

# A CLASSIFICATION OF LOCALLY CHERN HOMOGENEOUS HERMITIAN MANIFOLDS

LEI NI AND FANGYANG ZHENG

ABSTRACT. We apply the algebraic consideration of holonomy systems to study Hermitian manifolds whose Chern connection is Ambrose-Singer and prove structure theorems for such manifolds. The main result (Theorem 1.2) asserts that the universal cover of such a Hermitian manifold must be the product of a complex Lie group and Hermitian symmetric spaces. Previously this is known for dimensions up to 4. We also obtain partial results on Hermitian manifolds whose Bismut/Gauduchon connection is Ambrose-Singer, and discuss how do these results fit into a project of classifying such manifolds.

## CONTENTS

1. Introduction and main results	1
2. Preliminaries	4
3. Symmetric holonomy systems	5
4. Hermitian manifolds with a CAS structure	7
5. Generalized symmetric holonomy systems	9
6. The Bismut/Gauduchon Ambrose-Singer manifolds	11
References	17

## 1. INTRODUCTION AND MAIN RESULTS

In [5] Ambrose and Singer gave a necessary and sufficient condition for a simply-connected complete Riemannian manifold to admit a transitive group of isometric motions. These conditions were recognized later ([24]) as the existence of an affine connection which is *invariant under parallelism*, that is, the connection has parallel torsion and curvature with respect to itself. Namely

**Theorem 1.1** (Ambrose-Singer, Kostant). *Let  $(M, g)$  be a complete, simply-connected Riemannian manifold, and  $\nabla^g$  the Levi-Civita connection. Then  $M$  admits a transitive action of an isometric group (namely  $M$  is a Riemannian homogeneous space) if and only if there exists a metric connection  $D$  such that its torsion and curvature are parallel with respect to  $D$ .*

The conditions were originally expressed in terms of more convoluted PDEs involving the curvature tensor of the Levi-Civita connection  $\nabla^g$  and the torsion of the metric connection  $D$  as a characterization of simply-connected complete Riemannian homogeneous spaces (cf. [5]). It provides a generalization of Cartan's theorem on the symmetric (or locally symmetric) spaces.<sup>1</sup> Without the simply-connectedness the result holds locally. The metric connection  $D$  is called an *Ambrose-Singer connection* (abbreviated as an AS connection).

In this paper we study complete Hermitian manifolds that are *locally Chern homogeneous*, namely, its Chern connection  $\nabla$  satisfies the condition  $\nabla T = 0$  and  $\nabla R = 0$ , where  $T$  and  $R$  are respectively the torsion and curvature of  $\nabla$ . This notion is motivated by the above work of Ambrose-Singer [5] and others such as [21, 31, 24] (see also [9]). In particular, there is a corresponding result of Sekigawa [34] for Hermitian manifolds which asserts that a complete simply-connected Hermitian manifold is Hermitian homogeneous (namely Hermitian manifolds whose biholomorphic isometries act transitively) if and only

---

2010 *Mathematics Subject Classification.* 53C55 (Primary), 53C05 (Secondary).

*Key words and phrases.* Ambrose-Singer connection, Chern connection, holonomy algebra/group/system, Chern flat, locally homogeneous Hermitian manifolds.

Zheng is the corresponding author. The research is partially supported by NSFC grants # 12071050 and 12141101, Chongqing grant cstc2021ycjh-bgzxm0139 and CQYC2021059145, and is supported by the 111 Project D21024.

<sup>1</sup>In a private conversation, R. Hamilton suggested that the convergence of a Ricci flow solution to a homogeneous structure perhaps can be captured by the emergence of an Ambrose-Singer connection.

if there exists a Hermitian Ambrose-Singer connection, namely in addition to the assumptions of having parallel torsion and curvature, the metric connection also preserves the almost complex structure.

We refer the readers to [23, 36, 42, 44, 45] for other related studies on homogeneous complex manifolds (namely complex manifolds whose biholomorphism groups act transitively), and other homogeneous spaces. One should also consult [6] for some recent developments on local Hermitian homogeneous spaces and the flows on such spaces. Locally Chern homogeneous Hermitian manifolds forms a special class of homogenous complex manifolds as shown by the theorem below.

It is well known that on a Hermitian manifold there exists a unique canonical connection, namely the Chern connection [14] (in fact the existence, as well as the uniqueness, extends to any holomorphic vector bundle). For locally Chern homogeneous manifolds, namely when the Chern connection of a Hermitian manifold  $(M^n, g)$  is Ambrose-Singer, we will also say that the complex manifold  $M$  admits a *CAS structure*. In [30, Theorem 3.6] conditions are given when the universal cover of such a manifold admits some Kähler (hence Hermitian symmetric due to  $\nabla R = 0$ ) de Rham factors. Classification theorems were obtained in complex dimension 3 and 4. The examples of manifolds with CAS structure include complex Lie groups endowed with compatible left-invariant metrics and Hermitian symmetric spaces, their products and quotients of products. In this paper, we prove essentially the reverse holds by the following classification result for locally Chern homogeneous manifolds in general dimensions, which is the main result of this article:

**Theorem 1.2.** *Let  $(M^n, g)$  be a complete Hermitian manifold with a CAS structure, namely a locally Chern homogeneous Hermitian manifold. Then its universal cover  $\widetilde{M}$  splits into  $M_1 \times M_2$ , where  $M_1$  is a Chern flat Hermitian manifold with a complex Lie group structure, and  $M_2$  is the product of irreducible Hermitian symmetric spaces of dimension  $k$  with  $0 \leq k \leq n$ .*

In [30], it was shown that the above theorem holds when  $n \leq 4$  for the case that  $M$  is compact. The proof of the above theorem consists of three parts. The first part is to split off the Kähler de Rham factors (if any) in the universal cover of  $M$ , which correspond to the kernel distribution of the Chern torsion. Note that since the connection is *not* Levi-Civita (when the metric is not Kähler), the de Rham decomposition theorem no longer holds to its full generality, but the CAS assumption will enable us to get the desired splitting. The second part is to show that a CAS manifold without Kähler de Rham factors must be (first) Chern Ricci flat. This is achieved by constructing a holomorphic symplectic form on the manifold. These two parts were obtained in [30], and we will recall and give their outline in the next section. The third part is an algebraic analogue to the classic theorem of Alekseevskii and Kimelfel'd [4], where we show that for an abstract holonomy system tailored to our CAS situation, if it is Ricci flat the it will be flat. This is in the spirit of Simons' proof [35] of Berger's holonomy theorem, except here the holonomy system needs to be expanded since the connection considered here has a non-vanishing torsion.

Recall that Alekseevskii and Kimelfel'd proved in [4] that any Ricci flat complete Riemannian homogeneous manifold  $M^n$  must be flat, and in fact it is isometric to  $T^k \times \mathbb{R}^{n-k}$ , where  $T^k$  is a flat torus. They proved this result by the consideration of a volume entropy. It can also be derived from Cheeger-Gromoll's splitting theorem and the consideration of Clifford translation [47]. Since any Ricci flat Hermitian symmetric space must be flat, our Theorem 1.2 has the following immediate consequence which can be viewed as a Hermitian analogue to the aforementioned AK theorem:

**Corollary 1.3.** *Let  $(M^n, g)$  be a complete Hermitian manifold with a CAS structure. Assume that the Chern curvature is Ricci flat. Then it is Chern flat. In particular,  $M$  is covered by a complex Lie group.*

Note that one no longer has the Cheeger-Gromoll splitting theorem for Hermitian manifolds with Chern connection. Our proof here is mainly algebraic. The result answers in the case of CAS manifolds a question raised by Yau [49, Problem 87] where he asked if *one can say something nontrivial about the compact Hermitian manifolds whose holonomy is a proper subgroup of  $U(n)$* .

Recall that for any given metric connection  $D$  on a Riemannian manifold, its curvature tensor  $R^D$  is a covariant 4-tensor which is skew-symmetric with respect to its first two or last two positions. Its Ricci curvature  $Ric^D$  is the 2-tensor obtained by contracting the first and fourth (or equivalently, its second and third) positions of  $R^D$ . When the manifold is Hermitian and  $D$  is also a Hermitian connection, then there are three ways to contract its curvature tensor  $R^D$ , resulting in the *first*, *second*, and *third* Ricci of  $D$ , with  $Ric^D$  being the third Ricci.

In general, these three Ricci curvatures may not be equal, even for the Chern connection. But for CAS manifolds, the Chern curvature tensor obeys all Kähler symmetries (namely, CAS manifolds are *Chern Kähler-like*) hence all three Chern Ricci curvatures coincide.

**Remark 1.4.** *We remark that the naive way of extending AK theorem from Levi-Civita connection to Chern connection fails, namely, there are (locally) Hermitian homogeneous manifolds with vanishing (third) Chern Ricci curvature but are not Chern flat.*

For instance, some (compact quotients of) *almost abelian* Lie groups will be (third) Chern Ricci flat but not Chern flat. There are also example of compact locally homogeneous Hermitian manifolds with vanishing first and third Chern Ricci yet is not Chern flat. Such examples are given at the end of §4. From this, we see that the CAS assumption in Corollary 1.3 is stronger than just assuming the manifold to be (locally) Hermitian homogeneous, and that seems to be the proper way to generalize the AK theorem to the Chern connection case.

Besides Chern connection  $\nabla$ , another important canonical connection of a Hermitian manifold  $(M^n, g)$  is the *Bismut connection*  $\nabla^b$ , which is the unique Hermitian connection (meaning a metric connection which makes the almost complex structure  $J$  parallel) whose torsion is totally skew-symmetric. The Bismut connection is extensively studied by physicists in string theory, and is an essential component in the Hull-Strominger system which suggests that the hidden space is a non-Kähler Calabi-Yau manifold (cf. [19], [41]). In mathematics, Bismut [11] established the existence and uniqueness of this connection and used it in his study of local index theory. It is also related to the Hermitian Ricci flow theory (cf. [38], [39], [40] and references therein). While the geometric and complex analytic meaning of Chern curvature is relatively clear, the curvature of Bismut connection does not seem to enjoy the nice properties satisfied by the Chern connection. There is an interesting geometric explanation in the frame work of generalized Ricci flow developed by Garcia-Fernández and Streets [18] (see also the references therein).

Analogous to the Chern connection case, one could consider *Bismut Ambrose-Singer* manifolds (BAS for brevity): Hermitian manifolds whose Bismut connection has parallel torsion and curvature.

Unlike the Chern connection case, there are plenty of examples of BAS manifolds that are ‘non-trivial’, in the sense that they are neither Hermitian symmetric nor Bismut flat, so Theorem 1.2 fails completely when the Chern connection is replaced by the Bismut connection. The simplest such example is perhaps the Hopf manifold  $M^n = (\mathbb{C}^n \setminus \{0\})/\langle f \rangle$  with  $n \geq 3$ , where  $f(z) = 2z$ .  $z = (z_1, \dots, z_n)$ . Write  $|z|^2 = |z_1|^2 + \dots + |z_n|^2$ . Then direct calculation shows that the metric  $g$  with Kähler form  $\omega = \sqrt{-1} \frac{\partial\bar{\partial}|z|^2}{|z|^2}$  is BAS, non-balanced, and is NOT Bismut flat. There are plenty of other examples of BAS manifolds starting in dimension 3, and there are also examples of balanced (but non-Kähler) BAS manifolds.

Note that BAS manifolds form a subset of (locally) *naturally reductive homogeneous spaces*, which means homogeneous Riemannian manifolds with a metric connection which has parallel (with respect to itself) and totally skew-symmetric torsion. Such spaces were systematically studied by Agricola, Friedrich and others (see [1], [2], and the references therein). Classification of real dimensions up to 8 were obtained (see [3], [25], [37] and the references therein).

While it might not be feasible to classify naturally reductive spaces in general dimensions, it seems to be an interesting problem to classify or characterize all BAS manifolds. Hence we ask whether or not the analogue to Corollary 1.3 would hold for Bismut connection, namely, if a compact BAS manifold has vanishing (third) Bismut Ricci, then must it be Bismut flat? We suspect that the answer would be negative when  $n \geq 3$ , although we have not been able to construct a concrete counterexample yet. On the positive side, we would like to propose the following:

**Conjecture 1.5.** *Let  $(M^n, g)$  be a compact (or complete) Hermitian manifold such that its Bismut connection  $\nabla^b$  is Ambrose-Singer. If the first and third Bismut Ricci both vanish, then the Bismut curvature vanishes.*

Note that for BAS manifolds, the first and second Bismut Ricci are equal, and the Bismut flat manifolds have been fully classified [46] as being quotients of Samelson spaces. Hence the conjecture, if confirmed, would provide a classification for Ricci flat BAS manifolds. As a supporting evidence to the above conjecture, we show that it holds when  $n \leq 4$ :

**Proposition 1.6.** *Let  $(M^n, g)$  be a complete, non-Kähler BAS manifold with vanishing first and third Bismut Ricci. Then it is not balanced. Furthermore, it is Bismut flat if  $n \leq 4$ .*

Another partial result is to restrict to the case of *Bismut Kähler-like* metrics, which means that the Bismut curvature tensor obeys all Kähler symmetries. In this case all three Bismut Ricci coincides. The following is a weaker version of [54, Theorem 3], and we give an alternative proof here as an application of our algebraic result on holonomy system in §3 and §5:

**Proposition 1.7.** *Let  $(M^n, g)$  be a complete BAS manifold with vanishing first Bismut Ricci. Assume that it is Bismut Kähler-like. Then it must be Bismut flat.*

Besides Chern and Bismut connections, the family of Gauduchon connections, which is the line joining Chern and Bismut, have also been studied in the literature. For any real number  $t$ , the  $t$ -Gauduchon connection is defined by  $\nabla^{(t)} = (1 - \frac{t}{2})\nabla + \frac{t}{2}\nabla^b$ . So in particular,  $\nabla^{(0)} = \nabla$  is the Chern connection,  $\nabla^{(2)} = \nabla^b$  is the Bismut connection, while  $\nabla^{(1)}$  is the Hermitian projection of the Levi-Civita connection, which is called the *Lichnerowicz connection* in some literature. Note that when the metric  $g$  is Kähler, all  $\nabla^{(t)}$  coincide with the Levi-Civita connection  $\nabla^g$ . But when  $g$  is not Kähler,  $\nabla^{(t)} \neq \nabla^{(t')}$  whenever  $t \neq t'$ , and they can never be equal to  $\nabla^g$  as the latter is not a Hermitian connection in this case. Lafuente and Stanfield [26] proved the following beautiful theorem about Gauduchon flat manifolds:

**Theorem 1.8** (Lafuente and Stanfield [26]). *For any  $t \neq 0, 2$ , if a compact Hermitian manifold has flat  $t$ -Gauduchon connection  $\nabla^{(t)}$ , then it must be Kähler.*

In fact what they proved is much stronger: the result holds when the flatness is replaced by the Kähler-like condition, namely, when the curvature of  $\nabla^{(t)}$  obeys all Kähler symmetries (see for example [52] and references therein for more discussion on  $t$ -Gauduchon Kähler-like manifolds). Lafuente and Stanfield also showed that the metric must be Kähler when the compactness assumption is dropped but when  $t$  is not in  $\{\frac{2}{3}, \frac{4}{5}\}$ .

For any  $t \neq 0, 2$ , similar to the Chern or Bismut case, one could ask the question about  $t$ -GAS manifolds, namely those with  $\nabla^{(t)}$  being Ambrose-Singer. It turns out that there are examples of compact  $t$ -GAS manifolds that are ‘non-trivial’, in the sense that they are non-Kähler (thus can be neither Hermitian symmetric nor  $t$ -Gauduchon flat). For instance, take any compact quotient of a semi-simple complex Lie group, then it is  $t$ -GAS for any  $t \in \mathbb{R}$ . It seems that  $t$ -GAS manifolds form a highly restrictive class, so a full classification of them constitutes an interesting project for future study.

As a reduction one could also ask the AK type question for  $t$ -GAS manifolds when  $t \neq 0, 2$ , namely, *if  $\nabla^{(t)}$  has vanishing (third) Ricci curvature, then must  $\nabla^{(t)}$  be flat?* We suspect that the answer would be a YES but we are unable to prove it at this point. Instead, we have the following weaker statement:

**Proposition 1.9.** *Let  $t \neq 0, 2$  and  $(M^n, g)$  be a compact Hermitian manifold such that its  $t$ -Gauduchon connection  $\nabla^{(t)}$  is Ambrose-Singer. Then  $g$  must be balanced. Furthermore, if  $\nabla^{(t)}$  has vanishing first and third Ricci curvatures, then  $\nabla^{(t)}$  must be Kähler and flat, hence a complex torus.*

Note that in the aforementioned AK type question for  $t$ -GAS manifolds, one can not replace third Ricci by first Ricci along, as there are examples of non-Kähler  $t$ -GAS manifolds which has vanishing first  $t$ -Gauduchon Ricci, as pointed out by Lafuente and Stanfield in [26, page 2] (see also [43]), on a balanced Hermitian manifold  $M$  with vanishing first Chern Ricci form. On such a manifold,  $\nabla^{(t)}$  has vanishing first Ricci for all  $t$ , since the difference of the first Ricci form between two Gauduchon connections is given by the differential of Gauduchon’s torsion 1-form  $\eta$ , while the balanced condition means  $\eta = 0$ .

## 2. PRELIMINARIES

In this section we will collect some results from [30] which will be used in the proof of our main result, Theorem 1.2 stated in the introduction.

Let  $(M^n, g)$  be a complete Hermitian manifold. Denote by  $\nabla$  the Chern connection, by  $T$  and  $R$  its torsion and curvature. The following was proved in [30] (cf. also [48]):

**Proposition 2.1.** *Suppose that  $(M^n, g)$  is a Hermitian manifold with a CAS structure. Then the Chern curvature is Kähler-like. In particular, the Chern curvature satisfies the first Bianchi identity.*

In [30], we denoted by  $\mathcal{W}_p$  the subspace of  $T_p^{1,0}$  spanned by the image of  $T$ :

$$\mathcal{W} := \langle \{T(X, Y) \mid X, Y \in T^{1,0}M\} \rangle.$$

Let  $\mathcal{N} := \{Z \in T^{1,0}M \mid \langle T(X, Y), \bar{Z} \rangle = 0 \ \forall X, Y \in T^{1,0}M\}$ . The following was proved in [30].

**Proposition 2.2.** *The subbundles  $\mathcal{W}$  and  $\mathcal{N}$  of  $T^{1,0}M$  (also denoted as  $T'M$ ) are invariant under the parallel transport with respect to the Chern connection  $\nabla$ , and  $T'M = \mathcal{W} \oplus \mathcal{N}$  orthogonally. The curvature restriction  $R|_{\mathcal{W}} = 0$ . In particular, when  $M$  is not a locally Hermitian symmetric space, the action of the holonomy group  $G$  on  $T'M$  is reducible.*

The next result extends the construction, mainly Theorem 1.4 and its proof in Section 6 of [30]. Note that there was a typo in the statement of [30, Theorem 1.4], namely, the word ‘with  $k \geq 2$ ’ at the end should be deleted. We will recap the proof here for the sake of completeness.

**Theorem 2.3.** *Let  $(M^n, g)$  be a complete simply-connected Hermitian manifold with a CAS structure. Assume that it does not admit any Kähler de Rham factor. Then  $\mathcal{N}$  decomposes into  $\mathcal{N}_1 \oplus \mathcal{N}_2$  with  $R|_{\mathcal{W}+\mathcal{N}_2} = 0$  and  $\mathcal{N}_1$  decomposes further into  $\bigoplus_{j=1}^k \mathcal{K}_j$ , with each of  $\mathcal{K}_j$  being invariant and irreducible under the action of the holonomy group  $G$ . On each  $\mathcal{K}_j$  there exists a parallel  $(2, 0)$ -symplectic form which can be identified with the standard holomorphic symplectic  $(2, 0)$  form on  $\mathbb{C}^{2k_j}$  with  $2k_j = \dim_{\mathbb{C}}(\mathcal{K}_j)$ .*

*Proof.* We first recap the constructions in the proof of Theorem 1.4 in [30]. By the simply-connectedness assumption and the Ambrose-Singer holonomy theorem, namely Theorem 3.1 stated in the next section,  $\mathcal{W}$  is a trivial bundle which admits parallel (holomorphic) sections  $Z_1, \dots, Z_{\ell_1}$  where  $\ell_1 = \dim(\mathcal{W}_p)$ . Let  $\xi_i = \langle \cdot, \bar{Z}_i \rangle$  be the dual 1-forms correspondingly. Then let  $\tau_i(\cdot, \cdot) = d\xi_i = \partial\xi_i$ . It was proved in Lemma 6.2 of [30] that  $\tau_i(\cdot, \cdot) = \langle T(\cdot, \cdot), \bar{Z}_i \rangle$ , which also equals to  $-\iota_{\bar{Z}_i} \partial\omega$ . It was proved there, also can be seen by direct calculation, that  $\tau_i(\cdot, \cdot) = \partial\xi_i$  and  $\bar{\partial}\xi_i = 0$ , that  $\{\tau_i\}$  are parallel, holomorphic  $(2, 0)$ -forms.

The next key step in the proof of Theorem 1.4 of [30] is to pick one  $\tau$  in  $\text{span}\{\tau_i\}$  such that  $\tau|_{\mathcal{N}}$  is of maximum rank. Without the loss of generality we denote this one by  $\tau_1$ . It then derived from the assumption that  $M$  has no Kähler factor and  $\tau_1$  is of maximum rank, there exists a orthogonal decomposition of  $\mathcal{N}$  into  $\mathcal{N}_1 \oplus \mathcal{N}_2$  such that  $\mathcal{N}_2$  is generated by parallel global sections, hence  $R|_{\mathcal{N}_2} = 0$ . Moreover  $\mathcal{N}_1$  is of even dimension and  $\tau_1|_{\mathcal{N}_1}$  is non-degenerate. Let  $\ell_2 = \dim_{\mathbb{C}}(\mathcal{N}_2)$ . Then  $\dim_{\mathbb{C}}(\mathcal{N}_1) = 2k$  and  $2k + \ell_1 + \ell_2 = n$ . Since  $\mathcal{W}$  and  $\mathcal{N}_2$  are generated by parallel (holomorphic) sections/vector fields, both  $\mathcal{K}_0 := \mathcal{W} \oplus \mathcal{N}_2$  and  $\mathcal{N}_1$  are invariant under the parallel transport with respect to the Chern connection. Fix a point  $p \in M$  the holonomy action of any element  $h \in G$  splits as  $\tilde{h} \oplus \text{id} : (\mathcal{N}_1)_p \oplus (\mathcal{K}_0)_p \rightarrow (\mathcal{N}_1)_p \oplus (\mathcal{K}_0)_p$ .

Now consider  $\tau|_{\mathcal{N}_1}$ , a parallel  $(2, 0)$  form on this sub-bundle of  $T'M$ , which we shall still denote by  $\tau$ . With respect to a chosen unitary frame of  $\mathcal{N}_1$  at the point  $p$ ,  $\tau$  has the following matrix form:

$$A = \begin{bmatrix} a_1 E & & \\ & \ddots & \\ & & a_k E \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad a_1 \geq \dots \geq a_k > 0.$$

We may collect the  $a_i$  into distinct numbers of  $b_j$ . Namely we write

$$b_1 \tilde{E} = \begin{bmatrix} a_1 E & & \\ & \ddots & \\ & & a_1 E \end{bmatrix}, \quad A = \begin{bmatrix} b_1 \tilde{E} & & \\ & \ddots & \\ & & b_{\ell_3} \tilde{E} \end{bmatrix}, \quad b_1 > \dots > b_{\ell_3} > 0.$$

Namely  $-b_j^2$  are distinct eigenvalues of  $A\bar{A}$  if  $A$  is the matrix representation of  $\tau$  at  $p$ . Let  $2n_j$  denote the dimension of the corresponding eigenspaces  $\mathcal{V}_j$ . Clearly  $\sum_{j=1}^{\ell_3} n_j = k$ . Since  $\tau$  is parallel,  $\tilde{h}^* \tau = \tau$  for any  $h \in G$ . In terms of matrix we have that  $\tilde{h}^{tr} A \tilde{h} = A$ . Here  $\tilde{h}^{tr}$  denotes the transpose of  $\tilde{h}$ . Note that  $\tilde{h} \in U(2k)$ ,  $A\tilde{h} = \tilde{h}A$ . Since  $A$  is real we have  $\tilde{h}A = A\tilde{h}$ . From this we have that  $\tilde{h}A^2 = A^2\tilde{h}$ , which implies that  $\tilde{h}$  also keeps each eigenspace  $\mathcal{V}_j$  invariant. This then implies that  $\mathcal{N}_1$  further decomposes into orthogonal holonomy-invariant subbundles. The next statement follows from Weyl's completely reducibility theorem. Thus  $\tau$  is a constant multiple of the standard holomorphic symplectic  $(2, 0)$ -form on each  $\mathcal{V}_j$ .  $\square$

**Corollary 2.4.** *For any  $x, y \in T_p M$ ,  $(R_{x,y})|_{\mathcal{K}_j}$  has vanishing trace. In particular, the Ricci curvature of  $R|_{\mathcal{K}_j}$  vanishes.*

*Proof.* Since  $\tau|_{\mathcal{K}_j}$  is a holomorphic parallel  $(2, 0)$ -form, the restriction of the holonomy action on  $\mathcal{K}_j$  satisfies  $h \in SU(2k_j)$  with  $2k_j = \dim_{\mathbb{C}}(\mathcal{K}_j)$ . On the other hand,  $R_{x,y}|_{\mathcal{K}_j}$  can be obtained as the derivative of one parameter family  $h(t)$  (cf. Lemma 2.2 in [29] and its proof), thus with zero trace. The final statement is due to that  $\text{Ric}_{R|_{\mathcal{K}_j}}(v, \bar{v}) = \sum_{\ell} R(v, \bar{v}, E_{\ell}^j, \bar{E}_{\ell}^j)$  for  $v \in \Gamma(\mathcal{K}_j)$  where  $\{E_{\ell}^j\}$  is a unitary base of  $\mathcal{K}_j$ .  $\square$

### 3. SYMMETRIC HOLONOMY SYSTEMS

Here we work with real numbers and vector bundles over real numbers. To apply the discussion here to the previous section one can consider the realification of complex bundles. Note that the real part of a Hermitian metric on a complex bundle is a Riemannian metric on the realification of the complex bundle.

The concept of holonomy systems was introduced by J. Simons [35] in his intrinsic proof of Berger's holonomy theorem for the Riemannian holonomy groups with respect to the Levi-Civita connection.

The Riemannian holonomy system is a triple  $S = \{V, R, G\}$ , which consists of, a Euclidean space  $V$  of dimension  $\ell$  (we call it the degree of  $S$ ) endowed with an inner product  $\langle \cdot, \cdot \rangle$  (also denoted by a bilinear form  $H$ ), a connected compact subgroup  $G$  of  $\text{SO}(\ell)$ , and an algebraic curvature operator  $R$  (defined on  $V$ ) satisfying the 1st Bianchi identity and that  $R_{x,y} \in \mathfrak{g}$ ,  $\forall x, y \in V$  with  $\mathfrak{g} \subset \mathfrak{so}(\ell)$  being the Lie algebra of  $G$ . To see the relevance we first recall the Ambrose-Singer's holonomy theorem [22] (see also [29] for a simple proof of half of the result).

**Theorem 3.1** (Ambrose-Singer). *For any connection  $\nabla$  on a Riemannian vector bundle  $(E, h)$ , let  $R$  be its curvature. Given any path  $\gamma$  from  $q$  to  $p$ , let  $\gamma$  also denote the parallel transport along it. Then  $\{\gamma(R^q)\}$  generates the holonomy algebra, where  $\gamma(R^q) \in \mathfrak{so}(E_p)$  is defined  $\gamma \cdot (R_{x,y}|_{E_q}) \cdot \gamma^{-1}$ . Here  $x, y \in T_q M$ .*

In the case that  $(M, g)$  is a Hermitian manifold with a CAS structure, with respect to the Chern connection, which preserves the inner product of the underlying real tangent bundle, for any parallel invariant subbundle  $\mathcal{K} \subset T'M$ , one can apply the above to the bundle  $\mathcal{V} = \mathcal{K} \oplus \overline{\mathcal{K}}$ . Since  $\nabla R = 0$ , it is easy to see that  $\gamma(R^q) = R|_{\mathcal{V}_p}$ . Consequently we have

**Proposition 3.2.** *Let  $(M, g)$  be a Hermitian manifold whose Chern connection is Ambrose-Singer. Let  $\mathcal{V}$  be as above. Let  $G$  be the restricted holonomy group. Then  $S = (\mathcal{V}_p, R, G)$  is a holonomy system. In fact it is symmetric.*

*Proof.* The first part is a direct consequence of Theorem 3.1 and the observation that the curvature  $R$  satisfying the first Bianchi identity. The second statement follows from the fact that  $\nabla R = 0$ . Recall that a holonomy system is called symmetric if  $\gamma(R) = R$  for any  $\gamma \in G$ .  $\square$

The following result holds the key of the algebraic aspect of the proof.

**Proposition 3.3.** *Assume that  $S = \{V, R, G\}$  is an irreducible symmetric holonomy system. Then the Ricci flatness of  $R$  implies that  $R$  is flat.*

*Proof.* Here we follow the argument in Theorem 3.1 of [29]. First we set up the notations and conventions. Identify  $\mathfrak{so}(n)$  with  $\wedge^2 V$ . Here  $S^2(\wedge^2 V)$  denotes the symmetric transformations of  $\wedge^2 V$ . The  $S_B^2(\cdot)$  denotes the subspace satisfying the 1st Bianchi identity.

Define the metric on  $\mathfrak{g}(V)$  by

$$\langle A, B \rangle \doteq \frac{1}{2} \sum_i \langle A(e_i), B(e_i) \rangle = \frac{1}{2} \text{trace}(B^{tr} A)$$

In particular for  $A, B \in \mathfrak{g}$  the above inner product applies. Let  $P$  be the projection from  $\wedge^2(V)$  onto  $\mathfrak{g}$ , and let  $T : \mathfrak{g} \rightarrow \mathfrak{g}$  be the symmetric isomorphism corresponding to the negative definite bilinear form on  $\mathfrak{g}$

$$(3.1) \quad B(A, A') \doteq K(A, A') - 2\langle A, A' \rangle$$

with  $K$  being the Killing form of  $\mathfrak{g}$  (defined as  $K(A, A') = \text{trace}(\text{ad}_A \cdot \text{ad}_{A'})$ ). Namely  $T$  is defined by  $B(A, A') = \langle T(A), A' \rangle$ . Let  $\mathfrak{J} = \mathfrak{g} \oplus V$  (orthogonal sum with the inner product of  $V$  and  $\langle A, B \rangle$  on  $\mathfrak{g}$  as elements in  $\mathfrak{so}(n)$ ) and define a Lie algebra structure on  $\mathfrak{J}$  by letting

$$[A, A'] \doteq [A, A']; \quad [x, y] \doteq -R_{x,y}; \quad [A, x] \doteq A(x), \forall A, A' \in \mathfrak{g}, x, y \in V.$$

Since  $A(R) = 0, \forall A \in \mathfrak{g}$ , together with the first Bianchi identity, it is easy to check that the bracket so defined satisfies the Jacobi identity, namely  $\mathfrak{J}$  is a Lie algebra. Let  $B'$  be the Killing form of  $J$ . It is a basic result of Lie algebra that  $B'$  is  $\text{ad}_J$ -invariant.

*Claim 1:*  $B'|_{\mathfrak{g}}$  is given by  $B$  defined by (3.1), hence is negative definite. By the definition  $B'(A, B) = \text{trace}(\text{ad}_A \cdot \text{ad}_B) = \sum_{i=1}^n \langle \text{ad}_A \cdot \text{ad}_B(e_i), e_i \rangle + \sum_{\alpha} \langle \text{ad}_A \cdot \text{ad}_B(A_{\alpha}), A_{\alpha} \rangle$  where  $\{e_i\}$  ( $\{A_{\alpha}\}$ ) is an orthonormal frame of  $V$  ( $\mathfrak{g}$  respectively). The second summand is  $K(A, B)$ . By the definition of the Lie bracket the first summand is  $-\langle B(e_i), A(e_i) \rangle = -2\langle A, B \rangle$ . We first need the following computational results.

*Claim 2:*  $B'(A, x) = 0$  for  $A \in \mathfrak{g}$  and  $x \in V$ . Similarly

$$B'(A, x) = \text{trace}(\text{ad}_A \cdot \text{ad}_x) = \sum_{i=1}^n \langle \text{ad}_A \cdot \text{ad}_x(e_i), e_i \rangle + \sum_{\alpha} \langle \text{ad}_A \cdot \text{ad}_x(A_{\alpha}), A_{\alpha} \rangle$$

where  $\{e_i\}$  ( $\{A_{\alpha}\}$ ) is an orthonormal frame of  $V$  ( $\mathfrak{g}$  respectively). The first term vanishes since

$$\langle \text{ad}_A \cdot \text{ad}_x(e_i), e_i \rangle = \langle [A, R_{x,e_i}]_{\mathfrak{g}}, e_i \rangle = 0.$$

For the second term,  $\langle \text{ad}_A \cdot \text{ad}_x(A_{\alpha}), A_{\alpha} \rangle = \langle -A(A_{\alpha}(x)), A_{\alpha} \rangle = 0$ .

*Claim 3:*  $B'|_V = \lambda H$ , where  $H(x, y) := \langle x, y \rangle$ , and  $\lambda \neq 0$  if  $R \neq 0$ . Since  $B'|_V$  is  $\text{ad}_{\mathfrak{g}}$ -invariant, hence  $G$ -invariant. By the irreducibility of  $G$ -action on  $V$ , it implies that  $B'(x, y) = \lambda \langle x, y \rangle$  for some  $\lambda$ .

If  $\lambda = 0$ ,  $B'([x, y], [x, y]) = B'(x, [y, [x, y]]) = 0$  since  $[y, [x, y]] \in V$ . Now by *Claim 1*, which asserts that  $B'|_{\mathfrak{g}}$  is negative definite, we have that  $[x, y] = -R_{x, y} = 0, \forall x, y \in V$ . Hence  $R = 0$ .

Below we assume that  $\lambda \neq 0$ , otherwise we have proved the result by the above argument. By *Claim 3* we have that  $\langle [[x, y], z], w \rangle = \frac{1}{\lambda} B'([x, y], [z], w) = \frac{1}{\lambda} B'([x, y], [z, w])$ , which in turn equals to  $\frac{1}{\lambda} B([x, y], [z, w])$  by *Claim 1*. Putting them together we have the equation for  $x, y, z, w \in V$

$$(3.2) \quad -\langle R_{x, y} z, w \rangle = \langle [[x, y], z], w \rangle = \frac{1}{\lambda} B([x, y], [z, w]) = \frac{1}{\lambda} \langle [x, y], \mathbb{T}([z, w]) \rangle.$$

Now by (3.2) the Ricci curvature can be expressed as

$$\text{Ric}_R(x, x) = \sum_{i=1}^n \langle R_{x, e_i} e_i, x \rangle = \frac{1}{\lambda} \sum B([x, e_i], [x, e_i]).$$

Hence if  $\text{Ric}_R = 0$  it implies that  $[x, e_i] = -R_{x, e_i} = 0$  for any  $e_i, x$ , namely  $R = 0$ .

Note that a calculation shows that  $\langle [[x, y], z], w \rangle = -\langle R_{x, y}(z), w \rangle = \langle [x, y], (z \wedge w) \rangle$ . Hence by the above (3.2) we have that

$$\langle [x, y], (z \wedge w) \rangle = \frac{1}{\lambda} \langle [x, y], \mathbb{T}([z, w]) \rangle = -\frac{1}{\lambda} \langle [x, y], \mathbb{T}(R_{z, w}) \rangle.$$

This proves that  $R_{z, w} = -\lambda \mathbb{T}^{-1} \cdot P(z \wedge w)$ , an expression of curvature (due to Kostant) in terms of the Lie algebraic structure.  $\square$

The above proposition is the algebraic counter part of Theorem 8.6 of Vol II of [22].

#### 4. HERMITIAN MANIFOLDS WITH A CAS STRUCTURE

Recall that the subbundle  $\mathcal{N}_0$ , which is defined as

$$\mathcal{N}_0 := \{X \in \mathcal{N} \mid T(X, Y) = 0, \quad \forall Y \in T^{1,0}M\},$$

was introduced in [30] to capture any Kähler de Rham factor in the universal cover  $\widetilde{M}$  of a Hermitian manifold  $M$  with a CAS structure. It is easy to see that if there are de Rham Kähler factors in  $\widetilde{M}$ , then the corresponding holomorphic tangent subbundle must be  $\mathcal{N}_0$ . Applying it to the  $\widetilde{M}$  we have the following result.

**Proposition 4.1.** *Let  $(M, g)$  be a Hermitian manifold with a CAS structure. Let  $\widetilde{M}$  be its universal cover. Let  $\mathcal{N}_0$  be the subbundle of  $T\widetilde{M}$  defined above. Then  $\widetilde{M}$  splits as  $M_1 \times M_2$  with  $M_2$  being Kähler and  $T'M_2 = \mathcal{N}_0$ . Moreover  $M_1$  does not admit any Kähler factor.*

*Proof.* This is essentially Theorem 3.6 of [30].  $\square$

By combining Corollary 2.4, Propositions 3.2, 3.3, we get the proof of our main result, Theorem 1.2:

**Proof of Theorem 1.2.** First split the manifold  $\widetilde{M}$  into two factors as in the statement. By the proof of Lemma 2.2 in [29], the splitting of the holonomy action implies that splitting of  $R_{x, y}$ . Applying Corollary 2.4, Propositions 3.2, 3.3 to the irreducible  $G$ -invariant subbundle of  $T'M_1$  we conclude that the Chern curvature is flat on each irreducible summand, hence is totally flat. Now since  $M_1$  has zero curvature and parallel torsion, we may find global parallel vector fields  $X_1, \dots, X_k$  with  $k = \dim_{\mathbb{C}}(M_1)$  (which are holomorphic) such that

$$[X_i, X_j] = T(X_i, X_j) = T_{ij}^k X_k.$$

The fact that the torsion is parallel implies that  $\{T_{ij}^k\}$  are constants. By Lemma 3.1 of [30], they also satisfy the Jacobi identity in general. Now we can appeal the result of Cartan [13] (pp. 188-192) to assert that there is a complex Lie group structure on  $M_1$ . For a modern treatment of the existence of a complex Lie group structure one can see [15] pages 20-25, Section 3.1 of Ch1 for the existence of Lie group structure and [28], Theorem on page 212 of Section 11 of Ch IV for details on the existence of complex Lie group structure.  $\square$

**Proof of Corollary 1.3.** By Theorem 1.2, the universal cover of  $M$  is the product of a complex Lie group  $M_1$  with  $M_2$  which is products of irreducible simply-connected Hermitian symmetric spaces, namely  $M_2 = \mathbb{C}^{n_1} \times N_2 \times \cdots \times N_\ell$  with  $N_i$  being a non-flat irreducible simply-connected Hermitian symmetric space. However, the non-flat factor must be of semi-simple type hence have its Ricci curvature being a nonzero factor of the Kähler metric. Then the assumption implies that  $\widetilde{M} = M_1 \times \mathbb{C}^{n_1}$ , which is a complex Lie group itself.  $\square$

The corollary shows that  $T'M_1$  may miss some complex Lie group factors. To capture it let  $\mathcal{F}_p = \{Z \in T_p^{1,0}M \mid h(Z) = Z, \forall h \in G_p\}$ , where  $G$  is the holonomy group of the Chern connection at  $p$ . Recall the following result from [30].

**Proposition 4.2.** *Let  $(M, g)$  be a Hermitian manifold with a CAS structure. Let  $\mathcal{F} = \cup_x \mathcal{F}_x$  be the sub-bundle of  $T^{1,0}M$ . Then  $\mathcal{F}$  is a holomorphic integrable foliation. Moreover, if  $Z_1, \dots, Z_r$  is a parallel frame of  $\mathcal{F}$ , then*

$$[Z_i, Z_j] = c_{ij}^k Z_k$$

for some constant  $c_{ij}^k$ . In particular, when  $M$  is simply-connected, there exists a complex Lie group  $F$  acting almost freely, holomorphically on  $M$  such that  $T_x^{1,0}(F \cdot x) = \mathcal{F}_x$ .

Theorem 1.2 implies that on  $\widetilde{M}$ ,  $\mathcal{F} = T'M_1 \oplus \mathbb{C}^{n_1}$ . The main reason that our argument works is that for a manifold with CAS structure, at the curvature level, algebraically it is the same as a locally symmetric space.

We conclude this section by a discussion on Remark 1.4, which says that the naive way of generalizing AK type theorem from Levi-Civita connection to Chern connection would fail, namely, there are examples of compact locally homogeneous Hermitian manifold whose Chern connection has vanishing (third) Ricci curvature but is not Chern flat. This illustrates that the CAS assumption in Corollary 1.3 is necessary. We will give two such examples below.

The first example is some special type of *almost abelian manifolds*, namely, compact quotients of  $(G, J, g)$ , where  $G$  is an even-dimensional, connected and simply-connected, unimodular Lie group,  $J$  a left-invariant (integrable) complex structure on  $G$  and  $g$  a left-invariant metric on  $G$  compatible with  $J$ . Let  $\mathfrak{g}$  denote the Lie algebra of  $G$ .  $G$  is said to be *almost abelian* if  $\mathfrak{g}$  contains an abelian ideal  $\mathfrak{a}$  of codimension 1. The Hermitian structures on almost abelian Lie algebras were studied by a number of authors, and we refer the readers to [8, 12, 16, 17, 27] and the references therein for more information on this. We will follow the computation in §3 of [20] to serve our purpose here.

A unitary basis  $\{e_1, \dots, e_n\}$  of  $\mathfrak{g}^{1,0} = \{x - iJx \mid x \in \mathfrak{g}\}$  is called *admissible* if  $\mathfrak{a}$  is spanned by  $e_j + \bar{e}_j$ ,  $i(e_j - \bar{e}_j)$  for  $2 \leq j \leq n$  and  $i(e_1 - \bar{e}_1)$ . The structural constants  $C$  and  $D$  are defined by

$$[e_i, e_j] = \sum_k C_{ij}^k e_k, \quad [e_i, \bar{e}_j] = \sum_k (\overline{D_{kj}^i} e_k - D_{ki}^j \bar{e}_k).$$

When  $e$  is admissible, the only possibly non-zero components of  $C$  and  $D$  are

$$D_{11}^1 = \lambda \in \mathbb{R}, \quad D_{i1}^1 = v_i \in \mathbb{C}, \quad D_{i1}^j = A_{ij}, \quad C_{1i}^j = -\overline{A_{ji}}, \quad \forall 2 \leq i, j \leq n.$$

From the calculation in §3 of [20], we know that for the almost abelian Lie group  $(G, J, g)$ ,

- $G$  is unimodular  $\iff \lambda + \text{tr}(A) + \overline{\text{tr}(A)} = 0$ ;
- $g$  is Chern flat  $\iff \lambda = 0, v = 0$ , and  $[A, A^*] = 0$ ;
- $g$  has vanishing third Chern Ricci  $\iff \lambda = 0, v = 0$ , and  $\text{tr}(A) + \overline{\text{tr}(A)} = 0$ .

Here  $A^*$  stands for the conjugate transpose of  $A$ . With these notations and set-ups, our first example goes like the following:

For any  $n \geq 3$ , if we choose the  $(n-1) \times (n-1)$  matrix  $A$  so that  $\text{tr}(A) + \overline{\text{tr}(A)} = 0$  but  $[A, A^*] \neq 0$ , then the metric  $g$  would have vanishing third Chern Ricci but is not Chern flat. (Note that the first Chern Ricci vanishes here, but the second Ricci does not vanish).

Another (simpler) example is the (compact quotients of) nilpotent Lie groups with a left-invariant Hermitian structure that is balanced and with parallel Bismut torsion. Denote by  $G$  the Lie group and  $\mathfrak{g}$  its Lie algebra. Under some unitary basis  $e$  of  $\mathfrak{g}^{1,0}$  the structure equation takes the form:

$$(4.1) \quad \begin{cases} d\varphi_i = 0, & \forall 1 \leq i \leq r; \\ d\varphi_\alpha = \sum_{i=1}^r Y_{\alpha i} \varphi_i \wedge \bar{\varphi}_i, & \forall r+1 \leq \alpha \leq n, \end{cases}$$



where  $2n$  is the real dimension of  $G$ ,  $1 \leq r < n$ ,  $\varphi$  is the coframe dual to  $e$ , and  $Y_{\alpha i}$  are arbitrary complex constants such that  $\sum_{i=1}^r Y_{\alpha i} = 0$  for each  $\alpha$ .

When these  $Y_{\alpha i}$  are not all zero, the Chern connection is not flat. On the other hand, the metric is balanced, the first and third Chern Ricci both vanish (but the second Chern Ricci is not identically zero).

To conclude the discussion of this section, let us remark that we do not know any example which will serve as an negative answer to the following question:

**Question 4.3.** *Let  $(M, g)$  be a compact locally homogeneous Hermitian manifold. If the first, second, and third Chern Ricci all vanish, then must it be Chern flat?*

We do not even know the answer to this question for Lie-Hermitian manifolds, namely when the universal cover of  $M$  is a Lie group (not necessarily a complex Lie group) equipped with left-invariant complex structure and left-invariant metric.

## 5. GENERALIZED SYMMETRIC HOLONOMY SYSTEMS

In this section we consider the symmetric holonomy system for an affine connection with a torsion. Suppose now that we have a metric connection  $\nabla$  with torsion. For Hermitian manifold we also require that it satisfies  $\nabla J = 0$ , where  $J$  is the almost complex structure. For the situation that  $\nabla$  is Ambrose-Singer (or invariant under the parallelism), we have that  $g(R) = R$  and  $g(T) = T$  for any element  $g$  belonging to the restricted holonomy. Note that  $T : \wedge^2 T_p M \rightarrow T_p M$  for any given  $p \in M$ .

A generalized symmetric holonomy system is  $S = \{V, R, T, G\}$ , which consists of, a Euclidean space  $V$  of dimension  $n$  (we call it the degree of  $S$ ) endowed with an inner product  $\langle \cdot, \cdot \rangle$  (also denoted by a bilinear form  $H$ ), a connected compact subgroup  $G$  of  $\mathrm{SO}(\ell)$ , an algebraic curvature operator  $R : \wedge^2 V \rightarrow \wedge^2 V$  as a linear map, and a linear map  $T : \wedge^2 V \rightarrow V$ , together satisfying

$$(5.1) \quad R_{x,y} \in \mathfrak{g};$$

$$(5.2) \quad \mathfrak{S}\{R_{x,y}z - T(x, T(y, z))\} = 0;$$

$$(5.3) \quad \mathfrak{S}\{R_{x, T(y, z)}\} = 0;$$

$$(5.4) \quad g(R) = R, g(T) = T, \forall g \in G, \text{ or equivalently } A(T) = A(R) = 0, \forall A \in \mathfrak{g}.$$

Here  $\mathfrak{g}$  is the Lie algebra of  $G$ ,  $\mathfrak{S}$  is the cyclic permutation operation on the positions of  $x, y, z$ , the actions of  $A$  on  $T$  and  $R$  are the ones induced as derivations. Namely,  $A(T)(x, y) = AT(x, y) - T(Ax, y) - T(x, Ay)$  and  $(A(R))_{x,y} = A \cdot R_{x,y} - R_{Ax,y} - R_{x,Ay} - R_{x,y} \cdot A$ .

The following result, which is an immediate consequence of (5.2), (5.3) and (5.4), is due to Nomizu [31].

**Theorem 5.1** (Nomizu). *Let  $\mathfrak{J} = V \oplus \mathfrak{g}$ . Define a Lie algebra structure on  $\mathfrak{J}$  by letting*

$$[A, A'] \doteq [A, A']; \quad [x, y] \doteq -R_{x,y} \oplus (-T(x, y)); \quad [A, x] \doteq A(x), \forall A, A' \in \mathfrak{g}, x, y \in V.$$

*Then  $\mathfrak{J}$  is a Lie algebra.*

From the definitions in the theorem one can easily compute the Lie bracket of  $[x \oplus A, y \oplus A']$ . Let  $B'$  still denote the Killing form of the Lie algebra  $\mathfrak{J}$ , which is a bilinear form. We also denote the bilinear form  $-2\langle \cdot, \cdot \rangle + K(\cdot, \cdot)$  on  $\mathfrak{g}$  by  $B(\cdot, \cdot)$ , which is negative definite. Here  $K$  is the Killing form of  $\mathfrak{g}$  which is nonnegative due to the compactness of  $G$ . The results in Section 3 mostly no longer hold for the generalized symmetric holonomy system. Recall that we endow  $\mathfrak{J}$  with metrics from  $\mathfrak{g} \subset \mathfrak{so}(n)$  and  $V$ . However we still have the following result.

**Lemma 5.2.** *The  $B'|_{\mathfrak{g}}$  is the same as  $B$ , namely  $B'|_{\mathfrak{g}} = B$ . In particular  $B'|_{\mathfrak{g}}$  is negative definite. If we assume additionally that  $V$  is irreducible (namely as a  $G$ -module), then  $B'|_V(\cdot, \cdot) = \lambda H(\cdot, \cdot)$ .*

*Proof.* Direct calculation shows, for an orthonormal frame  $\{e_i\}_{i=1}^n$  of  $V$  and an orthonormal frame  $\{A_\alpha\}_{\alpha=1}^{\dim(\mathfrak{g})}$  of  $\mathfrak{g}$ , that

$$\begin{aligned} B'(C_1, C_2) &= \text{trace}(\text{ad}_{C_1} \cdot \text{ad}_{C_2}) \\ &= \sum_{i=1}^n \langle \text{ad}_{C_1} \cdot \text{ad}_{C_2}(e_i), e_i \rangle + \sum_{\alpha} \langle \text{ad}_{C_1} \cdot \text{ad}_{C_2}(A_\alpha), A_\alpha \rangle \\ &= -\sum_{i=1}^n \langle C_2(e_i), C_1(e_i) \rangle + K(C_1, C_2) \\ &= B(C_1, C_2). \end{aligned}$$

Here  $C_1, C_2 \in \mathfrak{g}$  are two arbitrary elements. This proves the claimed identity. The last claimed follows from Schur's lemma.  $\square$

For  $C \in \mathfrak{g}$ ,  $x \in V$ ,  $B'(C, x)$  only vanishes under additional conditions.

**Lemma 5.3.**  $B'(C, x) = 2\langle T(x, \cdot), C \rangle$ . Here we view  $T(x, \cdot)$  as an endomorphism of  $V$ . In particular, for a Ambrose-Singer connection it vanishes if  $R_{x,y} \cdot T = 0$ .

*Proof.* By Ambrose-Singer theorem and its specialization, Lemma 4 of [24] we have that the holonomy algebra  $\mathfrak{g}$  is generated by  $\sum_j R_{x_j, y_j}$  for some  $x_j, y_j \in V$ . Assume that we have  $B'(C, x) = 2\langle T(x, \cdot), C \rangle$ . Writing  $C = \sum_j R_{x_j, y_j}$  we have that

$$\begin{aligned} B'(C, x) &= \sum_{i=1}^n \sum_j \langle T(x, e_i), R_{x_j, y_j} e_i \rangle \\ &= - \sum_{i=1}^n \sum_j \langle R_{x_j, y_j} T(x, e_i), e_i \rangle \\ &= 0. \end{aligned}$$

This proves the second statement assuming the first. Now we prove the first part, namely the claimed identity in the lemma. Direct calculation shows that

$$\begin{aligned} B'(C, x) &= \sum_i \langle \text{ad}_C \text{ad}_x(e_i), e_i \rangle + \sum_\alpha \langle \text{ad}_C \text{ad}_x(A_\alpha), A_\alpha \rangle \\ &= - \sum_i \langle C(T(x, e_i)), e_i \rangle - \langle [C, R_{x, e_i}], e_i \rangle - \sum_\alpha \langle C(A_\alpha(x)), A_\alpha \rangle \\ &= \sum_i \langle T(x, e_i), C(e_i) \rangle. \end{aligned}$$

This proves the claimed equation.  $\square$

We say  $R|_{\text{image}(T)} = 0$  if  $R_{x,y} \cdot T = 0$  and  $R_{x, T(y, z)} = 0$ . Now we prove the following generalization of Proposition 3.3.

**Theorem 5.4.** Let  $S = \{V, R, T, G\}$  be a generalized symmetric holonomy system. Assume that it is irreducible. Assume further that  $R|_{\text{image}(T)} = 0$ , and either (i)  $T(x, \cdot)$  is skew-symmetric, or (ii)  $V$  is Hermitian with almost complex structure  $J$  and  $T$  splits as  $T : V^{1,0} \times V^{1,0} \rightarrow V^{1,0}$  and  $T : V^{0,1} \times V^{0,1} \rightarrow V^{0,1}$  with all other components being zero. Then  $\text{Ric}_R(x, x) = 0$  implies that  $R = 0$ .

*Proof.* Note that  $B'|_V(\cdot, \cdot) = \lambda H(\cdot, \cdot)$ . If  $\lambda = 0$ , then  $B'|_V \equiv 0$ . Observe, by the assumption  $R|_{\text{image}(T)} = 0$ , that

$$\begin{aligned} [y, [x, y]] &= [y, -R_{x,y} \oplus (-T(x, y))] \\ &= R_{y, T(x, y)} \oplus (R_{x,y} y + T(y, T(x, y))) \\ &= R_{x,y} y + T(y, T(x, y)) \in V. \end{aligned}$$

Hence from  $B'([x, y], [x, y]) = B'(x, [y, [x, y]]) = B'(x, R_{x,y} y + T(y, T(x, y))) = 0$  we have that

$$B'([x, y], [x, y]) = 0.$$

On the other hand by Lemma 5.3 we also have that

$$\begin{aligned} B'([x, y], [x, y]) &= B'(R_{x,y} \oplus T(x, y), R_{x,y} \oplus T(x, y)) \\ &= B'(R_{x,y}, R_{x,y}) < 0 \end{aligned}$$

unless  $R_{x,y} = 0$ . Here we have used that Lemma 5.3 implies  $B'((R_{x,y}, T(x, y))) = 0$ . This proves the theorem for the case  $\lambda = 0$ .

Now we assume that  $\lambda \neq 0$ . The first part argument above then shows that

$$(5.5) \quad \langle R_{x,y} y, x \rangle + \langle T(y, T(x, y)), x \rangle = \frac{1}{\lambda} B'(x, [y, [x, y]]) = \frac{1}{\lambda} B'([x, y], [x, y]).$$

Now using  $\text{Ric}_R(x, x) = \sum \langle R_{x, e_i} e_i, x \rangle$  we have that

$$\begin{aligned}
\langle T(x, e_i), T(x, e_i) \rangle + \text{Ric}_R(x, x) &= \langle T(e_i, T(x, e_i)), x \rangle + \text{Ric}_R(x, x) \\
&= \frac{1}{\lambda} \sum_{i=1}^n B'([x, e_i], [x, e_i]) \\
&= \frac{1}{\lambda} \sum_{i=1}^n B'(R_{x, e_i} \oplus T(x, e_i), R_{x, e_i} \oplus T(x, e_i)) \\
&= \frac{1}{\lambda} \sum_{i=1}^n B(R_{x, e_i}, R_{x, e_i}) + \sum_{i=1}^n \langle T(x, e_i), T(x, e_i) \rangle.
\end{aligned}$$

Canceling the same term on the both sides we have  $\text{Ric}_R(x, x) = \frac{1}{\lambda} \sum_{i=1}^n B(R_{x, e_i}, R_{x, e_i})$ , which implies that  $\frac{1}{\lambda} \sum_{i=1}^n B'(R_{x, e_i}, R_{x, e_i}) = 0$  if  $\text{Ric}_R = 0$ . The result then follows from Lemma 5.2, namely that  $B'|_{\mathfrak{g}}$  is negatively definite. This proves the result for case (i). Case (ii) is similar. This completes the proof of the theorem.  $\square$

## 6. THE BISMUT/GAUDUCHON AMBROSE-SINGER MANIFOLDS

From the discussion in the previous sections, we obtained a complete classification of compact Hermitian manifolds that is CAS, namely, whose Chern connection is Ambrose-Singer. Furthermore, we also know that Chern connection satisfies the Alekseevskii-Kimelfel'd type theorem, namely, a CAS metric will be flat if it is Ricci flat. It is natural to wonder what happens if we replace the Chern connection  $\nabla$  by the Bismut connection  $\nabla^b$ , or more generally by the  $t$ -Gauduchon connection  $\nabla^{(t)}$ , which is the linear combination  $(1 - \frac{t}{2})\nabla + \frac{t}{2}\nabla^b$  where  $t \in \mathbb{R}$ .

Recall that in §1 we called a Hermitian metric *BAS* if its  $\nabla^b$  is Ambrose-Singer, namely, both the torsion and curvature of  $\nabla^b$  are parallel with respect to  $\nabla^b$  itself. Similarly, we called a Hermitian metric *t-GAS* if its  $t$ -Gauduchon connection  $\nabla^{(t)}$  has parallel torsion and curvature with respect to  $\nabla^{(t)}$  itself. We would like to understand the set of all BAS (or more generally, all *t-GAS*) manifolds, and know in particular whether or not the Alekseevskii-Kimelfel'd type theorem will hold for  $\nabla^b$  (or  $\nabla^{(t)}$ ).

First let us clarify the meaning of Ricci curvature since we are dealing with general metric connections now. Recall that the Ricci curvature tensor of a metric connection  $D$  on a Riemannian manifold  $(M, g)$  is defined by

$$(6.1) \quad \text{Ric}^D(x, y) = \sum_{i=1}^m R^D(\varepsilon_i, x, y, \varepsilon_i)$$

where  $R^D(x, y, z, w) = \langle D_x D_y z - D_y D_x z - D_{[x, y]} z, w \rangle$  is the curvature of  $D$ ,  $\{\varepsilon_i\}$  is a local orthonormal frame of  $M$ , while  $x, y, z, w$  are tangent vectors. In general  $\text{Ric}^D$  might not be symmetric as  $D$  may have torsion. When  $(M^n, g)$  is Hermitian, we may choose the orthonormal frame  $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$  so that

$$\sqrt{2}\varepsilon_i = e_i + \bar{e}_i, \quad \sqrt{2}\varepsilon_{n+i} = \sqrt{-1}(e_i - \bar{e}_i); \quad i = 1, \dots, n,$$

where  $\{e_1, \dots, e_n\}$  is a local unitary frame. Plugging into (6.1), we get

$$(6.2) \quad \text{Ric}^D(x, y) = \sum_{i=1}^n \{R^D(e_i, x, y, \bar{e}_i) + R^D(\bar{e}_i, x, y, e_i)\}.$$

Now suppose  $D$  is Hermitian, namely,  $Dg = 0$  and  $DJ = 0$ . Then  $R^D(*, *, e_i, e_j) = 0$ . So for any type  $(1, 0)$  tangent vectors  $X, Y$  we have

$$(6.3) \quad \text{Ric}^D(X, Y) = \sum_{i=1}^n R^D(e_i, X, Y, \bar{e}_i), \quad \text{Ric}^D(\bar{X}, Y) = \sum_{i=1}^n R^D(e_i, \bar{X}, Y, \bar{e}_i).$$

This is often called the *third Ricci curvature* of the Hermitian connection  $D$ , also denoted as  $\text{Ric}^{D(3)}$ , while the *first* and *second* Ricci are defined respectively by

$$\text{Ric}^{D(1)}(x, y) = \sum_{i=1}^n R^D(x, y, e_i, \bar{e}_i), \quad \text{Ric}^{D(2)}(X, \bar{Y}) = \sum_{i=1}^n R^D(e_i, \bar{e}_i, X, \bar{Y}).$$

For the Bismut connection  $\nabla^b$  we will denote them as  $\text{Ric}^{b(i)}$ ,  $i = 1, 2, 3$ . In §1, we raised the following (= Conjecture 1.5):

**Conjecture 6.1.** *Suppose  $(M^n, g)$  is a compact Hermitian manifold that is BAS. Assume that the first and third Bismut Ricci both vanish. Then it must be Bismut flat.*

In complex dimension 2, the answer to Conjecture 6.1 is positive. As a direct consequence of [51, Theorem 2], we have the following

**Proposition 6.2.** *A compact, non-Kähler Hermitian surface  $(M^2, g)$  is BAS if and only if it is Vaisman with constant scalar curvature.*

Recall that a Hermitian manifold is said to be *Vaisman* if it is locally conformally Kähler and its Lee form is parallel under the Levi-Civita connection. Compact non-Kähler Vaisman surfaces are fully classified by Belgun in the beautiful work [10]. They are either non-Kähler properly elliptic surfaces, or Kodaira surfaces, or Class 1 or elliptic Hopf surfaces.

We remark that for a non-Kähler Vaisman surface, the scalar curvature of Levi-Civita or Chern or Bismut connection (or the trace of first or second Ricci of Chern or Bismut connection) all differ by constants. So the word ‘constant scalar curvature’ here simply means any one of them (hence all of them) is constant. Also, such a surface always admits local unitary frame  $\{e_1, e_2\}$  under which the only possibly non-zero component of the Bismut curvature tensor is  $R_{1\bar{1}1\bar{1}}^b$ . So clearly if it is Bismut Ricci flat (or equivalently Bismut scalar flat) then it will be Bismut flat:

**Corollary 6.3.** *If a compact BAS surface is Bismut Ricci flat, then it is Bismut flat.*

Here we could drop the adjective ‘non-Kähler’ since any locally Hermitian symmetric space would be flat if Ricci flat. Note that the only compact, non-Kähler Bismut flat surfaces are the isosceles Hopf surfaces ([46]). The above statement confirms Conjecture 6.1 in the 2-dimensional case.

Now let us move on to higher dimensions. Recall that *Bismut torsion-parallel* (or *BTP* for short) manifolds are Hermitian manifolds whose Bismut connection has parallel torsion. Such manifolds form a rather interesting class, and contains both our BAS manifolds and the BKL (Bismut Kähler-like) manifolds. Compact, non-Kähler BTP manifolds in dimension 3 were characterized in [53], from which we could pick out those with vanishing first and third Bismut Ricci, and they all turn out to be Bismut flat, proving Conjecture 6.1 in the  $n = 3$  case.

Recall that the  $B$ -tensor of a Hermitian manifold  $(M^n, g)$  is defined by  $B_{i\bar{j}} = \sum_{r,s=1}^n T_{rs}^j \overline{T_{rs}^i}$  under any unitary frame  $e$ , where  $T_{ik}^j$  are components of the Chern torsion. Then  $\sigma_B = \sqrt{-1} \sum_{i,j=1}^n B_{i\bar{j}} \varphi_i \wedge \overline{\varphi_j}$  is a globally defined non-negative  $(1, 1)$ -form on  $M$ . Here  $\varphi$  is the coframe dual to  $e$ . Also recall that the (first) Chern Ricci form is the global  $(1, 1)$ -form given by

$$\rho = \sqrt{-1} \sum_{i,j=1}^n \text{Ric}_{i\bar{j}}^{(1)} \varphi_i \wedge \overline{\varphi_j} = \sqrt{-1} \partial \bar{\partial} \log \omega^n,$$

where  $\omega$  is the Kähler form of  $g$ . The following technical lemma is useful in picking out Bismut Ricci flat BAS metrics:

**Lemma 6.4.** *Suppose  $(M^n, g)$  is a BTP manifold that is (third) Bismut Ricci flat. Then the (first) Chern Ricci form is equal to  $\sigma_B$ .*

*Proof.* Under any local unitary frame  $e$ , let us denote by  $T_{ik}^j$ ,  $R_{i\bar{j}k\bar{\ell}}$ , and  $R_{i\bar{j}k\bar{\ell}}^b$  the components of the Chern torsion, Chern curvature, and Bismut curvature. Here we used the convention  $T(e_i, e_k) = \sum_j T_{ik}^j e_j$  so our  $T_{ik}^j$  equals twice of that in [53]. Under the BTP assumption, we have  $\nabla^b T = 0$ , so by formula (3.2) and (3.4) in Lemma 3.1 of [53], we get the following

$$(6.4) \quad R_{i\bar{j}k\bar{\ell}}^b - R_{k\bar{j}i\bar{\ell}}^b = -T_{ik}^r \overline{T_{j\bar{\ell}}^r} - T_{ir}^j \overline{T_{\ell r}^k} - T_{kr}^\ell \overline{T_{j\bar{r}}^i} + T_{ir}^\ell \overline{T_{j\bar{r}}^k} + T_{kr}^j \overline{T_{\ell r}^i}$$

$$(6.5) \quad R_{i\bar{j}k\bar{\ell}}^b - R_{i\bar{j}\ell\bar{k}}^b = -T_{ik}^r \overline{T_{j\bar{\ell}}^r} - T_{ir}^j \overline{T_{\ell r}^k} - T_{kr}^\ell \overline{T_{j\bar{r}}^i} + T_{ir}^\ell \overline{T_{j\bar{r}}^k}$$

for any  $1 \leq i, j, k, \ell \leq n$ , where  $r$  is summed from 1 to  $n$  on the right hand side of the above two equations. Taking the difference, we get  $R_{i\bar{j}k\bar{\ell}}^b - R_{k\bar{j}i\bar{\ell}}^b = T_{kr}^j \overline{T_{\ell r}^i}$ . Letting  $k = \ell$  and sum it up, we get  $\text{Ric}^{(1)} - \text{Ric}^{b(3)} = B$ . So  $\rho = \sigma_B$  when  $\text{Ric}^{b(3)} = 0$ . This proves the lemma.  $\square$

In particular, we have  $d\sigma_B = 0$  since  $d\rho = 0$ , which gives us the following:

**Lemma 6.5.** *Let  $(M^n, g)$  be a (third) Bismut Ricci flat BTP manifold. Then under any local unitary frame it holds that*

$$(6.6) \quad \sum_r (T_{rk}^j B_{i\bar{r}} + T_{ir}^j B_{k\bar{r}} - T_{ik}^r B_{r\bar{j}}) = 0, \quad \forall 1 \leq i, j, k, \leq n.$$

In particular,  $M$  has non-negative first Chern class but is not Fano.

*Proof.* Equation (6.6) is a direct consequence of the structure equations and the fact that  $d\sigma_B = 0$ , so we just need to prove the ‘not Fano’ part. Denote by  $b_1 \geq \dots \geq b_n \geq 0$  the eigenvalues of  $B$ . Under the BTP assumption, we have  $\nabla^b B = 0$ , so each  $b_i$  is a global constant. If we choose  $e$  so that  $B$  is diagonal, then (6.6) becomes

$$(6.7) \quad (b_i + b_k - b_j)T_{ik}^j = 0, \quad \forall 1 \leq i, j, k, \leq n.$$

We claim that  $b_n$  must be zero. Assume otherwise, by letting  $j = n$  in (6.7), we get  $T_{ik}^n = 0$  for any  $i, k$ , which leads to  $b_n = B_{n\bar{n}} = \sum_{r,s} |T_{rs}^n|^2 = 0$ , a contradiction. So  $B \not\equiv 0$  thus  $M$  is not Fano.  $\square$

Now let us prove the following statement (= Proposition 1.6):

**Proposition 6.6.** *Any complete, non-Kähler BAS manifold with vanishing first and third Bismut Ricci must be non-balanced. Moreover, if  $n \leq 4$  it must be Bismut flat.*

*Proof.* Let  $(M^n, g)$  be a complete BAS manifold with vanishing first and third Bismut Ricci. Any Kähler de Rham factor will be Hermitian symmetric and Ricci flat hence flat, so we may assume that  $M$  is without any Kähler de Rham factor. Recall that the  $\phi$ -tensor is defined by  $\phi_i^j = \sum_r T_{ir}^j \bar{\eta}_r$  under any unitary frame, where  $\eta$  is the Gauduchon torsion 1-form. By formula (6.4), (6.5) in the proof of Lemma 6.4, we see that vanishing of first and third Bismut Ricci imply that  $\text{Ric}^{(1)} = B$  and  $B = \phi + \phi^*$ . Since the trace of the  $B$  tensor is equal to the square norm of the Chern torsion, which does not vanish identically as  $g$  is not Kähler, so we know that  $\phi \neq 0$  hence  $\eta \neq 0$ , namely,  $g$  is not balanced.

By [53, Proposition 5.2], around any given point in  $M$  there always exists an *admissible* frame, which is a local unitary frame  $e$  under which

$$\eta_1 = \dots = \eta_{n-1} = 0, \quad \eta_n = \lambda > 0, \quad T_{in}^j = \delta_{ij} a_i, \quad \forall 1 \leq i, j \leq n,$$

where  $a_n = 0$  and  $a_1 + \dots + a_{n-1} = \lambda$ . Since  $\eta, B, \phi$  are all parallel under  $\nabla^b$ , the number  $\lambda = |\eta| > 0$  and each  $a_i$  are global constants on  $M$ , as these  $\lambda a_i$  are eigenvalues of  $\phi$ . Denote by  $\theta^b, \Theta^b$  the matrix of connection and curvature of the Bismut connection  $\nabla^b$  under an admissible frame  $e$ . Since  $e_n$  is lined up with  $\eta$ , we know that  $\nabla^b e_n = 0$  thus  $\theta_{n*}^b = 0$  and  $\Theta_{n*}^b = 0$ . By  $\nabla^b T = 0$ , we get

$$(6.8) \quad 0 = d(\delta_{ij} a_i) = dT_{in}^j = \sum_r (T_{rn}^j \theta_{ir}^b + T_{ir}^j \theta_{nr}^b - T_{in}^r \theta_{rj}^b) = (a_j - a_i) \theta_{ij}^b.$$

Write  $b_i = B_{i\bar{i}} = \sum_{r,s} |T_{rs}^i|^2$ , we have

$$(6.9) \quad b_i = \lambda(a_i + \bar{a}_i) = 2|a_i|^2 + 2 \sum_{1 \leq j < k < n} |T_{jk}^i|^2 = 2|a_i|^2 + 2\delta_i.$$

The second part follows from the following claim and establishment of the assumption in the claim afterwards.

**Claim:** *If there exists an admissible frame under which  $\theta^b$  is diagonal, and  $n \leq 4$ , then  $R^b = 0$ .*

To prove the claim, note that by [53, Theorem 1.1] any BTP manifold always satisfies the symmetry condition  $R_{i\bar{j}k\bar{\ell}}^b = R_{k\bar{\ell}i\bar{j}}^b$ , so the only possibly non-trivial Bismut curvature components are given by

$$\Theta_{ii}^b = \sum_{k < n} S_{ik} \varphi_k \wedge \bar{\varphi}_k, \quad \forall i < n,$$

where  $S$  is a symmetric real  $(n-1) \times (n-1)$  matrix. The vanishing of the first and third Bismut Ricci implies that the diagonal entries of  $S$  vanish, also the sum of each row vanishes. When  $n \leq 4$ , this will force  $S = 0$ , hence  $R^b = 0$ . This establishes the Claim.

Now we prove the assumption of the Claim holds when  $n \leq 4$ . Let us assume  $n = 4$  (the  $n = 3$  case is analogous but much easier). When  $\{a_1, a_2, a_3\}$  are all distinct, by (6.8) we know that  $\theta^b$  is diagonal, so by Claim 1 we get  $R^b = 0$ . If  $a_1 = a_2 = a_3$ , then  $a_1 = \frac{\lambda}{3}$  as their sum needs to be  $\lambda$ , hence  $b_1 = b_2 = b_3 = \frac{2}{3}\lambda^2$ . On the other hand, by (6.7) we know that  $T_{ik}^j = 0$  for any  $1 \leq i, j, k \leq 3$ , thus (6.9) would yield  $\frac{2}{3}\lambda^2 = \frac{2}{9}\lambda^2 + 0$ , which is a contradiction. We are therefore left with the case when  $a_1 = a_2 \neq a_3$ . By (6.8), we know that  $\theta^b$  is block-diagonal, with a  $2 \times 2$  block on the upper left corner. Let us write  $a_1 = a$  and  $b_1 = b$ , then by (6.9) we get

$$\begin{aligned} b &= 2\lambda \text{Re}(a) = 2|a|^2 + 2\delta; \\ b_3 &= 2\lambda \text{Re}(a_3) = 2|a_3|^2 + 2\delta_3; \\ \delta &= |T_{12}^1|^2 + |T_{13}^1|^2 + |T_{23}^1|^2 = |T_{12}^2|^2 + |T_{13}^2|^2 + |T_{23}^2|^2; \\ \delta_3 &= |T_{12}^3|^2 + |T_{13}^3|^2 + |T_{23}^3|^2. \end{aligned}$$

Since  $a_3 = \lambda - 2a$  and  $b_3 = 2\lambda^2 - 2b$ , the first two equations above give

$$(6.10) \quad b = 2\lambda \operatorname{Re}(a) = 2|a|^2 + 2\delta = 4|a|^2 + \delta_3.$$

If  $a = 0$ , then  $b = 0$  and  $\delta = \delta_3 = 0$ , so  $T_{**}^i = T_{i*}^* = 0$  for  $i = 1, 2$ . This means that  $E = \operatorname{span}\{e_1, e_2\}$  is contained in the kernel of  $T$ , thus is parallel under the Levi-Civita connection. So  $E$  gives us a Kähler de Rham factor, contradicting with our assumption at the beginning of the proof. Therefore we may assume that  $a \neq 0$ , or equivalently,  $b > 0$ .

If  $b < \lambda^2$ , or equivalently, if  $b_3 > 0$ , then by (6.7) we know that the only possibly non-zero  $T_{ik}^j$  for  $1 \leq i, j, k \leq 3$  is  $T_{12}^3$ . In particular,  $\delta = 0$  and  $\delta_3 = |T_{12}^3|^2$ . By (6.10),  $\delta_3 = -2|a|^2 < 0$  which is impossible.

Now assume that  $b = \lambda^2$ . The three eigenvalues of the  $B$  tensors (we always ignore the direction  $e_n = e_4$  here) are  $\{\lambda^2, \lambda^2, 0\}$ . In this case  $b_3 = 0$ , hence  $a_3 = 0$  and  $a = \frac{\lambda}{2}$ ,  $\delta_3 = 0$ ,  $\delta = \frac{\lambda^2}{4}$ . By (6.7) we know that the only possibly non-zero  $T_{ik}^j$  for  $1 \leq i, j, k \leq 3$  are  $T_{i3}^j = P_{i\bar{j}}$ , where  $1 \leq i, j \leq 2$ . The trace of the  $2 \times 2$  matrix  $P$  is zero, as  $T_{13}^1 + T_{23}^2 = \eta_3 = 0$ . Also, by (6.10) we know that the sum of norm square of each column of  $P$  is equal to  $\delta = \frac{\lambda^2}{4}$ . In particular, the two off diagonal elements of  $P$  have equal norm. Let us write

$$P = \begin{bmatrix} x & y \\ \rho y & -x \end{bmatrix},$$

where  $|\rho| = 1$  and  $|x|^2 + |y|^2 = \frac{\lambda^2}{4}$ . Note that for any unitary change of  $\{e_1, e_2\}$ , the new frame  $\{\tilde{e}_1, \tilde{e}_2, e_3, e_4\}$  is still admissible, and  $P$  is changed to  $\tilde{P} = UPU^*$  where  $U$  is the  $U(2)$ -valued function changing  $\{e_1, e_2\}$  to  $\{\tilde{e}_1, \tilde{e}_2\}$ . If  $y \neq 0$ , then in a small neighborhood we may chose  $U$  so that the upper right entry  $\tilde{y}$  of  $\tilde{P}$  is zero. The lower left entry of  $\tilde{P}$  is then zero as  $\tilde{P}$  is again tracefree and with equal sum of norm square for each column. In other words, by performing a unitary change of  $\{e_1, e_2\}$  if necessary, then under the new admissible frame we may assume that  $P$  is diagonal, with  $|T_{13}^1| = |x| = \frac{\lambda}{2}$  being a global constant. Replace  $e_3$  by  $\rho_3 e_3$  for a suitable function with  $|\rho_3| = 1$ , we may further assume that  $T_{13}^1 = |T_{13}^1|$ , hence is a global positive constant. In summary, we may choose admissible frame so that the only non-zero torsion components are

$$T_{14}^1 = T_{24}^2 = a, \quad T_{13}^1 = -T_{23}^2 = a, \quad a = \frac{\lambda}{2}.$$

We already know that  $\theta^b$  is block diagonal, with blocks of size 2, 1, 1, and  $\theta_{44}^b = 0$ . Since  $\nabla^b T = 0$ , we have

$$0 = dT_{13}^1 = \sum_r (T_{r3}^1 \theta_{1r}^b + T_{1r}^1 \theta_{3r}^b - T_{13}^r \theta_{r1}^b) = T_{13}^1 \theta_{33}^b,$$

hence  $\theta_{33}^b = 0$ . In other words, the fixing of the gauge  $T_{13}^1 = |T_{13}^1| > 0$  made our admissible frame  $e$  to satisfy  $\nabla^b e_3 = 0$ . Similarly, by  $T_{13}^2 = 0$ , we have

$$0 = dT_{13}^2 = \sum_r (T_{r3}^2 \theta_{1r}^b + T_{1r}^2 \theta_{3r}^b - T_{13}^r \theta_{r2}^b) = (T_{23}^2 - T_{13}^1) \theta_{12}^b = -2T_{13}^1 \theta_{12}^b,$$

which implies that  $\theta_{12}^b = 0$ , therefore  $\theta^b$  is diagonal. Hence by the Claim we get  $R^b = 0$ . This completes the proof of Proposition 6.6, which is Proposition 1.6.  $\square$

The above proof of the  $n = 4$  case of Conjecture 6.1 is basically by brute force, which relies heavily on the relatively small dimension.

Recall that a Hermitian manifold is said to be *Bismut Kähler-like* (or *BKL* for short, see [7] and [48]), if the Bismut curvature tensor  $R^b$  obeys all the Kähler symmetries:  $R_{xyz\bar{w}}^b = 0$  and  $R_{x\bar{y}z\bar{w}}^b = R_{z\bar{y}x\bar{w}}^b$  for any type  $(1, 0)$  tangent vectors  $x, y, z, w$ . When  $n \geq 2$ , there are examples of non-Kähler manifolds which are Bismut Kähler-like, and such metrics were classified in complex dimensions  $n \leq 5$  ([50, 51, 54]). By the main result of [51], the BKL condition is equivalent to BTP plus pluriclosedness. (On the other hand, BKL and BAS are two sets that neither one is contained in the other, although they are both contained in the BTP set).

Now let us prove the following proposition stated in the introduction (= Proposition 1.7), which is a special case of [54, Theorem 3]. Here we give an alternative proof as an application of Proposition 3.3 (or the more general version Theorem 5.4):

**Proposition 6.7.** *Let  $(M^n, g)$  be a complete Hermitian manifold whose Bismut connection is Ambrose-Singer (BAS) and with vanishing first Bismut Ricci curvature. If  $g$  is BKL, then  $g$  is Bismut flat.*

*Proof.* Fix any  $p \in M$  and let  $V$  be the tangent space of  $M$  at  $p$ . Note that the Bismut Kähler-like assumption guarantees that  $R^b$  obeys the first Bianchi identity, so the Bismut connection  $\nabla^b$  gives a holonomy system, which is clearly a symmetric one. In order to apply Proposition 3.3, we need to verify that the Ricci curvature vanishes for each irreducible component. Decompose the tangent bundle into subbundles where the Bismut holonomy group acts irreducibly. Writing in complex frames, say  $T^{1,0}M = \oplus_{j=1}^r E_j$ , then the Bismut curvature form  $\Theta^b$  is block-diagonal:

$$\Theta^b = \begin{bmatrix} \Theta_1^b & & \\ & \ddots & \\ & & \Theta_r^b \end{bmatrix}, \quad \Theta_j^b = (\Theta_{ik}^b), \quad e_i, e_k \in E_j.$$

Since  $\nabla^b$  is assumed to be Kähler-like, the entries of  $\Theta_j^b$  are all combinations of  $\varphi_i \bar{\varphi}_k$  for  $e_i, e_k \in E_j$  only. In particular,  $\text{tr}(\Theta^b) = 0$  if and only if  $\text{tr}(\Theta_j^b) = 0$  for each  $j$ . So if we assume that the Bismut curvature has vanishing Ricci, then each irreducible component will also have vanishing Ricci, thus one can apply Proposition 3.3 to conclude that the curvature vanishes. This completes the proof. Note that since the curvature is assumed to be parallel, so any Kähler de Rham factor will be locally Hermitian symmetric, hence it will be flat if it is Ricci flat.  $\square$

Recall that Bismut flat manifolds were fully classified [46]. They are quotients of Samelson spaces, namely, Lie groups with bi-invariant metrics and compatible left-invariant complex structures. Hermitian metrics with vanishing first Bismut Ricci are said to be *Calabi-Yau with torsion* (or *CYT* in short) in the literature. It is equivalent to the condition when the restricted holonomy group of  $\nabla^b$  is contained in  $SU(n)$ .

An interesting question closely related to Conjecture 6.1 is the following one raised by Garcia-Fernández and Streets [18, Question 3.58]:

**Question 6.8** (Garcia-Fernández and Streets). *Given a homogeneous Riemannian manifold  $(M, g)$  with an invariant 3-form  $H$ , denote by  $\nabla^H$  the metric connection with totally skew-symmetric torsion  $H$ . If  $\nabla^H$  has vanishing Ricci, then must it be flat?*

Such a  $(g, H)$  is called a *Bismut Ricci Flat* pair, or a *BRF pair* in short. This happens when and only when  $H$  is harmonic and the Ricci curvature  $\text{Ric}^g$  of the Levi-Civita connection  $\nabla^g$  is given by

$$\text{Ric}^g(x, y) = \frac{1}{4} \langle \iota_x H, \iota_y H \rangle$$

for any tangent vector  $x, y$ , where  $\iota_x$  stands for contraction. This concept is important in generalized Riemannian geometry which corresponds to special types of generalized Einstein structures.

Garcia-Fernández and Streets answered the question positively for real dimension 3 in (a special case of) [18, Proposition 3.55]. In [32] and [33], Podestà and Raffero investigated the question and answered it positively in (real) dimension 4, but in dimension 5 or higher, they constructed counterexamples. To be more precise, in [32] they constructed an explicit sequence  $M_{p,q}$  of compact homogeneous 5-manifolds with invariant closed 3-form  $H$  so that  $\nabla^H$  is Ricci flat but not flat. Here  $p$  and  $q$  are any positive integers that are relatively prime.

Note that Question 6.8 is closely related to Conjecture 6.1 but different. The latter assumes the manifold to be Hermitian and the Bismut connection to be Ambrose-Singer, but it does not require the torsion 3-form to be closed. It would be interesting to understand all BRF pairs, and in particular all BRF pairs where  $\nabla^H$  is also Ambrose-Singer.

Next we consider  $t$ -GAS manifolds: compact Hermitian manifolds whose  $t$ -Gauduchon connection  $\nabla^{(t)}$  is Ambrose-Singer (i.e., having parallel torsion and curvature). Assume  $t \neq 0, 2$ . Let us give a couple of quick remarks. The first is that there are non-Kähler (hence not  $\nabla^{(t)}$ -flat by the theorem of Lafuente and Stanfield [26]) such manifolds. The simplest example would be any compact quotient of a complex semi-simple Lie group. Such a manifold is actually  $t$ -GAS for any  $t \in \mathbb{R}$ . The second remark is that, for any  $t \neq 2$ , a compact  $t$ -GAS manifold is always balanced:

**Lemma 6.9.** *Let  $(M^n, g)$  be a compact Hermitian manifold that is  $t$ -GAS for some  $t \neq 2$ . Then  $g$  is balanced.*

*Proof.* Under any local unitary frame  $e$ , let us denote by  $\theta^{(t)}$  the matrix of the  $t$ -Gauduchon connection  $\nabla^{(t)}$ , and write  $\theta^{(0)} = \theta$  for the Chern connection. Then  $\theta^{(t)} = \theta + \frac{t}{2} \gamma$  where  $\gamma'_{ij} = \sum_k T_{ik}^j \varphi_k$  and  $\gamma = \gamma' - {}^t \gamma'$ . Here as before we denoted by  $\varphi$  the coframe dual to  $e$ , and by  $T_{ik}^j$  the components of the Chern torsion under  $e$ , namely,  $T(e_i, e_k) = \sum_j T_{ik}^j e_j$ . Denote by  $\eta = \sum_i \eta_i \varphi_i$  the Gauduchon torsion

1-form, defined by  $\eta_i = \sum_k T_{ki}^k$ . Then one has  $\partial(\omega^{n-1}) = -\eta \wedge \omega^{n-1}$ , where  $\omega = \sqrt{-1} \sum_k \varphi_k \wedge \bar{\varphi}_k$  is the Kähler form of  $g$ .

The structure equation is  $d\varphi = -\theta \wedge \varphi + \tau$ , where  $\tau = \frac{1}{2} \iota\gamma' \wedge \varphi$  is the column vector of Chern torsion. For any fixed  $p \in M$ , let us choose our local unitary frame  $e$  so that  $\theta^{(t)}$  vanishes at  $p$ . Then at  $p$  we have  $\theta = -\frac{t}{2}\gamma$ , thus by the structure equation we know that  $\bar{\partial}\varphi_r = \frac{t}{2} \sum_{i,j} T_{ri}^j \varphi_i \wedge \bar{\varphi}_j$  at  $p$ . We compute that

$$\bar{\partial}\eta = \sum_{i,j} \{-\eta_{i,\bar{j}} + \frac{t}{2} \sum_r \eta_r \overline{T_{rj}^i}\} \varphi_i \wedge \bar{\varphi}_j,$$

where the index after comma stands for covariant derivative with respect to  $\nabla^{(t)}$ . From this we get

$$n\sqrt{-1} \bar{\partial}\eta \wedge \omega^{n-1} = -\left(\sum_i \eta_{i,\bar{i}} + \frac{t}{2} |\eta|^2\right) \omega^n,$$

where  $|\eta|^2 = \sum_i |\eta_i|^2$ . Similarly,  $n\sqrt{-1} \eta \wedge \bar{\eta} \wedge \omega^{n-1} = |\eta|^2 \omega^n$ . On the other hand,

$$\partial\bar{\partial}(\omega^{n-1}) = \bar{\partial}(\eta \wedge \omega^{n-1}) = (\bar{\partial}\eta + \eta \wedge \bar{\eta}) \wedge \omega^{n-1} = \frac{\sqrt{-1}}{n} \left\{ \sum_i \eta_{i,\bar{i}} + \left(\frac{t}{2} - 1\right) |\eta|^2 \right\} \omega^n.$$

When the metric is  $t$ -GAS, we have  $\nabla^{(t)}T = 0$  hence  $\eta_{i,\bar{j}} = 0$ . Integrating the last equation above over the compact manifold  $M$ , we get  $(\frac{t}{2} - 1) \int_M |\eta|^2 \omega^n = 0$ . Hence for any  $t \neq 2$  we would have  $\eta = 0$ , which means that  $g$  is balanced.  $\square$

Denote by  $T, R$  the torsion and curvature of the Chern connection  $\nabla$ , and the components of  $T$  under a unitary frame  $e$  are given by  $T(e_i, e_k) = \sum_j T_{ik}^j e_j$ . Denote by  $R^{(t)}$  the curvature of the  $t$ -Gauduchon connection  $\nabla^{(t)}$ . We have the following

**Lemma 6.10.** *Let  $(M^n, g)$  be a Hermitian manifold. Then under any local unitary frame  $e$  it holds:*

$$(6.11) \quad T_{ik,\bar{j}}^\ell + R_{ij\bar{k}\bar{\ell}} - R_{kj\bar{i}\bar{\ell}} + \frac{t}{2} \sum_r \{T_{ir}^\ell \overline{T_{jr}^k} - T_{kr}^\ell \overline{T_{jr}^i} - T_{ik}^r \overline{T_{j\ell}^r}\} = 0$$

$$(6.12) \quad R_{ij\bar{k}\bar{\ell}}^{(t)} - R_{i\bar{j}\bar{k}\bar{\ell}} - \frac{t}{2} \{T_{ik,\bar{j}}^\ell + \overline{T_{j\ell,\bar{i}}^k}\} + \frac{t^2}{4} \sum_r \{T_{ir}^j \overline{T_{\ell r}^k} + T_{kr}^\ell \overline{T_{jr}^i} + T_{ik}^r \overline{T_{j\ell}^r} - T_{ir}^\ell \overline{T_{jr}^k}\} = 0$$

for any  $1 \leq i, j, k, \ell \leq n$ , where index after comma stands for covariant derivative with respect to  $\nabla^{(t)}$ .

*Proof.* Let  $\theta, \theta^{(t)}, \gamma, \tau$  be as in the proof of Lemma 6.9. Denote by  $\Theta, \Theta^{(t)}$  the matrix of Chern curvature and  $t$ -Gauduchon curvature, respectively. We have  $\theta^{(t)} = \theta + \frac{t}{2}\gamma$ , and  $\iota\gamma' \wedge \varphi = 2\tau$ . Take the exterior differential on the structure equation  $d\varphi = -\theta \wedge \varphi + \tau$ , we get the first Bianchi identity:

$$(6.13) \quad d\tau = -\iota\theta \wedge \tau + \iota\Theta \wedge \varphi.$$

Fix any point  $p \in M$ . Choose the local unitary frame  $e$  near  $p$  so that  $\theta^{(t)}|_p = 0$ . Then at  $p$  we have  $\theta = -\frac{t}{2}\gamma$ , hence

$$\partial\varphi = (1-t)\tau = \frac{1}{2}(1-t) \iota\gamma' \wedge \varphi \quad \text{and} \quad \bar{\partial}\varphi = -\frac{t}{2} \bar{\gamma}' \wedge \varphi \quad \text{at } p.$$

Now take the  $(2, 1)$ -part in (6.13), at  $p$  we have

$$\sum_k \{\bar{\partial}\gamma'_{k\ell} + \frac{t}{2} \sum_r (\bar{\gamma}'_{\ell r} \wedge \gamma'_{kr} + \gamma'_{r\ell} \wedge \bar{\gamma}'_{rk}) + 2\Theta_{k\ell}\} \wedge \varphi_k = 0$$

for any  $1 \leq \ell \leq n$ . This leads us to the identity (6.11). On the other hand, by definition we have  $\Theta = d\theta - \theta \wedge \theta$  and  $\Theta^{(t)} = d\theta^{(t)} - \theta^{(t)} \wedge \theta^{(t)}$ . So at  $p$  we get

$$(\Theta^{(t)})^{1,1} - \Theta = \frac{t}{2} \{\bar{\partial}\gamma' - \partial\bar{\gamma}'\} - \frac{t^2}{4} \{\gamma' \wedge \bar{\gamma}' + \bar{\gamma}' \wedge \gamma'\}.$$

This leads to the identity (6.12), and the lemma is proved.  $\square$

**Proof of Proposition 1.9.** Suppose  $t \neq 0, 2$  and  $(M^n, g)$  is a compact  $t$ -GAS manifold with vanishing first and third  $t$ -Gauduchon Ricci. By Lemma 6.9 we know that  $g$  is balanced. For  $1 \leq i \leq 3$ , denote by  $Ric^{(t)(i)}$  the  $i$ -th Ricci of  $\nabla^{(t)}$ , and by  $Ric^{(i)}$  the  $i$ -th Chern Ricci. Since  $\eta = 0$  and  $Ric^{(t)(1)} = 0$ , by letting  $k = \ell$  and sum up in (6.12), we get  $Ric^{(1)} = 0$ . Similarly, since  $\eta = 0$  and  $Ric^{(t)(3)} = 0$ , by letting  $i = \ell$  and sum up in (6.12), we get  $Ric^{(3)} = \frac{t^2}{4} B$ , where  $B_{k\bar{j}} = \sum_{r,s} T_{rs}^j \overline{T_{rs}^k}$ . Now if we let  $k = \ell$  and sum up in (6.11), we get  $Ric^{(1)} - Ric^{(3)} = 0$ . Putting these together, we get  $B = 0$ , hence  $\text{tr}(B) = |T|^2 = 0$ . Therefore  $T = 0$  and  $g$  is Kähler. This completes the proof of Proposition 1.9.  $\square$



Note that if we drop the assumption of  $Ric^{(t)(1)} = 0$  in Proposition 1.9, then we only get  $Ric^{(1)} = \frac{t^2}{4}B$ . This implies that the global  $(1, 1)$ -form  $\sigma_B = \sqrt{-1} \sum_{i,j} B_{i\bar{j}} \varphi_i \wedge \bar{\varphi}_j$  is equal to a positive constant multiple of the first Chern Ricci form, hence is  $d$ -closed. Since  $\nabla^{(t)}B = 0$ , the eigenvalues of  $B$  are all constants. Choose  $e$  so that  $B$  is diagonal, then by  $d\sigma_B = 0$  we get

$$(B_{i\bar{i}} + B_{k\bar{k}} - B_{j\bar{j}})T_{ik}^j = 0 \quad \forall 1 \leq i, j, k \leq n.$$

While this is certainly highly restrictive, we do not know how to proceed from here to prove that  $g$  must be Kähler, so we added the assumption that the first  $t$ -Gauduchon Ricci also vanishes. One could certainly replace that by the vanishing of the second  $t$ -Gauduchon Ricci, or simply the vanishing of the trace of either of them, namely, the vanishing of the first (= second)  $t$ -Gauduchon scalar curvature.

**Acknowledgments.** We would like to thank C. Böhm for his interests and bringing our attention to a result of Alekseevskii and B. N. Kimel'fel'd, P. Petersen and H. Wu for their enthusiastic interests and encouragement, J. Stanfield and S. Gindi for their interests in extending the result on Chern connection to the BAS manifolds, Y. Ustinovskiy for his interest in extending results to the torsion connection. The first author would like to thank T. Colding and MIT mathematics for the hospitality during his visit when the first draft of the paper was completed, B. Kleiner for insightful questions, D. Yang and G. Zhang for their hospitality during his visit of CIMS at NYU. We are also grateful to J. Streets for his interests on the results related to the Bismut connection and for the references [18], [33], A. Raffero for correcting some inaccurate referencing. Finally we also thank R. Lafuente and a referee for helpful comments on various presentation improvements.

#### REFERENCES

- [1] I. Agricola, *Connections on naturally reductive spaces, their Dirac operator and homogeneous models in string theory*. Comm. Math. Phys. **232** (2003), 535–563. (cited on page 3)
- [2] I. Agricola and T. Friedrich, *On the holonomy of connections with skew-symmetric torsion*. Math. Ann. **328** (2004), 711–748. (cited on page 3)
- [3] I. Agricola, A.C. Ferreira, and T. Friedrich, *The classification of naturally reductive homogeneous spaces in dimensions  $n \leq 6$* . Differential Geom. Appl. **39** (2015), 59–92. (cited on page 3)
- [4] D. V. Alekseevskii and B. N. Kimel'fel'd, *Structure of homogeneous Riemannian spaces with zero Ricci curvature (Russian)*. Funkcional. Anal. i Priložen. **9** (1975), no.2, 5–11. (cited on page 2)
- [5] W. Ambrose and I.M. Singer, *On homogeneous Riemannian manifolds*. Duke Math. J. **25** (1958), 647–669. (cited on page 1)
- [6] D. Angella, F. Pediconi, *A survey on locally homogeneous almost-Hermitian spaces (English summary)*. Riv. Math. Univ. Parma (N.S.) **13** (2022), no.2, 373–418. (cited on page 2)
- [7] D. Angella, A. Otal, L. Ugarte, R. Villacampa, *On Gauduchon connections with Kähler-like curvature*. Comm. Anal. Geom. **30** (2022), no.5, 961–1006. (cited on page 14)
- [8] R.M. Arroyo and R.A. Lafuente, *The long-time behavior of the homogeneous pluriclosed flow*. Proc. Lond. Math. Soc.(3) **119** (2019), no.1, 266–289. (cited on page 8)
- [9] W. Ballmann, *Homogeneous structures*. Notes on geometry, available at author's website. (cited on page 1)
- [10] F. Belgun, *On the metric structure of non-Kähler complex surfaces*. Math. Ann. **317** (2000), 1–40. (cited on page 12)
- [11] J.-M. Bismut, *A local index theorem for non-Kähler manifolds*. Math. Ann. **284** (1989), no.4, 681–699. (cited on page 3)
- [12] C. Bock, *On low dimensional solvmanifolds*. Asian J. Math. **20** (2016), 199–262. (cited on page 8)
- [13] E. Cartan, *La théorie des groupes finis et continus et la géométrie différentielle traitées par méthode du repère mobile*. Cahiers Scientifiques, no.18, Gauthier-Villars, Paris, 1937. (cited on page 7)
- [14] S.-S. Chern, *Complex manifolds without potential theory*. With an appendix on the geometry of characteristic classes. Second edition. Universitext Springer-Verlag, New York-Heidelberg, 1979. iii+152 pp. (cited on page 2)
- [15] B. A. Dubrovin, A. T. Fomenko and S. P. Novikov, *Modern geometry-methods and applications. Part II. The geometry and topology of manifolds*. Translated from Russian by Robert G. Burns. Graduate Texts in Mathematics, 104, Sprng-Verlag, New York, 1985. xv+430 pp. ISBN: 0-387-96162-3-53-01 (cited on page 7)
- [16] A. Fino and F. Paradiso, *Generalized Kähler almost abelian Lie groups*. Ann. Mat. Pura Appl.(4) **200** (2021), 1781–1812. (cited on page 8)
- [17] A. Fino and F. Paradiso, *Balanced Hermitian structures on almost abelian Lie algebras*. J. Pure Applied Algebra **227** (2023), no.2, Pape No. 107186. (cited on page 8)
- [18] M. Garcia-Fernández and J. Streets, *Generalized ricci flow*. AMS University Lecture Series **76**, 2021. (cited on pages 3, 15, 17)
- [19] S.J. Gates, C.M. Hull and M. Rocěk, *Twisted multiplets and new supersymmetric nonlinear sigma models*. Nuc. Phys. B **248** (1984), 157-186. (cited on page 3)
- [20] Y. Guo and F. Zheng, *Hermitian geometry of Lie algebras with abelian ideals of codimension 2*. Math Zeit **304** (2023) no.3, Paper No 51, 24pp. (cited on page 8)
- [21] S. Kobayashi, *Espaces à connexions affines et Riemanniennes symétriques (French)*. Nagoya Math. J. **9** (1955), 25–37. (cited on page 1)

- [22] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. I, II*. Reprint of the 1963 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996. xii+329 pp. (cited on pages 6, 7)
- [23] B. Kostant, *Holonomy and the Lie algebra of infinitesimal motions of a Riemannian manifold*. Trans. Amer. Math. Soc. **80** (1955), 528–542. (cited on page 2)
- [24] B. Kostant, *A characterization of invariant affine connections*. Nagoya Math. J. **16** (1960), 35–50. (cited on pages 1, 10)
- [25] O. Kowalski and L. Vanhecke, *Classification of five-dimensional naturally reductive spaces*. Math. Proc. Cambridge Philos. Soc. **97** (1985), no.3, 445–463. (cited on page 3)
- [26] R. Lafuente and J. Stanfield, *Hermitian manifolds with flat Gauduchon connections*. arXiv: 2204.08170. (cited on pages 4, 15)
- [27] J. Lauret and C. Will, *On the symplectic curvature flow for locally homogeneous manifolds*. J. Symplectic Geom. **15** (2017), no.1, 1–49. (cited on page 8)
- [28] Y. Matsushima, *Differentiable manifolds*. Translated from the Japanese by E. T. Kobayashi. Pure and Applied Mathematics, **9**. Marcel Dekker, Inc., New York, 1972. vii+303 pp. (cited on page 7)
- [29] L. Ni, *An alternate induction argument in Simons’ proof of holonomy theorem*. Analysis and Partial Differential Equations on Manifolds, Fractals and Graphs (Nankai, 2019, A. Grigoryan, Y Sun, Eds.) Advances in Analysis and Geometry, **3** (2021), 443–458. (cited on pages 5, 6, 7)
- [30] L. Ni and F. Zheng, *On Hermitian manifolds whose Chern connection is Ambrose-Singer*. Trans. Amer. Math. Soc. **376** (2023), no.9, 6681–6707. (cited on pages 2, 4, 5, 7, 8)
- [31] K. Nomizu, *Invariant affine connections on homogeneous spaces*. Amer. J. Math. **76** (1954), 33–65. (cited on pages 1, 9)
- [32] F. Podestà and A. Raffero, *Bismut Ricci flat manifolds with symmetries*. Proceedings of the Royal Society of Edinburgh Section A: Mathematics **153** (2023), no.4, 1371–1390. (cited on page 15)
- [33] F. Podestà and A. Raffero, *Infinite families of homoheneous Bismut Ricci flat manifolds*. Comm. Contemp. Math. doi: 10.1142/S0219199722500754. (cited on pages 15, 17)
- [34] K. Sekigawa, *Notes on homogeneous almost Hermitian manifolds*. Hokkaido Math. J. **7** (1978), 206–213. (cited on page 1)
- [35] J. Simons, *On the transitivity of holonomy systems*. Ann. of Math. **76** (1962), 213–234. (cited on pages 2, 5)
- [36] I.M. Singer, *Infinitesimally homogeneous spaces*. Comm. Pure Appl. Math. **13** (1960), 685–697. (cited on page 2)
- [37] R. Storm, *The classification of 7- and 8-dimensional naturally redictive spaces*. Canadian J. Math. **72** (2020), no.5, 1246–1255. (cited on page 3)
- [38] J. Streets, *Pluriclosed flow and the geometrization of complex surfaces*. Geometric Analysis, Progress in Mathmematics vol. **333** (2020), 471–510. (cited on page 3)
- [39] J. Streets and G. Tian, *A parabolic flow of pluriclosed metrics*. Int. Math. Res. Notices **16** (2010), 3101–3133. (cited on page 3)
- [40] J. Streets and G. Tian, *Regularity results for pluriclosed flow*. Geometry and Topology **17** (2013), 2389–2429. (cited on page 3)
- [41] A. Strominger, *Superstrings with torsion*. Nuclear Phys. B. **274** (1986), 253–284. (cited on page 3)
- [42] F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*. London Math. Soc. Lect. Notes, vol.**83**. Cambridge Univ. Press, Cambridge (1983). (cited on page 2)
- [43] L. Vezzoni, *A note on canonical Ricci forms on a 2-step nilmanifolds*. Proc. Amer. Math. Soc. **141** (2013), no. 1, 325–333. (cited on page 4)
- [44] H.-C. Wang, *Closed manifolds with homogeneous complex structure*. Amer. J. Math. **76** (1954), 1–32. (cited on page 2)
- [45] H.-C. Wang, *Complex parallisiable manifolds*. Proc. Amer. Math. Soc. **5** (1954), 771–776. (cited on page 2)
- [46] Q. Wang, B. Yang, F. Zheng, *On Bismut flat manifolds*. Trans. Amer. Math. Soc. **373** (2020), 5747–5772. (cited on pages 3, 12, 15)
- [47] J.-A. Wolf, *Spaces of constant curvature*. Fifth edition. Publish or Perish, Inc., Houston, TX, 1984. xviii+412 pp. ISBN: 0-914098-07-1 53-02. (cited on page 2)
- [48] B. Yang and F. Zheng, *On curvature tensors of Hermitian manifolds*. Comm. Anal. Geom. **26** (2018), no.5, 1195–1222. (cited on pages 4, 14)
- [49] S.-T. Yau, *Open problems in geometry*. Differential Geometry: partial differential equations on manifolds, Proc. Sympos. Pure. Math., vol. **54**, Amer. Math. Soc., Providence, RI, 1993, pp. 1–28. (cited on page 2)
- [50] S.-T. Yau, Q. Zhao and F. Zheng, *On Strominger Kähler-like manifolds with degenerate torsion*. Trans. Amer. Math. Soc. **376** (2023), no.5, 3063–3085. (cited on page 14)
- [51] Q. Zhao, F. Zheng, *Srominger connection and pluriclosed metrics*. J. Reine Angew Math. (Crelles). **796** (2023), 245–267. (cited on pages 12, 14)
- [52] Q. Zhao, F. Zheng, *On Gauduchon Kähler-like manifolds*. J. Geom. Anal. **32** (2022), no.4, Paper No. 110, 27pp. (cited on page 4)
- [53] Q. Zhao, F. Zheng, *On Hermitian manifolds with Bismut-Strominger parallel torsion*. arXiv: 2208.03071. (cited on pages 12, 13)
- [54] Q. Zhao, F. Zheng, *Bismut Kähler-like manifolds in dimension 4 and 5*. arXiv: 2303.09267. (cited on pages 3, 14)

LEI NI. DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO, LA JOLLA, CA 92093, USA  
 Email address: leni@ucsd.edu

FANGYANG ZHENG. SCHOOL OF MATHEMATICAL SCIENCES, CHONGQING NORMAL UNIVERSITY, CHONGQING 401331, CHINA  
 Email address: 20190045@cqnu.edu.cn; franciszheng@yahoo.com