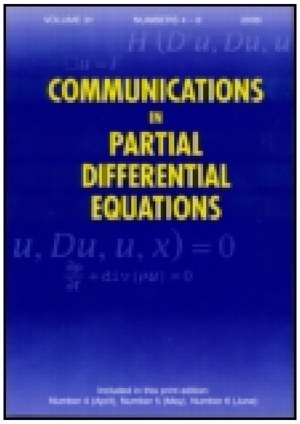

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We prove a comparison theorem on the modulus of continuity of the solution of a heat equation with a drifting term on Bakry-Emery manifolds. A direct consequence of the result is an alternate proof of an eigenvalue comparison result of Bakry-Qian. Examples are given to show that the estimate is sharp. Discussions on an explicit lower estimate for the corresponding ODE and an application to the diameter lower bound for gradient shrinking solitons are also included.

Keywords Bakry-Emery manifolds; Eigenvalue estimates; Harmonic oscillator; Heat equation; Modulus of continuity; Ricci solitons; Weber's equation.

Mathematics Subject Classification Primary 35P15, 35J10; Secondary 35K05, 58J35.

1. A Lower Bound for the First Eigenvalue of the Drift Laplacian

Recall that (M, g, f), a triple consisting of a manifold M, a Riemannian metric g and a smooth function f, is called a gradient Ricci soliton if the Ricci curvature and the Hessian of f satisfy:

$$\mathrm{Rc}_{ij} + f_{ij} = a g_{ij}. \tag{1.1}$$

It is called shrinking, steady, or expanding soliton if $a > 0$, $a = 0$ or $a < 0$ respectively. More generally (M, g, f) is called a Bakry-Emery manifold if the so-called Bakry-Emery Ricci tensor $\mathrm{Rc}_{ij} + f_{ij} \geq a g_{ij}$ for some $a \in \mathbb{R}$. In this paper we apply the modulus of continuity estimates developed in [1–3] to give a different proof of an eigenvalue comparison estimate on Bakry-Emery manifolds for the operator $\Delta_f \doteqdot \Delta - \langle \nabla(\cdot), \nabla f\rangle$ on strictly convex $\Omega \subset M$ with diameter D and smooth boundary. This result was first proved in [5, Theorem 14], which serves as a

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generalization to the earlier works of Payne-Weinberger[10], Li-Yau[7] and Zhong-Yang [12]. An more recent result of this kind was obtained in [6].

Here in fact in Theorem 1.2 we extend a comparison theorem of [3] on the modulus of continuity to manifolds with lower bound on the Bakry-Emery Ricci tensor. This new result gives sharp modulus of continuity comparison between the solution to $\frac{\partial}{\partial t} - \Delta_f$ and the corresponding sup-solution to a heat equation on a certain interval. This implies the eigenvalue comparison result of Bakry-Qian, Theorem 1.1, since first eigenvalue determines the rate of convergence to equilibrium. As in [6] applying to the soliton setting, this implies a lower diameter estimate for nontrivial gradient shrinking solitons (which improves [6]). We remark here that the eigenvalue estimate we obtain is sharp for (M, g, f) satisfying the Bakry-Emery-Ricci lower bound $\mathrm{Rc}_{ij} + f_{ij} \geq ag_{ij}$, but presumably is not so for Ricci solitons where the Bakry-Emery-Ricci tensor is constant, and so we expect that our diameter bound is also not sharp. We discuss the sharpness of the eigenvalue inequality in Section 2.

Before we state the result, we define a corresponding 1-dimensional eigenvalue problem. On $[-\frac{D}{2}, \frac{D}{2}]$ we consider the functionals

$$\mathscr{F}(\psi) = \int_{\frac{D}{2}}^{\frac{D}{2}} e^{-\frac{a}{2}s^2}(\psi')^2 ds, \quad \text{and} \quad \mathscr{R}(\psi) = \frac{\mathscr{F}(\psi)}{\int e^{-\frac{a}{2}s^2}\psi^2 ds},$$

namely the Dirichlet energy with weight $e^{-\frac{a}{2}s^2}$ and its Rayleigh quotient. The associated elliptic operator is $\mathscr{L}_a = \frac{d^2}{ds^2} - as\frac{d}{ds}$. Let $\bar{\lambda}_{a,D}$ be the first non-zero Neumann eigenvalue of $\mathscr{L}_a$, which is the minimum of $\mathscr{R}$ among $W^{1,2}$-functions with zero average.

Theorem 1.1 (Bakry-Qian). *Let Ω be a compact manifold M, or a bounded strictly convex domain inside a complete manifold M, satisfying that $\mathrm{Rc}_{ij} + f_{ij} \geq a\, g_{ij}$. Assume that D is the diameter of Ω. Then the first non-zero Neumann eigenvalue $\widetilde{\lambda}_1$ of the operator Δ_f is at least $\bar{\lambda}_{a,D}$.*

Proof. The key is to adapt Theorem 2.1 of [3] to this setting. Recall that ω is a modulus of continuity for a function f on M if for all x and y in M, $|f(y) - f(x)| \leq 2\omega\left(\frac{d_M(x,y)}{2}\right)$.

Theorem 1.2. *Let $v(x, t)$ be a solution to*

$$\frac{\partial v}{\partial t} = \Delta v - 2\langle X, \nabla v\rangle \tag{1.2}$$

with $2X = \nabla f$. Assume also $v(x, t)$ satisfies the Neumann boundary condition. Suppose that $v(x, 0)$ has a modulus of continuity $\varphi_0(s) : [0, \frac{D}{2}] \to \mathbb{R}$ with $\varphi_0(0) = 0$ and $\varphi_0' > 0$ on $[0, \frac{D}{2}]$. Assume further that there exists a function $\varphi(s, t) : [0, \frac{D}{2}] \times \mathbb{R}_+ \to \mathbb{R}$ such that

(i) $\varphi(s, 0) = \varphi_0(s)$ *on* $[0, D/2]$;
(ii) $\frac{\partial \varphi}{\partial t} \geq \varphi'' - a\, s\, \varphi'$ *on* $[0, D/2] \times \mathbb{R}_+$;
(iii) $\varphi'(s, t) > 0$ *on* $[0, \frac{D}{2}]$;
(iv) $\varphi(0, t) \geq 0$ *for each* $t \geq 0$.

Here $\varphi' = \frac{\partial}{\partial s}\varphi(s,t)$, $\varphi'' = \frac{\partial^2}{\partial s^2}\varphi(s,t)$. Then $\varphi(s,t)$ is a modulus of the continuity of $v(x,t)$ for all $t > 0$.

Proof. The proof of Theorem 1.2 is a modification of the argument of Theorem 2.1 in [3]. Precisely, consider

$$\mathscr{C}_\epsilon(x,y,t) \doteqdot v(y,t) - v(x,t) - 2\varphi\left(\frac{r(x,y)}{2}, t\right) - \epsilon e^t.$$

It suffices to show that $\mathscr{C}_\epsilon \le 0$ for any $\epsilon > 0$. The proof of this claim is via *reductio ad absurdum*. Assume that there exists (x_0, y_0, t_0) such that $\mathscr{C}_\epsilon(\cdot,\cdot,t) = 0$ for the first time. Namely $\mathscr{C}_\epsilon(x,y,t)$ achieves its maximum over $\Omega\times\Omega\times[0,t_0]$ at (x_0,y_0,t_0). The strictly convexity, the Neumann boundary condition satisfied by $v(x,t)$, and the positivity of φ' rule out the possibility that the maximum can be attained at $(x_0,y_0)\in\partial(\Omega\times\Omega)$. For the interior pair (x_0,y_0) where the maximum of $\mathscr{C}$ is attained, pick a frame $\{e_i\}$ at x_0 and parallel translate it along a minimizing geodesic $\gamma(s):[0,d]\to M$ joining x_0 with y_0. Still denote it by $\{e_i\}$. We may also arrange $e_n = \gamma'(s)$. Let $\{E_i\}$ be the frame at (x_0,y_0) (in $T_{(x_0,y_0)}\Omega\times\Omega$) defined as $E_i = e_i \oplus e_i$ for $1 \le i \le n-1$ and $E_n = e_n \oplus (-e_n)$. Direct calculations show that at (x_0,y_0,t_0),

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \sum_{j=1}^{n}\nabla^2_{E_jE_j}\right)\mathscr{C}_\epsilon(x,y,t) &= -\left(\langle\nabla f(y),\gamma'\rangle - \langle\nabla f(x),\gamma'\rangle\right)\varphi' \\ &\quad + \varphi'\sum_{i=1}^{n-1}\nabla^2_{E_iE_i}r(x,y) - 2\varphi_t + 2\varphi'' - \epsilon e^t.\end{aligned}$$

Here we have used the first variation $\nabla\mathscr{C}(\cdot,\cdot,t_0) = 0$ at (x_0,y_0) which implies the identities

$$(\nabla v)(y,t_0) = \varphi'\gamma'(d) \quad (\nabla v)(x,t_0) = \varphi'\gamma'(0).$$

Now choose the variational vector field $V_i(s) = e_i(s)$, the parallel transport of e_i along $\gamma(s)$, the second variation computation gives that

$$\sum_{i=1}^{n-1}\nabla^2_{E_iE_i}r(x,y) \le -\int_0^d \mathrm{Rc}(\gamma',\gamma')ds. \tag{1.3}$$

Hence at (x_0,y_0,t_0) we have that

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \sum_{j=1}^{n}\nabla^2_{E_jE_j}\right)\mathscr{C}(x,y,t) &\le -\varphi'\int_0^d(\nabla^2 f + \mathrm{Rc})(\gamma',\gamma')ds - 2\varphi_t + 2\varphi'' - \epsilon e^t \\ &\le -\varphi' a r(x,y) - 2\varphi_t + 2\varphi'' - \epsilon e^t \\ &< 0.\end{aligned}$$

Here we have used $d = r(x, y)$ and $s = \frac{r(x,y)}{2}$. This contradicts with that at (x_0, y_0, t_0) since it is the first time $\mathscr{C}(x, y, t) = 0$,

$$\left.\frac{\partial \mathscr{C}}{\partial t}\right|_{(x_0,y_0,t_0)} \geq 0,$$

$$\left.\nabla^2_{E_iE_i}\mathscr{C}\right|_{(x_0,y_0,t_0)} \leq 0.$$

The above argument works well as long as (x_0, y_0) is not conjugate to each other along the minimizing geodesic $\gamma(s)$ since we need this condition in establishing (1.3). For the case (x_0, y_0) is a pair of points conjugate to each other, we can evoke a trick similar to the one of Calabi to work through the argument. Let $e_i(s)$, with $1 \leq i \leq n-1$ and $0 \leq s \leq d$, be the parallel transport of e_i along a minimizing geodesic $\gamma(s)$ joining $x_0 = \gamma(0)$, $y_0 = \gamma(d)$. Let

$$\tilde{\gamma}(\eta_1, \eta_2, \ldots, \eta_{n-1}, s) = exp_{\gamma(s)}(\eta_1 e_1(s) + \cdots \eta_{n-1}e_{n-1}(s)).$$

Let $L(\eta_1, \ldots, \eta_{n-1})$ be the arc length of $\tilde{\gamma}(\eta_1, \ldots, \eta_i, \cdot)$, which is greater or equal to the distance between $\gamma(\eta_1, \ldots, \eta_{n-1}, 0)$ and $\gamma(\eta_1, \ldots, \eta_{n-1}, d)$. Let

$$\begin{aligned}\widetilde{\mathscr{C}}(\eta_1, \ldots, \eta_{n-1}) \doteqdot\ & v(\gamma(\eta_1, \ldots, \eta_{n-1}, d), t_0) - v(\gamma(\eta_1, \ldots, \eta_{n-1}, 0), t_0)\\ & - 2\varphi\left(\frac{L(\eta_1, \ldots, \eta_{n-1})}{2}, t\right) - \epsilon e^{t_0}.\end{aligned}$$

Then in view of the monotonicity of $\varphi(s, t)$ in s,

$$\begin{aligned}\widetilde{\mathscr{C}}(0, \ldots, 0) &= \mathscr{C}(x_0, y_0, t_0)\\ &\geq \mathscr{C}(\gamma(\eta_1, \ldots, \eta_{n-1}, 0), \gamma(\eta_1, \ldots, \eta_{n-1}, d), t_0)\\ &\geq \widetilde{\mathscr{C}}(\eta_1, \ldots, \eta_{n-1}).\end{aligned}$$

In the previous computation if we replace $\mathscr{C}$ by $\widetilde{\mathscr{C}}$, we can still apply the maximum principle to get a contradiction verbatim. $\square$

To prove Theorem 1.1, let $\bar{\omega}(s)$ be the first non-constant eigenfunction for $\mathscr{L}_a$, which can be chosen to be positive on $(0, \frac{D}{2})$. To apply Theorem 1.2 as in [9] we consider $\bar{\omega}^{D'}(s)$, the eigenfunction on $[-\frac{D'}{2}, \frac{D'}{2}]$ with the corresponding eigenvalue $\bar{\lambda}_{a,D'}$. Let $\varphi(x, t) = Ce^{-\bar{\lambda}_{a,D'}t}\bar{\omega}^{D'}(s)$. Let $w(x)$ be the first non-constant eigenfunction of Δ_f and let $v(x, t) = e^{-\widetilde{\lambda}_1 t}w(x)$. Since $\bar{\omega}^{D'}(s)$ is an odd function (by adding an eigenfunction $\psi(s)$ with $\psi(-s)$ one can always obtain one), we do have $\varphi(0, t) = 0$. The possibility of choosing $(\bar{\omega}^{D'})'(s) > 0$ on $[0, \frac{D}{2}]$ can be proved as follows. By the uniqueness, we have that $(\bar{\omega}^{D'})'(0) > 0$. It suffices to show that $(\bar{\omega}^{D'})'(s) > 0$ for $s \in (0, \frac{D'}{2})$. For this, observe that $e^{-\frac{a}{2}s^2}(\bar{\omega}^{D'})'(s)$ is non-increasing in s and vanishes at $s = \frac{D'}{2}$.

Finally Theorem 1.2 implies that for sufficient large C, $\varphi(s, t)$ is a modulus of continuity of $v(x, t)$ for all $t > 0$. Hence $\widetilde{\lambda}_1 \geq \bar{\lambda}_{a,D'}$. The claimed result of Theorem 1.1 follows by letting $D' \to D$. $\square$

2. Sharpness of the Lower Bound

In this section we show that (for $n \geq 3$ for any a or for $n \geq 2$ for $a \leq 0$) the lower bound $\tilde{\lambda}_1 \geq \bar{\lambda}_a$ given in Theorem 1.1 is sharp: precisely, for each $\varepsilon > 0$ we construct a Bakry-Emery manifold (M, g, f) with diameter D and $\tilde{\lambda}_1 < \bar{\lambda}_{a,D} + \varepsilon$.

We will construct a smooth manifold M which is approximately a thin cylinder with hemispherical caps at each end. Let γ be the curve in $\mathbb{R}^2$ with curvature k given as function of arc length as follows for suitably small positive r and $\delta > 0$ small compared to r:

$$k(s) = \begin{cases} \dfrac{1}{r}, & s \in \left[0, \dfrac{\pi r}{2} - \delta\right]; \\ \varphi\left(\dfrac{s - \frac{\pi r}{2}}{\delta}\right)\dfrac{1}{r}, & s \in \left[\dfrac{\pi r}{2} - \delta, \dfrac{\pi r}{2} + \delta\right]; \\ 0, & s \in \left[\dfrac{\pi r}{2} + \delta, \dfrac{D}{2}\right], \end{cases} \tag{2.1}$$

extended to be even under reflection in both $s = 0$ and $s = D/2$. This corresponds to a pair of line segments parallel to the x axis, capped by semicircles of radius r and smoothed at the joins. We write the corresponding embedding $(x(s), y(s))$. Here φ is a smooth nonincreasing function with $\varphi(s) = 1$ for $s \leq -1$, $\varphi(s) = 0$ for $s \geq 1$, and satisfying $\varphi(s) + \varphi(-s) = 1$. We choose the point corresponding to $s = 0$ to have $y(0) = 0$ and $y'(0) = 1$. The manifold M will then be the hypersurface of rotation in $\mathbb{R}^{n+1}$ given by $\{(x(s), y(s)z) : s \in \mathbb{R}, z \in \mathbb{S}^{n-1}\}$. On M we choose the function f to be a function of s only, such that

$$f'' = \begin{cases} a\left(1 - \dfrac{D}{\pi r}\right), & s \in \left[0, \dfrac{\pi r}{2} - \delta\right]; \\ \varphi\left(\dfrac{s - \frac{\pi r}{2}}{\delta}\right) a\left(1 - \dfrac{D}{\pi r}\right) + \left(1 - \varphi\left(\dfrac{s - \frac{\pi r}{2}}{\delta}\right)\right) a, & s \in \left[\dfrac{\pi r}{2} - \delta, \dfrac{\pi r}{2} + \delta\right]; \\ a, & s \in \left[\dfrac{\pi r}{2} + \delta, \dfrac{D}{2}\right], \end{cases} \tag{2.2}$$

with $f'(0) = 0$ (the value of $f(0)$ is immaterial). Note that this choice gives $f'(D/2) = 0$. We extend f to be even under reflection in $s = 0$ and $s = D/2$.

With these choices we compute the Bakry-Emery-Ricci tensor and verify that the eigenvalues are no less than a for suitable choice of r. The eigenvalues of the second fundamental form are $k(s)$ (in the s direction) and $\frac{\sqrt{1-(y')^2}}{y}$ in the orthogonal directions. Therefore the Ricci tensor has eigenvalues $(n-1)k\frac{\sqrt{1-(y')^2}}{y}$ in the s direction, and $k\frac{\sqrt{1-(y')^2}}{y} + (n-2)\frac{1-(y')^2}{y^2}$ in the orthogonal directions. We can also compute the eigenvalues of the Hessian of f: The curves of fixed z in M are geodesics parametrized by s, the Hessian in this direction is just f'' as given above. Since f depends only on s we also have that $\nabla^2 f(\partial_s, e_i) = 0$ for e_i tangent to $\mathbb{S}^{n-1}$, and $\nabla^2 f(e_i, e_j) = \frac{y'}{y} f' \delta_{ij}$.

The identities $y(s) = \int_0^s \cos(\theta(\tau))\, d\tau$ and $y'(s) = \cos(\theta(s))$ where $\theta(s) = \int_0^s k(\tau)\, d\tau$ applied to (2.1) imply that

$$y = \begin{cases} r\sin(s/r), & s \in \left[0, \frac{\pi r}{2} - \delta\right]; \\ r(1 + o(\delta)), & s \in \left[\frac{\pi r}{2} - \delta, \frac{D}{2}\right], \end{cases} \qquad y' = \begin{cases} \cos(s/r), & s \in \left[0, \frac{\pi r}{2} - \delta\right]; \\ o(\delta), & s \in \left[\frac{\pi r}{2} - \delta, \frac{\pi r}{2} + \delta\right]; \\ 0, & s \in \left[\frac{\pi r}{2} + \delta, \frac{D}{2}\right], \end{cases}$$

as δ approaches zero. This gives the following expressions for the Bakry-Emery Ricci tensor $\mathrm{Rc}_f = \mathrm{Rc} + \nabla^2 f$:

$$\mathrm{Rc}_f(\partial_s, \partial_s) = \begin{cases} a + \frac{n-1}{r^2} - \frac{aD}{\pi r}, & s \in \left[0, \frac{\pi r}{2} - \delta\right]; \\ a + \varphi\left(\frac{2 - \frac{\pi r}{2}}{\delta}\right)\left(\frac{n-1}{r^2}(1 + o(\delta)) - \frac{aD}{\pi r}\right) & s \in \left[\frac{\pi r}{2} - \delta, \frac{\pi r}{2} + \delta\right]; \\ a, & s \in \left[\frac{\pi r}{2} + \delta, \frac{D}{2}\right], \end{cases}$$

and

$$\mathrm{Rc}_f(e, e) = \begin{cases} \frac{n-1}{r^2} + \frac{as\left(1 - \frac{D}{\pi r}\right)}{r\tan(s/r)}, & s \in \left[0, \frac{\pi r}{2} - \delta\right]; \\ \frac{n-2}{r^2} + o(\delta), & s \in \left[\frac{\pi r}{2} - \delta, \frac{\pi r}{2} + \delta\right]; \\ \frac{n-2}{r^2}(1 + o(\delta)) & s \in \left[\frac{\pi r}{2} + \delta, \frac{D}{2}\right], \end{cases}$$

while $\mathrm{Rc}_f(\partial_s, e) = 0$, for any unit vector e tangent to $\mathbb{S}^{n-1}$. In particular we have $\mathrm{Rc}_f \geq ag$ for sufficiently small r and δ for any $a \in \mathbb{R}$ if $n \geq 3$, and for $a < 0$ if $n = 2$. Note also that the diameter of the manifold M is $D(1 + o(\delta))$.

Having constructed the manifold M, we now prove that for this example the first non-trivial eigenvalue $\tilde{\lambda}_1$ of Δ_f can be made as close as desired to $\bar{\lambda}_{a,D}$ by choosing r and δ small. Theorem 1.1 gives the upper bound $\tilde{\lambda}_1 \geq \bar{\lambda}_{a,D(1+o(\delta))} = \bar{\lambda}_{a,D} + o(\delta)$. To prove an upper bound we can simply find a suitable test function to substitute into the Rayleigh quotient which defines $\tilde{\lambda}_1$. We set

$$\psi(s, z) = \begin{cases} w(s - D/2), & \frac{\pi r}{2} + \delta \leq s \leq D - \left(\frac{\pi r}{2} + \delta\right); \\ w\left(\frac{D}{2} - \frac{\pi r}{2} - \delta\right), & 0 \leq s \leq \frac{\pi r}{2} + \delta, \text{ and } D - \frac{\pi r}{2} - \delta \leq s \leq D, \end{cases}$$

where w is the solution of $w'' - asw' + \bar{\lambda}_{a,D-\pi r-2\delta} w = 0$ with $w(0) = 0$ and $w'\left(\frac{D}{2} - \frac{\pi r}{2} - \delta\right) = 0$ and $w'(0) = 1$. This choice gives

$$\mathscr{R}(\psi) = \frac{\bar{\lambda}_{a,D-\pi r-2\delta} \int_{\{|s-D/2| \le \frac{D}{2} - \frac{\pi}{2r} - \frac{\delta}{2}\}} \psi^2 \mathrm{e}^{-f} dVol(g)}{\int_{\{|s-D/2| \le D/2\}} \psi^2 \mathrm{e}^{-f} dVol(g)} \le \bar{\lambda}_{a,D-\pi r-2\delta}.$$

It follows that $\tilde{\lambda}_1 \to \bar{\lambda}_{a,D}$ as r and δ approach zero, proving the sharpness of the lower bound in Theorem 1.1.

Remark 2.1. If we allow manifolds with boundary the construction is rather simpler: simply take a cylinder $r\mathbb{S}^{n-2} \times [-D/2, D/2]$ for small r, with quadratic potential $f = \frac{a}{2}s^2$, and substitute the test function $\psi(z, s) = w(s)$ defined above.

3. A Linear Lower Bound

Here we use the notation from Section 1. Concerning the lower estimate of $\bar{\lambda}_{a,D}$, at least for $a \ge 0$, if we apply Theorem 1.2 to the trivial case $M = \mathbb{R}$ with $\varphi(x, t) = \mathrm{e}^{-(\frac{\pi}{D'})^2 t} \sin(\frac{\pi}{D'} s)$, and letting $D' \to D$ we get $\bar{\lambda}_{a,D} \ge \frac{\pi^2}{D^2}$.

For a more precise estimate, observe that $y = (\bar{\omega}^{D'})'$ satisfies

$$\left(\mathrm{e}^{-\frac{a}{2}s^2} y'\right)' = (a - \bar{\lambda}_{a,D}) \mathrm{e}^{-\frac{a}{2}s^2} y. \tag{3.1}$$

This immediately implies that $\bar{\lambda}_{a,D} \ge a + \mu_{a,D}$, where $\mu_{a,D}$ is the Dirichlet eigenvalue of the Hermite operator $\frac{d^2}{ds^2} - 2a\, s \frac{d}{ds}$. On the other hand, if y is the first Dirichlet eigenfunction on $[-D/2, D/2]$, which is an even function, let $\omega = \int_0^s y$. Direct calculation also shows that ω is a Neumann eigenfunction, hence we have the upper bound $\bar{\lambda}_{a,D} \le a + \mu_{a,D}$. This establishes the identity:

$$\bar{\lambda}_{a,D} = a + \mu_{a,D}. \tag{3.2}$$

The following result and its consequence improve the main results of [6]. It applies to a compact Riemannian manifold (M, g) satisfying $\mathrm{Rc} \ge (n-1)K$ for some $K > 0$, and concludes that $\lambda_1(M) \ge \frac{n-1}{2}K + \frac{\pi^2}{D^2}$ holds with D being the diameter of the manifold M. This improves earlier works of [8, 11], etc.

Proposition 3.1. *When $a > 0$, the first nonzero Neumann eigenvalue $\bar{\lambda}_{a,D}$ is bounded from below by $\frac{a}{2} + \frac{\pi^2}{D^2}$. Moreover, the $\frac{a}{2}$ in the lower bound is the largest possible. In particular $\widetilde{\lambda}_1(\Omega)$, with Ω being a convex domain in any Riemannian manifold with $\mathrm{Rc}_{ij} + f_{ij} \ge a g_{ij}$, is bounded from below by $\frac{a}{2} + \frac{\pi^2}{D^2}$.*

Proof. We present two proofs for the lower bound via two types of normalization. The first is to reduce the problem to finding the first nontrivial Neumann eigenvalue $\bar{\lambda}_{2,\sqrt{\frac{a}{2}}D}$ for the Hermite equation: $\frac{d^2}{ds^2} - 2s\frac{d}{ds}$ on the interval $\left[-\sqrt{\frac{a}{2}}\frac{D}{2}, \sqrt{\frac{a}{2}}\frac{D}{2}\right]$ since $\bar{\lambda}_{a,D} = \frac{a}{2}\bar{\lambda}_{2,\sqrt{\frac{a}{2}}D}$. Namely the first one normalizes the a.

For the operator $\frac{d^2}{ds^2} - 2s\frac{d}{ds}$ on interval $[-\frac{\overline{D}}{2}, \frac{\overline{D}}{2}]$ the equation (3.2) implies that $\bar{\lambda}_{2,\overline{D}} = 2 + \mu_{2,\overline{D}}$, with μ being the first Dirichlet eigenvalue. Now we may introduce the transformation $w = e^{-\frac{s^2}{2}}\varphi$. Direct calculation shows that φ is the first Dirichlet eigenfunction of $\frac{d^2}{ds^2} - 2s\frac{d}{ds}$ if and only if w is the eigenfunction of the harmonic oscillator:

$$\frac{d^2}{ds^2}w - s^2 w = -(\mu + 1)w$$

with w vanishing on the boundary. By Corollary 6.4 of [9] we have that

$$\mu + 1 \geq \frac{\pi^2}{\overline{D}^2}.$$

Combining them together we have that $\bar{\lambda}_{2,\overline{D}} \geq 1 + \frac{\pi^2}{\overline{D}^2}$. Letting $\overline{D} = \sqrt{\frac{a}{2}}D$ we get the claimed lower bound for $\bar{\lambda}_{a,D}$.

The second one is via the normalization on D. Precisely by the change of variable, $\frac{D^2}{4}\mu_{a,D} = \mu_{\frac{D^2}{4}a,2}$. Now let $\bar{a} = \frac{D^2}{4}a$ and consider the operator $\frac{d^2}{ds^2} - \bar{a}s\frac{d}{ds}$. The transformation $\varphi \to e^{-\frac{\bar{a}}{4}s^2}\varphi$ relates $\mu_{\bar{a},2}$ with the first eigenvalue $\mu(b)$ of the harmonic oscillator $\frac{d^2}{ds^2} - bs^2$ with $b = \frac{\bar{a}^2}{4}$ via the equation:

$$\mu(b) = \mu_{\bar{a},2} + \frac{\bar{a}}{2}. \tag{3.3}$$

For the harmonic oscillator $\frac{d^2}{ds^2} - bs^2$ on $(-1, 1)$,

$$\mu(b) = \inf_{\int \psi^2 = 1, \psi(\pm 1) = 0} \int_{-1}^{1} (\psi')^2 + bs^2\psi^2 ds.$$

Note that the functional is increasing and concave in b. Hence we have that

$$\frac{\pi^2}{4} = \mu(0) \leq \mu(b) \leq \mu(0) + \mu'(0)b = \frac{\pi^2}{4} + \left(\frac{1}{3} - \frac{2}{\pi^2}\right)b.$$

Combining the above with (3.3) and the equation $\mu_{a,D} = \frac{4}{D^2}\mu_{\bar{a},2}$, we have that $\bar{\lambda}_{a,D} = \mu_{a,D} + a$ satisfies

$$\frac{a}{2} + \frac{\pi^2}{D^2} \leq \bar{\lambda}_{a,D} \leq \frac{a}{2} + \frac{\pi^2}{D^2} + \left(\frac{\pi^2}{12} - \frac{1}{2}\right)\frac{a^2D^2}{\pi^2}. \qquad \square$$

Corollary 3.1. *If* (M, g, f) *is a nontrivial* (*namely with nonconstant* f) *gradient shrinking soliton satisfying* (1.1) *with* $a > 0$. *Then*

$$\text{Diameter}(M, g) \geq \sqrt{\frac{2}{3a}}\pi.$$

Proof. The result follows from the above lower estimate on the first Neumann eigenvalue, applying to the case that $\Omega = M$, and the observation, Lemma 2.1 of [6], that $2a$ is an eigenvalue of the operator $\Delta - \langle \nabla f, \nabla(\cdot)\rangle$. $\square$

This result clearly is not sharp. See [4] for a related upper bound on the diameter. A better eigenvalue lower bound (and hence a better diameter lower bound) will follow from a better understanding of the first Dirichlet eigenvalue of the harmonic oscillator. We investigate this in the next section.

4. The Harmonic Oscillator on Bounded Symmetric Intervals

In this section we will investigate the sharp lower bound given by the eigenvalue of the one-dimensional harmonic oscillator on a bounded symmetric interval: recall from section 3 that the first Neumann eigenvalue $\bar{\lambda}_{2,D}$ is equal to $\hat{\lambda}_{1,D} + 1$, where $\hat{\lambda}_{b,D}$ is defined by the existence of a solution of the eigenvalue problem

$$\begin{aligned} w'' + \left(\hat{\lambda}_{b,D} - bs^2\right) w = 0, &\quad s \in [-D/2, D/2]; \\ w(D/2) = w(-D/2) = 0; & \\ w(x) > 0, &\quad s \in (-D/2, D/2). \end{aligned}$$

The solution of the ordinary differential equation $w'' - s^2 w + \lambda w = 0$ (with $w'(0) = 0$), which is also called Weber's equation, can be written in terms of confluent hypergeometric functions: We have

$$w(s) = \mathrm{e}^{-s^2} U\left(\frac{1}{4} - \frac{\lambda}{8}, \frac{1}{2}, 2s^2\right)$$

where U is the confluent hypergeometric function of the first kind. Thus $\hat{\lambda}_{1,D}$ is the first root of the equation $U\left(\frac{1}{4} - \frac{\lambda}{8}, \frac{1}{2}, \frac{D^2}{2}\right) = 0$. Since U is strictly monotone in the first argument, the solution is an analytic function of D.

Noting that $\hat{\lambda}_{1,D} = \frac{\pi^2}{D^2}\hat{\lambda}_{\frac{D^4}{\pi^4},\pi}$, we use a perturbation argument to compute the Taylor expansion for $\hat{\lambda}_{b,\pi}$ as a function of $b = \frac{a^2}{4}$ about $b = 0$ (this provides an expansion for $\hat{\lambda}_{1,D}$ about $D = 0$). That is, we consider the solution u of the eigenvalue problem

$$\begin{aligned} u'' + \left(\lambda - bs^2\right) u = 0, &\quad s \in [-\pi/2, \pi/2]; \\ u(\pi/2) = u(-\pi/2) = 0; & \\ u(s) > 0, &\quad s \in (-\pi/2, \pi/2). \end{aligned}$$

The solution for $b = 0$ is of course given by $u(s) = \cos(s)$. The perturbation expansion produces a solution of the form

$$u(s, b) = \sum_{k=0}^{\infty} b^k \sum_{j=0}^{3k} \left(\alpha_{k,j} s^j \cos s + \beta_{k,j} s^k \sin s\right),$$

with $\lambda = \sum_{k=0}^{\infty} \lambda_k b^k$. This expansion is unique provided we specify that u is even, $\alpha_{0,0} = 1$, $\beta_{0,1} = 0$, and $\alpha_{k,0} = \beta_{k,0} = 0$ for $k > 0$. The first few terms in the expansion for λ are given by

$$\begin{aligned} \hat{\lambda}_{b,\pi} = 1 + \left(\frac{\pi^2}{12} - \frac{1}{2}\right) b + \left(\frac{\pi^4}{720} - \frac{5\pi^2}{48} + \frac{7}{8}\right) b^2 + \left(\frac{\pi^6}{30240} - \frac{\pi^4}{48} + \frac{31\pi^2}{32} - \frac{121}{16}\right) b^3 \\ + \left(\frac{\pi^8}{362880} - \frac{\pi^6}{270} + \frac{683\pi^4}{1280} - \frac{14573\pi^2}{768} + \frac{17771}{128}\right) b^4 + O(b^5). \end{aligned}$$

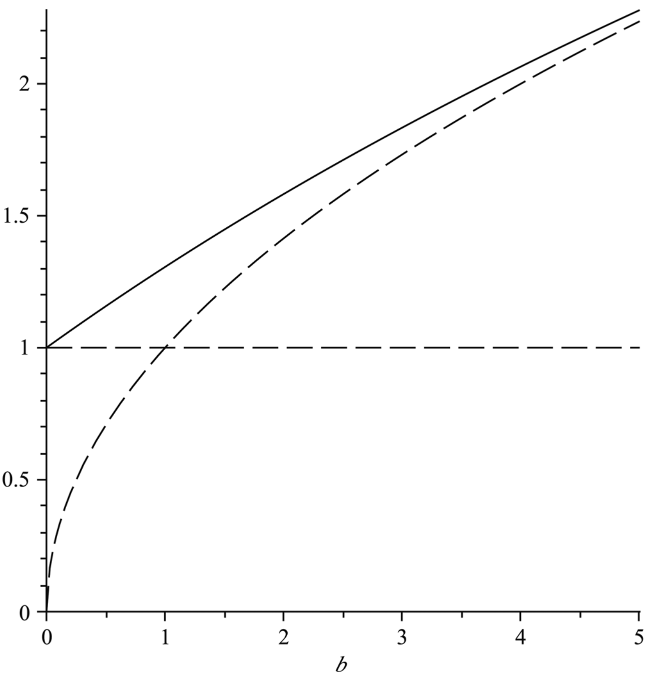

Figure 1. The eigenvalue $\hat{\lambda}_{b,\pi}$ for Weber's equation $y'' + (\lambda - bs^2)y = 0$, $y(\pm\pi/2) = 0$ (solid curve); shown also are the lower bounds $\hat{\lambda} \geq 1$ and $\hat{\lambda} \geq \sqrt{b}$ (dashed curves).

We note that there is also a useful lower bound for $\hat{\lambda}_{b,\pi}$, which we can arrive at as follows: The inclusion of $[-D/2, D/2]$ in $\mathbb{R}$ implies $\hat{\lambda}_{1,D} \geq \lim_{d\to\infty} \hat{\lambda}_{1,d} = 1$, with eigenfunction $u(s) = e^{-s^2/2}$. Therefore we also have

$$\hat{\lambda}_{b,\pi} = \sqrt{b}\hat{\lambda}_{1,\pi b^{1/4}} \geq \sqrt{b}.$$

Figure 1 gives an illustration of $\hat{\lambda}_{b,\pi}$ in terms of b, with the lower bounds indicated. This translates to an estimate for the drift eigenvalue $\bar{\lambda}_{a,\pi}$ appearing in Theorem 1.1: We have $\bar{\lambda}_{a,\pi} = \frac{a}{2} + \hat{\lambda}_{a^2/4,\pi}$, giving the following Taylor expansion:

$$\begin{aligned}\bar{\lambda}_{a,\pi} = 1 + \frac{a}{2} + \left(\frac{\pi^2}{12} - \frac{1}{2}\right)\frac{a^2}{4} + \left(\frac{\pi^4}{720} - \frac{5\pi^2}{48} + \frac{7}{8}\right)\frac{a^4}{16} \\ + \left(\frac{\pi^6}{30240} - \frac{\pi^4}{48} + \frac{31\pi^2}{32} - \frac{121}{16}\right)\frac{a^6}{64} \\ + \left(\frac{\pi^8}{362880} - \frac{\pi^6}{270} + \frac{683\pi^4}{1280} - \frac{14573\pi^2}{768} + \frac{17771}{128}\right)\frac{a^8}{256} + O(a^{10}).\end{aligned}$$

In particular the lower bound $\hat{\lambda} \geq 1$ translates to $\bar{\lambda} \geq 1 + a/2$, and the lower bound $\hat{\lambda} \geq \sqrt{b}$ translates to $\bar{\lambda} \geq a/2 + \sqrt{a^2/4} = a$. Finally, by scaling we obtain the following:

$$\bar{\lambda}_{a,D} = \frac{\pi^2}{D^2} + \frac{a}{2} + \left(\frac{\pi^2}{12} - \frac{1}{2}\right)\frac{D^2a^2}{4\pi^2} + \left(\frac{\pi^4}{720} - \frac{5\pi^2}{48} + \frac{7}{8}\right)\frac{D^6a^4}{16\pi^6}$$

$$+\left(\frac{\pi^6}{30240}-\frac{\pi^4}{48}+\frac{31\pi^2}{32}-\frac{121}{16}\right)\frac{D^{10}a^6}{64\pi^{1}0}$$
$$+\left(\frac{\pi^8}{362880}-\frac{\pi^6}{270}+\frac{683\pi^4}{1280}-\frac{14573\pi^2}{768}+\frac{17771}{128}\right)\frac{D^{14}a^8}{256\pi^{14}}$$
$$+\,O(D^{18}a^{10}) \quad \text{as } D^2a\to 0.$$

An interesting consequence of the Taylor expansion (combined with the fact that the estimate $\tilde{\lambda}_1 \geq \bar{\lambda}_{a,D}$ is sharp as proved in section 2) is the following:

Proposition 4.1. *The constant $a/2$ in the lower bound $\tilde{\lambda}_1 \geq \frac{\pi^2}{D^2}+\frac{a}{2}$ is the largest possible.*

This follows from the Taylor expansion for small values of aD^2.

We note that the sharp diameter bound (given by the value of a where the dotted line $\lambda = 2a$ intersects with the solid curve in Figure 2) is not dramatically different from the one given in Corollary 3.1 (where the dotted line intersects the dashed line $\lambda = 1 + a/2$). Since the eigenvalue estimate $\tilde{\lambda}_1 \geq \bar{\lambda}_{a,D}$ appears from the examples in Section 2 to be sharp only in situations which are far from gradient solitons, we expect that neither of these diameter bounds is close to the sharp lower diameter bound for a nontrivial gradient Ricci soliton.

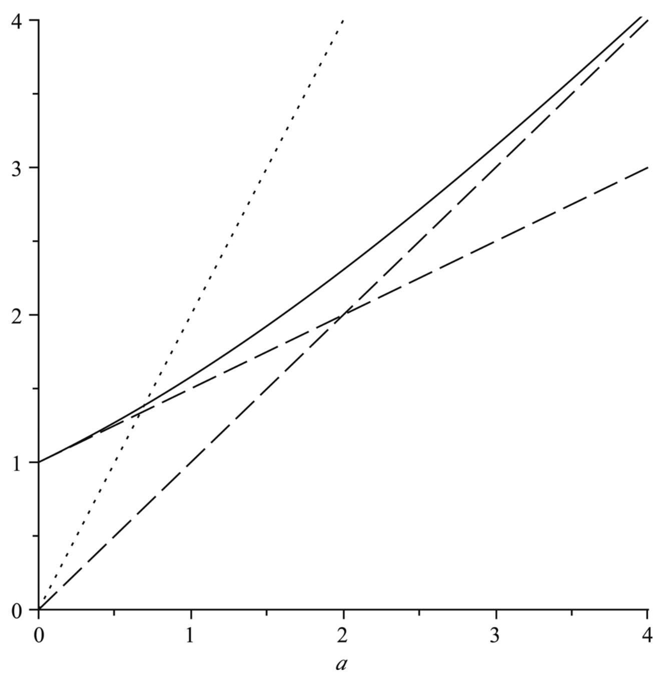

Figure 2. The eigenvalue $\bar{\lambda}_{a,\pi}$ for the drift Laplacian equation $y'' - asy' + \lambda y = 0$, $y'(\pm\pi/2) = 0$ (solid curve); shown also are the lower bounds $\bar{\lambda} \geq 1 + \frac{a}{2}$ and $\bar{\lambda} \geq a$ (dashed lines), and the line $\bar{\lambda} = 2a$ corresponding to non-Einstein gradient Ricci solitons (dotted line).

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