_\epsilon,$$

$$\omega_\infty := -\operatorname{Ric}(\omega) + \sqrt{-1}\partial\bar\partial u_\infty > 0,$$

then it is easy to see that (3.2) becomes

$$(-\operatorname{Ric}(\omega) + \sqrt{-1}\partial\bar\partial u_\infty)^n = e^{u_\infty + \frac{k-1}{2(n-k)}u_\infty}\omega^n.$$

Taking $\partial\bar\partial\log(\cdot)$ on both sides of the above equation we have that

$$\operatorname{Ric}(\omega_\infty) = -\omega_\infty - \frac{k-1}{2(n-k)}\sqrt{-1}\partial\bar\partial u_\infty.$$

This implies that K_N is ample. The existence of a Kähler–Einstein metric is known by Aubin–Yau's theorem.

We also remark that the argument can be easily modified to prove the same result under the assumption:

$$\alpha|X|^2\operatorname{Ric}(X,\overline{X}) + \beta R(X,\overline{X},X,\overline{X}) \le -\sigma|X|^4, \forall X \text{ of } (1,0)\text{-type},$$

for some positive constants α, β and $\sigma \ge 0$. The existing literature (e.g. [17]) is enough to extend Theorem 1.1 to the case that Ric_k is quasi-negative (as well as σ above is quasi-positive) proving that K_N is big. We leave the details to the interested readers.

We are grateful to Professor McKernan for showing us an example of a smooth algebraic variety N^n with ample canonical line bundle, which admits a linear hypersurface $\mathbb{P}^{n-1}$. Together with [10, Theorem 1.3], the example and Theorem 1.1 show that the class of manifolds with a Kähler metric of $\operatorname{Ric}_k < 0$, for $1 \le k \le n-1$, is a strictly smaller class than that of manifolds with $c_1 < 0$ (equivalently those admitting a Kähler metric with $\operatorname{Ric} < 0$). Note that on this example with $\mathbb{P}^{n-1}$, by [10, Theorem 1.3] a Kähler metric with $S_{n-1} \le 0$ is not possible either. By taking product with a curve of high genus repeatedly this gives an example of Kähler manifold which has $\operatorname{Ric}_k < 0$, but $S_{k-1} > 0$ somewhere, since if $S_{k-1} \le 0$ everywhere, [10, Theorem 1.3] implies the impossibility of an embedded $\mathbb{P}^{k-1}$ in such manifold. This picture contrasts sharply with the fact that the class of Kähler manifolds with $\operatorname{Ric}_k > 0$ (for some $k < n$) is strictly larger than that of the Fano manifolds [9]. It also suggests an interesting question, namely if a compact Kähler manifold with $\operatorname{Ric}_k < 0$ admits a Kähler metric with $\operatorname{Ric}_{k+\ell} < 0$ for $\ell \ge 1$ (with $k + \ell < n$ since the case $k + \ell = n$ has been proven)?

Appendix A

In this appendix first we show that the averaging technique in Sec. 2 (cf. [10, Appendix], which was suggested by F. Zheng) also gives a quick proof of a result of Demailly-Skoda [5] on the relation between the Nakano positivity and Griffiths positivity of holomorphic vector bundles. The original proof used an action of $\mathbb{Z}_q^r$ (Royden used a similar action in [12]).

Proposition A.1 (Demailly-Skoda). *Let (E, h) be a holomorphic vector bundle with $\operatorname{rank}(E) = r$ over a complex manifold N^n. Let $(\det(E), \det(h))$ be the determinant line bundle. Assume that (E, h) is Griffiths positive. Then $E \otimes \det(E)$ (equipped with the induced metric) is Nakano positive.*

Before the proof recall that Nakano positivity means that for any section nonzero section $\tau = \sum_{i=1}^r \sum_{\alpha=1}^n \tau^{i\alpha} \frac{\partial}{\partial z^\alpha} \otimes e_i$ (abbreviated as $\tau^{i\alpha} \frac{\partial}{\partial z^\alpha} \otimes e_i$) of $T'N \otimes E$,

$$\sum_{\alpha,\beta=1}^{n} \sum_{i,k=1}^{r} \Theta_{\alpha\bar{\beta}i\bar{k}} \tau^{i\alpha} \overline{\tau^{k\beta}} > 0,$$

where Θ denotes the curvature of E with $\Theta_{\alpha\bar{\beta}i\bar{k}} = \langle \Theta_{\frac{\partial}{\partial z^\alpha} \frac{\partial}{\partial \bar{z}^\beta}}(e_i), \overline{e_k} \rangle$. Below we assume that $\{\frac{\partial}{\partial z^\alpha}\}$ and $\{e_i\}$ are normal frames at a point $p \in N$.

Proof. By direct calculation on the metric of the tensor product, the Nakano positivity of $E \otimes \det(E)$ amounts to (Einstein convention applied) showing that for any $\tau \neq 0$

$$\Theta_{\alpha\bar{\beta}k\bar{k}} \tau^{i\alpha} \overline{\tau^{i\beta}} + \Theta_{\alpha\bar{\beta}i\bar{k}} \tau^{i\alpha} \overline{\tau^{k\beta}} > 0. \tag{A.1}$$

For a section τ as above and $w = (w^1, \ldots, w^r) \in \mathbb{S}^{2r-1} \subset E_p$ (identified as $\mathbb{C}^r$), let $W = \sum_{\alpha=1}^n (\sum_{i=1}^r \tau^{i\alpha} w^i) \frac{\partial}{\partial z^\alpha}$ and $u = \sum_{k=1}^r \bar{w}^k e_k$ be elements of $T_p'N$ and E_p. The Griffiths positivity implies that $\langle \Theta_{W\overline{W}}(u), \bar{u} \rangle > 0$ for generic $w \in \mathbb{S}^{2r-1}$. As in [11], taking the integration average $⨍$ over $\mathbb{S}^{2r-1}$, the Berger's lemma implies that

$$\begin{aligned} r(r+1) ⨍ &\langle \Theta_{W\overline{W}}(u), \bar{u} \rangle d\mu(w) \\ &= r(r+1) ⨍ \Theta_{\alpha\bar{\beta}l\bar{k}} \tau^{i\alpha} w^i \overline{\tau^{j\beta} w^j} \bar{w}^k w^l \, d\mu(w) \\ &= \sum_{i \neq k} \Theta_{\alpha\bar{\beta}k\bar{k}} \tau^{i\alpha} \overline{\tau^{i\beta}} + \sum_{i \neq j} \Theta_{\alpha\bar{\beta}i\bar{j}} \tau^{i\alpha} \overline{\tau^{j\beta}} + 2\Theta_{\alpha\bar{\beta}i\bar{i}} \tau^{i\alpha} \overline{\tau^{i\beta}} \\ &= \Theta_{\alpha\bar{\beta}k\bar{k}} \tau^{i\alpha} \overline{\tau^{i\beta}} + \Theta_{\alpha\bar{\beta}i\bar{k}} \tau^{i\alpha} \overline{\tau^{k\beta}}. \end{aligned}$$

This proves (A.1), hence the proposition. □

Secondly we include McKernan's construction of the algebraic manifold mentioned in the previous section. The result and the proof are all due to him.

Proposition A.2 (McKernan). *Fix a positive integer n. There is a smooth projective variety X of dimension n with the following two properties*:

(1) *K_X is ample and*
(2) *X contains a copy of $\mathbb{P}^{n-1}$.*

Lemma A.1. *Let X be the cone over the d-uple embedding of $\mathbb{P}^{n-1}$ in $\mathbb{P}^N$.*[a] *Let $\pi : Y \to X$ be the blow-up of the vertex p, let E be the exceptional divisor, let $H \subset E \simeq \mathbb{P}^{n-1}$ be a hyperplane, let $D = K_Y|_E$, the restriction of the canonical divisor of Y to E and let $m = d - n$. We have*

(1) $K_Y = \pi^* K_X - \frac{m}{d} E$ *and*
(2) $D = mH$.

In particular, we have

(i) *If $d < n$ then $-D$ is ample.*
(ii) *If $d = n$ then D is numerically trivial.*
(iii) *If $d > n$ then D is ample.*

Proof. For (1), we start with the equation

$$K_Y + E = \pi^* K_X + aE,$$

where the rational number a, known as the log discrepancy, is to be determined. If we restrict both sides to E we get

$$\begin{aligned} -nH &= K_E \\ &= (K_Y + E)|_E \\ &= (\pi^* K_X + aE)|_E \\ &= \pi^* K_X|_E + aE|_E \\ &= -adH. \end{aligned}$$

Here the first line is the usual formula for the canonical divisor of projective space and we apply adjunction to get from the first line to the second line. It follows that $a = \frac{n}{d}$. This gives (1) and restricting to E gives (2). □

Now we prove Proposition A.2.

Proof. We start with $W \subset \mathbb{P}^{N+1}$, the closure of the cone from Lemma A.1, for any $d > n$. Then W is a projective variety with an isolated singularity p. Pick an ample divisor H. Let $\pi : V \to W$ blow-up the point p. By Lemma A.1 we have $K_V = \pi^* K_W - \frac{m}{d} E$. Fix a positive integer k and consider the divisor $K_W + kH$. Let $G = \pi^* H$. We have

$$\begin{aligned} K_V + kG &= \pi^*(K_W + kH) - \frac{m}{d} E \\ &= \pi^*(kH) + \pi^* K_W - \frac{m}{d} E. \end{aligned}$$

[a] Here we take the cone over a hyperplane section in $\mathbb{P}^{N+1}$.

As $\pi^*K_W - \frac{m}{d}E$ is relatively ample, it follows that $K_V + kG$ is ample if k is sufficiently large (cf. [6, Chap. II, Proposition 7.10(b)]).

Pick $B \in |2kH|$, a general element of the linear system $|2kH|$. Then Bertini implies that B is a smooth divisor that does not contain p. The $\mathbb{Q}$-divisor $\frac{1}{2}B - H$ defines a double cover $\sigma : Y \to W$ with branch locus B (cf. [7]). Then Y has two isolated singularities q and r lying over p and is otherwise smooth. Both singularities are analytically isomorphic to the cone singularity of Lemma A.1.

The Riemann–Hurwitz formula reads as

$$K_V = \sigma^*(K_W + \frac{1}{2}B).$$

As

$$K_W + \frac{1}{2}B \sim_{\mathbb{Q}} K_W + kH$$

is ample, it follows that K_V is ample, as σ is finite.

Let $\psi : X \to Y$ be the blow-up of q and r. Then X is a smooth projective variety and

$$K_X = \psi^*K_Y - \frac{m}{d}(E_q + E_r),$$

where E_q is the exceptional divisor over q and E_r is the exceptional divisor over r. Note that X is also double cover $\tau : X \to V$ of V branched over the divisor $C - 2H$, where $C = \pi^*B$ is the strict transform of B. Observe that via the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\sigma} \\ V & \xrightarrow{\pi} & W \end{array}$$

we have

$$\begin{aligned} K_X &= \psi^*K_Y - \frac{m}{d}(E_q + E_r) \\ &= \psi^*\sigma^*\left(K_W + \frac{1}{2}B\right) - \frac{m}{d}(E_q + E_r) \\ &= \tau^*\pi^*\left(K_W + \frac{1}{2}B\right) - \frac{m}{d}\tau^*E \\ &= \tau^*\left(\pi^*\left(K_W + \frac{1}{2}B\right) - \frac{m}{d}E\right). \end{aligned}$$

We already saw that

$$\pi^*\left(K_W + \frac{1}{2}B\right) - \frac{m}{d}E$$

is ample, for k sufficiently large. It follows that K_X is ample as τ is a finite morphism. This proves the claim (1).

On the other hand both E_q and E_r are copies of projective space and this gives (2). □

Let $\mathcal{M}^-_{n,k}$ be the set of n-dimensional compact manifolds with a Kähler metric such that its $\mathrm{Ric}_k < 0$. Let $\mathcal{M}^-_n$ be the set of n-dimensional compact manifolds with ample canonical line bundle. Let $\mathcal{S}^-_{n,k}$ be the set of n-dimensional compact manifolds with a Kähler metric such that its kth scalar $S_k < 0$. Clearly $\mathcal{M}^-_{n,k} \subset \mathcal{S}^-_{n,k}$.

Corollary A.1. *The following relation holds*:

(1) $\mathcal{M}^-_{n,k} \subsetneq \mathcal{M}^-_n$, $\forall\, 1 \le k < n$, *and*
(2) $\mathcal{M}^-_{n,k} \nsubseteq \mathcal{S}^-_{n,k-1}$, $\forall\, 2 \le k \le n$.

Proof. The result follows by combining Theorem 1.3 of [10], the example above and Theorem 1.1, after taking a suitable product with high genus curves multiple times. □

The relation (2) for $k = 2$ in particular implies that Theorem 1.1 provides a new result beyond what the main result of [16] can possibly cover.

Acknowledgments

The first author's research is supported by China Postdoctoral Grant No. BX20200356. The research of the second author is partially supported by "Capacity Building for Sci-Tech Innovation-Fundamental Research Funds".

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