COMPUTATION ON THE LARGE TIME ASYMPTOTICS OF THE ENTROPY

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In this note, due to various requests, we supply the details of the computation on the asymptotics of the entropy stated in [**N1**], Corollary 4.3, and Proposition 1.1 of [**N2**]. The similar computation has already appeared in my earlier paper [**N3**], page 935-936, which is the main reason we skipped the details in the original paper.

In this note we assume that (M, g) is a complete Riemannian manifold with nonnegative Ricci curvature. We also further assume that it has maximum volume growth. Let $H(x, y, t)$ be the heat kernel. For this note we fix x and write simply $H(y, t)$. Also we simply denote the distance between x and y by $r(y)$. Recall the definition of the entropy

$$\mathcal{W}(H,t) := \int_M \left(t|\nabla f|^2 + f - n\right) H\, d\mu$$

where $f = -\log H - \frac{n}{2}\log(4\pi t)$. Also recall the *Nash entropy*

$$\mathcal{N}(H,t) = -\int_M H\log H\, d\mu - \frac{n}{2}\log(4\pi t) - \frac{n}{2}.$$

Let ν_∞ be the cone angle at infinity which can be defined by

$$\nu_\infty := \lim_{r\to\infty} \theta(r)\omega_n$$

where $\theta(r) := \frac{V(r)}{r^n}$, $V(r)$ is the volume of the ball $B_x(r)$ centered at x and ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Recall that in [**N2**], page 331, we have proved that

$$\lim_{t\to\infty} \mathcal{W}(H,t) = \lim_{t\to\infty} \mathcal{N}(H,t).$$

The main purpose here is to supply the detail of the claim that

$$\lim_{t\to\infty} \mathcal{N}(H,t) = \log \nu_\infty.$$

Under the above notation, let us recall a result of Li, Tam and Wang [**LTW**].

Theorem 0.1 (Li-Tam-Wang)**.** *Let $(\mathcal{M}^n, g)$ be a complete Riemannian manifold with nonnegative Ricci curvature and maximum volume growth. For any $\delta > 0$, the heat kernel of $(\mathcal{M}^n, g)$ satisfies*

$$\frac{\omega_n}{\theta(\delta r(y))}(4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{1+9\delta}{4t} r^2(y)\right) \le H(y,t)$$
$$\le (1 + C(n,\theta_\infty)(\delta+\beta))\,\frac{\omega_n}{\theta_\infty}(4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{1-\delta}{4t} r^2(y)\right),$$

where $\theta_\infty = \lim_{r\to\infty}\theta(r)$,

$$\beta := \delta^{-2n} \max_{r\ge(1-\delta)r(y)} \left(1 - \frac{\theta_x(r)}{\theta_x(\delta^{2n+1} r)}\right).$$

Note that β is a function of $r(y)$ and

$$\lim_{r(y)\to\infty} \beta = 0.$$

Therefore, for any $\epsilon > 0$, there exists a B sufficiently large such that if $r(y) \geq B$ we have

$$\begin{aligned} \tfrac{\omega_n}{\theta_\infty}(1-\epsilon)(4\pi t)^{-\frac{n}{2}} \exp\left(-\tfrac{1+9\delta}{4t} r^2(y)\right) &\leq H(y,t) \\ &\leq (1 + C(n,\theta_\infty)(\delta+\epsilon))\, \tfrac{\omega_n}{\theta_\infty}(4\pi t)^{-\frac{n}{2}} \exp\left(-\tfrac{1-\delta}{4t} r^2(y)\right). \end{aligned} \tag{0.1}$$

We can also require that $\theta(\delta^{2n+1} r) \leq (1+\epsilon)\theta_\infty$.

The upper estimates:

First by the lower estimate of Li-Tam-Wang,

$$\begin{aligned} \mathcal{N}(H,t) &\leq -\int_M H \log\left(\frac{\omega_n}{\theta(\delta r(y))}\right) d\mu + \int_M H \left(\frac{1+9\delta}{4t} r^2(y)\right) d\mu - \frac{n}{2} \\ &= I + II - \frac{n}{2}. \end{aligned}$$

We shall estimate I and II below as in [**N3**].

Split

$$I = \int_0^B + \int_B^\infty \left(\int_{\partial B(s)} H \log\left(\frac{\omega_n}{\theta(\delta r(y))}\right) dA \right) ds = I_1 + I_2.$$

It is easy to see that

$$\lim_{t\to\infty} I_1 \leq 0.$$

To compute II_2 we now make use of the lower estimate in (0.1) to have that

$$\begin{aligned} I_2 &\leq (1-\epsilon)\frac{\omega_n}{\theta_\infty}(4\pi t)^{-\frac{n}{2}} \int_B^\infty \int_{\partial B(s)} \exp\left(-\frac{1+9\delta}{4t}s^2\right) \log\left(\frac{\omega_n}{\theta(\delta s)}\right) dA\, ds \\ &\leq \log\left(\frac{(1+\epsilon)\theta_\infty}{\omega_n}\right)(1-\epsilon) n\omega_n (4\pi t)^{-\frac{n}{2}} \int_B^\infty \exp\left(-\frac{1+9\delta}{4t}s^2\right) s^{n-1}\, ds. \end{aligned}$$

Here we have used that $\delta(\delta r(y)) \leq \theta_\infty(1+\epsilon)$ and the surface area of $\partial B(s)$ satisfies $A(s) \geq n\theta_\infty s^{n-1}$. Computing the integral via the change of variable $\tau = \frac{1+9\delta}{4t}s^2$ and taking $t \to \infty$ we have that

$$\lim_{t\to\infty} I_2 \leq \log\left(\frac{(1+\epsilon)\theta_\infty}{\omega_n}\right)(1-\epsilon)\left(1+9\delta\right)^{-n/2}.$$

The estimate of II is very similar. Using the Gamma function identity

$$\Gamma(\frac{n}{2}+1) = \Gamma(\frac{n}{2})\frac{n}{2}$$

we can have that

$$\lim_{t\to\infty} II \leq (1+\epsilon)(1+C(n,\theta_\infty)(\delta+\epsilon))(1+9\delta)(1-\delta)^{n/2+1}\frac{n}{2}.$$

Summarizing we have that

$$\begin{aligned} \lim_{t\to\infty} \mathcal{N}(H,t) &\leq \log\left(\frac{(1+\epsilon)\theta_\infty}{\omega_n}\right)(1-\epsilon)\left(1+9\delta\right)^{-n/2} \\ &\quad + (1+\epsilon)(1+C(n,\theta_\infty)(\delta+\epsilon))(1+9\delta)(1-\delta)^{n/2+1}\frac{n}{2} - \frac{n}{2}. \end{aligned}$$

Letting $\epsilon \to 0$, then $\delta \to 0$ we have what

$$\lim_{t\to\infty} \mathcal{N}(H,t) \leq \log \nu_\infty.$$

The lower estimate works very similarly.

References

[LTW] P. Li, L.-F. Tam and J. Wang, *Sharp bounds for the Green's function and the heat kernel,* Math. Res. Lett. **4** (1997), no. 4, 589–602.

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[N3] L. Ni, *A monotonicity formula on complete Kaehler manifolds with nonnegative bisectional curvature,* J. Amer. Math. Soc. **17** (2004), 909–946.

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