Liouville Theorems and a Schwarz Lemma for Holomorphic Mappings Between Kähler Manifolds

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Abstract

We derive some consequences of the Liouville theorem for plurisubharmonic functions of L.-F. Tam and the author. The first result provides a nonlinear version of the complex splitting theorem (which splits off a factor of $\mathbb{C}$ isometrically from the simply connected Kähler manifold with nonnegative bisectional curvature and a linear growth holomorphic function) of L.-F. Tam and the author. The second set of results concerns the so-called k-hyperbolicity and its connection with the negativity of the k-scalar curvature (when $k = 1$ they are the negativity of holomorphic sectional curvature and Kobayashi hyperbolicity) introduced recently in [33] by F. Zheng and the author. We lastly prove a new Schwarz-lemma-type estimate in terms of *only the holomorphic sectional curvatures of both domain and target manifolds*. © 2020 Wiley Periodicals LLC.

1 Introduction

The first goal of this paper is to derive some consequences of the Liouville theorem proved in [28], which asserts that *any continuous plurisubharmonic function $u(x)$ defined on a Kähler manifold with nonnegative bisectional curvature satisfying that $u(x) = o(\log(r(x)))$, where $r(x)$ denotes the distance function to a fixed point p, must be a constant.* This result was recently generalized in [18, 27] with weaker assumptions on the curvature.

THEOREM 1.1. *Let M^m and N^n be two complete Kähler manifolds. Assume that the bisectional curvature of M is nonnegative and the bisectional curvature of N is nonpositive. (i) Then any holomorphic map $f : M \to N$ satisfying*

$$\limsup_{r(x)\to\infty} \frac{\|df\|(x)}{r^{\epsilon}(x)} = 0 \tag{1.1}$$

for any $\epsilon > 0$, where $r(x)$ is the distance of x to a fixed point $p \in M$, is totally geodesic. (ii) If $m = 1$, then the same result holds under a weaker assumption that N has nonpositive holomorphic sectional curvature.

In particular, if M is irreducible, then f is either a constant or a holomorphic isometry.

Communications on Pure and Applied Mathematics, 0001–0026 (PREPRINT)
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The result above can be viewed as a nonlinear version of the (holomorphic) splitting theorem proved in [28], which asserts that *on a simply connected M with nonnegative bisectional curvature, any nonconstant linear growth holomorphic function splits off a* $\mathbb{C}$, since a holomorphic function can be viewed as the holomorphic map into $\mathbb{C}$, and being totally geodesic implies that ∇f is parallel, hence the splitting. The linear growth of a holomorphic function implies the boundedness of its gradient by [4].

A result with the same conclusion was proved earlier for harmonic maps from compact quotients of symmetric irreducible spaces into manifolds with nonpositive complex sectional curvature, namely the celebrated geometric super-rigidity in [19] (see also [13]). But the result here is different in nature from the result of Mok-Siu-Yeung in the sense that while allowing the domain manifold to be noncompact, we imposed a curvature condition on the domain instead. The result of [19] concerns harmonic maps, which is a considerably larger class than holomorphic maps. On the other hand, [19] is restricted for domain manifolds being compact quotients of symmetric spaces, and at the same time poses a stronger curvature condition on the target than the result above.

By Cheng's gradient estimate for harmonic maps into a Cartan–Hadamard manifold [3] (see also page 339 of [23] for an alternate proof without using the gradient estimate) we have the following corollary.

COROLLARY 1.2. *Let M^m and N^n be two complete Kähler manifolds. Assume that the bisectional curvature of M is nonnegative, and N is a Cartan-Hadamard Kähler manifold. Then any holomorphic map $f : M \to N$, whose differential df satisfies*

$$\limsup_{r(x)\to\infty} \frac{d^N(p', f(x))}{r^{1+\epsilon}(x)} = 0$$

for any $\epsilon > 0$, where $r(x)$ is the distance of x to a fixed point $p \in M$, and $d^N(p', \cdot)$ is the distance function of N to a point $p' \in N$, is totally geodesic.

The proof utilizes a $\partial\bar{\partial}$-Bochner formula for holomorphic maps, which implies the plurisubharmonicity of $\log(A + \|\partial f\|^2)$ (for any $A > 0$). The results above also hold for pluriharmonic maps. It was proved in [28] that a harmonic function of sub-quadratic growth is pluriharmonic under the assumption that the quadratic orthogonal bisectional curvature of M is nonnegative. We have not been able to prove a nonlinear analogue of this result for harmonic maps yet.

Various concepts of hyperbolicity arise in conjunction with the Schwarz lemma [15]. Applying a $\partial\bar{\partial}$-lemma (which is collected in Section 2) on the logarithmic of k-dimensional volume we derive, in Section 3, results related to the *k-hyperbolicity of a Kähler manifold* in conjunction with the so-called k^{th} scalar curvature. Below we shall recall and define these concepts after proper motivations.

Recall that in [33], (N^n, h) is defined to have negative (positive) k-scalar curvature if

$$S_k(y,\Sigma) \doteqdot \frac{k(k+1)}{2\,\mathrm{Vol}(\mathbb{S}^{2k-1})}\int_{|Z|=1, Z\in\Sigma} H(Z)d\theta(Z) < 0 \quad (>0)$$

for any $y \in N$ and any k-dimensional subspaces $\Sigma \subset T'_y N$. Here H denotes the holomorphic sectional curvature of N, namely $H(Z) = R^N(Z, \bar{Z}, Z, \bar{Z})$. We say $S_k(y) < 0$ if $S_k(y, \Sigma) < 0$ for every k-dimensional Σ. N is called with negative k-scalar curvature if $S_k(y) < 0$ everywhere. Regarding compact Kähler manifolds with $S_k < 0$, in view of the recent result of [33], an interesting question is "*when is a compact Kähler manifold with $S_2 < 0$ projective*?"

The celebrated Brody criterion [14, 15] asserts that N^n is Kobayashi hyperbolic if and only if any holomorphic map from complex plane $\mathbb{C}$ into N^n must be a constant map. Motivated by this criterion of the Kobayashi hyperbolicity (which amounts to 1-hyperbolicity as illustrated below) and the work of [6], [15] (see also [41] for the extension to the meromorphic mappings and another definition of an intrinsic k-measure) we *define a compact Kähler manifold N^n to be k-hyperbolic if and only if any holomorphic map $f : \mathbb{C}^k \to N$ must be degenerate (namely the image of f must be of dimension less than k)*. This provides a natural generalization of Kobayashi hyperbolicity (namely 1-hyperbolicity), and is equivalent to that a pseudo norm on $\wedge^k T_x N$ (on k-dimensional parallelepiped) defined via holomorphic mappings from $\mathbb{D}^k \to N$ is indeed a norm (see the Appendix for details and a proof of this equivalence).

In the meantime, recall that the classical Schwarz lemma of Yau-Royden [35,42] for holomorphic maps from Riemann surfaces into compact Kähler manifolds with negative holomorphic sectional curvature implies, via the above Brody's criterion, that any compact Kähler manifold N^n with negative holomorphic sectional curvature must be 1-hyperbolic. In view of that S_k defined above coincides with the holomorphic sectional curvature H for $k = 1$; it is hence natural to ask the question (Q): *whether any holomorphic map from $\mathbb{C}^k$ into a compact (N^n, h) with $S_k < 0$ must be degenerate.* Namely, *whether any compact Kähler manifold N^n with $S_k < 0$ is k-hyperbolic* (in the sense defined above). The following result provides a strong indication of a positive answer to the question (Q), by answering it affirmatively when the map is from a compact quotient of $\mathbb{C}^k$ or (N^n, h) has $\mathrm{Ric}_k < 0$. We define $\mathrm{Ric}(x, \Sigma)$ as the Ricci curvature of the curvature tensor restricted to the k-dimensional subspace $\Sigma \subset T'_x M$. Precisely for any $v \in \Sigma$, $\mathrm{Ric}(x,\Sigma)(v, \bar{v}) \doteqdot \sum_{i=1}^k R(E_i, \overline{E}_i, v, \bar{v})$ with $\{E_i\}$ being a unitary basis of Σ. We say that $\mathrm{Ric}_k(x) < 0$ if $\mathrm{Ric}(x, \Sigma) < 0$ for every k-dimensional subspace Σ. Clearly $\mathrm{Ric}_k(x) < 0$ implies that $S_k(x) < 0$, and it coincides with H when $k = 1$, with Ric when $k = n$.

Since Ric_k coincides with H for $k = 1$, part (ii) of Theorem 1.3 below provides a generalization of the above-mentioned consequence of Royden-Yau.

THEOREM 1.3. *Assume that* $\dim_{\mathbb{C}} M = m \le n = \dim_{\mathbb{C}} N$.

(i) *Let* (M, g) *be a compact Kähler manifold such that* $\mathrm{Ric}^M \geq 0$. *Let* (N^n, h) *be a complete Kähler manifold such that* $S_m^N(y) < 0$. *Then any holomorphic map* $f : M \to N$ *must be degenerate. The same result holds if* $\mathrm{Ric}^M > 0$ *and* $S_m^N \le 0$.

(ii) *Let* (M^m, g) *be noncompact complete Kähler manifold with nonnegative scalar curvature and* Ric^M *is bounded from below. Assume that* (N^n, h) *has* $\mathrm{Ric}_m^N \le -\kappa < 0$ *(which holds if* $\mathrm{Ric}_m^N < 0$ *and* N *is compact). Then any holomorphic map* $f : M \to N$ *must be degenerate. In particular,* (N^n, h) *is* k*-hyperbolic if* $\mathrm{Ric}_k^N < 0$.

(iii) *Let* (M^m, g) *be a noncompact complete Kähler manifold with* $\mathrm{Ric}^M \ge 0$. *Assume that* (N^n, h) *is noncompact and has the m-Ricci curvature* $\mathrm{Ric}_m^N < 0$*, and* $f : M \to N$ *is a holomorphic map. If* $D(x) = \frac{(f^*\omega_h)^m(x)}{\omega_g^m(x)}$ *(with* ω_g *and* ω_h *being the Kähler forms of* M *and* N*) satisfies that*

$$\limsup_{x\to\infty} \frac{D(x)}{r^{\epsilon}(x)} = 0$$

for any $\epsilon > 0$*, then* f *must be degenerate.*

Note that the condition $\mathrm{Ric}_k < 0$ (or > 0) is independent of each other for different k, unlike $\{S_k < 0\}$, which becomes less restrictive as k increases. A natural question related to Ric_k is *whether* $\mathrm{Ric}_k < 0$ *(or* $\mathrm{Ric}_k > 0$*) implies the projectivity of the manifold*. For $k = 1$, the answer has been known to be positive for both $\mathrm{Ric}_k > 0$ and $\mathrm{Ric}_k < 0$ (cf. [38–40]). The result of [33] shows that for $k = 2$, $\mathrm{Ric}_k > 0$ does imply the projectivity. The projectivity and the rational connectedness of M with $\mathrm{Ric}_k(M) > 0$ for some $k \in \{1, \ldots, n\}$ have been proved in a recent preprint of the author [25]. The projectivity has also been proved recently for compact manifolds with $\mathrm{Ric}_k < 0$ by Chu-Lee-Tam [16]. The method of [17] also implies that *a Kähler manifold with* $\mathrm{Ric}_k \ge K > 0$ *must be compact.*

We should remark that even for the case of the equal dimension (namely $n = m$), the result of part (i) of above theorem seems new (at least the author is not aware of any such statement in the literature). Note that part (i) can be applied to m-dimensional tori. Hence if a map from $\mathbb{C}^m$ factors through a compact quotient of $\mathbb{C}^m$, the result here provides a positive answer to the question (Q). Part (ii) is known for the equal-dimensional case [14]. Part (ii) of the above theorem in particular implies that *a compact Kähler manifold* (N^n, h) *with* $\mathrm{Ric}_k^N < 0$ is k-hyperbolic. Moreover, a recent joint paper [16] illustrates the existence of a closed n-dimensional algebraic manifold with $\mathrm{Ric}_k < 0$ admitting a holomorphioc embedding of $\mathbb{P}^{k-1}$, via a construction of M^c^Kernan. This implies that the results of parts (i) and (ii) are sharp.

For part (iii) of Theorem 1.3, clearly the negativity of Ric_m^N is needed, since there are nondegenerate linear maps with bounded D between complex Euclidean spaces. In general, it is still unknown whether the Liouville theorem for the plurisubharmonic functions holds on a Kähler manifold M^m with nonnegative Ricci curvature. Note that in [18] the Liouville theorem for plurisubharmonic functions was proved for Kähler manifolds with nonnegative holomorphic sectional curvature, and in [27] for Kähler manifolds with nonnegative orthogonal bisectional curvature. On the other hand, under the nonnegativity of Ricci curvature there is a partial result proved in [22] that asserts that *for any plurisubharmonic function $u(x)$ with $o(\log(r(x)))$ growth,* $(\sqrt{-1}\partial\bar{\partial}u)^m \equiv 0$ holds. Part (iii) of the above theorem uses this statement.

In Section 4 we also prove some extension of the above result, which in particular implies that $f : \mathbb{C}^m \to N$ is asymptotically degenerate if $S^N_m < 0$, and it is degenerate if $\sigma_{m-1}(\partial f)$ is of order $o(r^2(x))$. Here $\sigma_{m-1}(\partial f)$ is the $(m-1)^{\text{th}}$ symmetric function of the singular values of $\partial f : T'_x M \to T'_{f(x)} N$.

There are many generalizations of the classical Schwarz lemma on holomorphic maps between unit balls via the work of Ahlfors, Chen-Cheng-Look, Lu, Mok-Yau, Royden, Yau, and others (see [15,44] and references therein). In Section 5 we prove a new version that only involves the holomorphic sectional curvature of domain and target manifolds. Hence it is perhaps the most natural high-dimensional generalization of the classical result of Ahlfors. To state the result, we introduce the following: For the tangent map $\partial f : T'_x M \to T'_{f(x)} N$, we define its maximum norm square to be

$$\|\partial f\|^2_m(x) \doteqdot \sup_{v \neq 0} \frac{|\partial f(v)|^2}{|v|^2}.$$

THEOREM 1.4. *Let (M, g) be a complete Kähler manifold such that the holomorphic sectional curvature $H^M(X)/|X|^4 \geq -K$ for some $K \geq 0$. Let (N^n, h) be a Kähler manifold such that $H^N(Y) < -\kappa|Y|^4$ for some $\kappa > 0$. Let $f : M \to N$ be a holomorphic map. Then*

$$\|\partial f\|^2_m(x) \leq \frac{K}{\kappa} \tag{1.2}$$

provided that the bisectional curvature of M is bounded from below. In particular, if $K = 0$, any holomorphic map $f : M \to N$ must be a constant map.

The proof uses a viscosity consideration from PDE theory (cf. [30] for another such application concerning the isoperimetric inequalities in Riemannian geometry). It is also reminiscent of Pogorelov's lemma (cf. lemma 4.1.1 of [10]) for the Monge-Ampère equation, since the maximum eigenvalue of $\nabla^2 u$ is the $\|\cdot\|_m$ for the normal map ∇u for any smooth function u. The assumption on the bisectional curvature lower bound can be replaced with the existence of an exhaustion function $\rho(x)$, which satisfies that

$$\limsup_{\rho\to\infty}\left(\frac{|\partial\rho| + [\sqrt{-1}\partial\bar{\partial}\rho]_+}{\rho}\right) = 0.$$

In view of the ample applications of the classical Schwarz lemma, we expect some implications of Theorem 1.4. A consequence of Theorem 1.4 asserts that *the equivalence of the negativities of the holomorphic sectional curvature implies the equivalence of the metrics*. Namely, if M^m admits two Kähler metrics g_1 and g_2 satisfying

$$-L_1|X|^4_{g_1} \le H_{g_1}(X) \le -U_1|X|^4_{g_1}, \quad -L_2|X|^4_{g_2} \le H_{g_2}(X) \le -U_2|X|^4_{g_2},$$

then for any $v \in T'_x M$ we have the estimates

$$|v|^2_{g_2} \le \frac{L_1}{U_2}|v|^2_{g_1}, \quad |v|^2_{g_1} \le \frac{L_2}{U_1}|v|^2_{g_2}.$$

Note that in this case the bisectional curvature lower bound can be easily checked via the polarization formula (cf. [32], formula in the proof of corollary 2.1), and the result can be stated locally given that the global result is derived from a local estimate. This result can be viewed as a stability statement of the classical result asserting that a complete Kähler manifold with the negative constant holomorphic sectional curvature must be a quotient of the complex hyperbolic space form. A natural question is *whether a compact Kähler manifold M with its homomorphic sectional curvature being close to -1 is biholomorphic to a quotient of the complex hyperbolic space.* See important works [8, 9] for the Riemannian case. A similar question for the positive holomorphic sectional curvature turns out to have an affirmative answer thanks to the resolution of Hartshorne's conjecture [20] (cf. [37] for a proof of the Kähler case using only Kähler geometry). The algebraic result of Berger (cf. [7, theorem G.6.3]) only provides an almost $1/4$ sectional curvature pinching, which is a bit shy from the assumptions needed to apply the previously known results (cf. [11, 31, 43]).

There are estimates, sometimes even local estimates, associated with Theorems 1.1, 1.3, 1.4, and 4.1. These are collected in Corollaries 3.1, 3.4, and 4.2 and estimates (4.2), (4.4), and (5.4). These estimates can be conveniently applied to local settings.

We also obtained three statements (in Corollaries 3.6 and 5.5) asserting the amount of "energy" (in terms of the curvature ratio) needed to have nondegenerate or nonconstant holomorphic maps between two Kähler manifolds with curvature constraints. These are in some sense dual versions of the Schwarz lemma for positively curved manifolds.

Finally, we show an easy application of the Liouville theorems proved in [18, 27] to d-closed $(1,1)$-forms. We call η subharmonic if $\Delta_{\bar\partial}\eta \le 0$ as a $(1,1)$-form. Here $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ is the Hodge-Laplace operator. The following is a Liouville theorem for $(1,1)$-forms.

PROPOSITION 1.5. *Assume that M is a complete noncompact Kähler manifold with either $H \ge 0$, or its orthogonal bisectional $B^\perp \ge 0$ and* Ric ≥ 0*. Let η be a d-closed subharmonic $(1,1)$-form on M such that $\|\eta\|(x) = o(\log r(x))$. Then η*

is $\overline{\partial}$-harmonic. Namely, subharmonic $(1,1)$-forms of sublogarithmic growth must be harmonic.

PROOF. By lemma 2.1 of [29], which is proved via Kähler identities,

$$\sqrt{-1}\,\partial\overline{\partial}\Lambda\eta = -\Delta_{\overline{\partial}}\eta \geq 0.$$

Hence $\Lambda\eta$ is a plurisubharmonic function. Clearly $|\Lambda\eta| \leq \|\eta\|$. The Liouville theorem of [18] and [27] implies that $\Lambda\eta \equiv C$, for a constant C. From this the claimed result follows easily from the above formula. $\square$

As illustrated in the next two sections, the main step in proving Theorem 1.1 is to show that $f^*\omega_h$ is a subharmonic $(1,1)$-form.

2 $\partial\overline{\partial}$-Bochner Formulae for Holomorphic Mappings

Let $f : M^m \to N^n$ be a holomorphic map between Kähler manifolds. Choose holomorphic normal coordinate $(z_1, z_2, \ldots, z_m)$ near a point p on the domain manifold M, correspondingly $(w_1, w_2, \ldots, w_n)$ near $f(p)$ in the target. The Kähler forms of M and N are $\omega_g = \sqrt{-1}\,g_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta$ and $\omega_h = \sqrt{-1}\,h_{i\bar{j}}dw^i \wedge d\bar{w}^j$, respectively. Correspondingly, the Christoffel symbols are given by

$${}^M\Gamma^\beta_{\alpha\gamma} = \frac{\partial g_{\alpha\bar{\delta}}}{\partial z^\gamma}g^{\bar{\delta}\beta} = \Gamma^\beta_{\gamma\alpha}, \qquad {}^N\Gamma^j_{ik} = \frac{\partial h_{i\bar{l}}}{\partial w^k}h^{\bar{l}k} = \Gamma^j_{ki}.$$

We adapt the Einstein's convention when there is any repeated index. The symmetry in the Christoffel symbols is due to Kählerity. If the appearance of the indices can distinguish the manifolds we omit the superscripts ^{M} and ^{N}. Correspondingly, the curvatures are given by

$${}^M R^\beta_{\alpha\bar{\delta}\gamma} = -\frac{\partial}{\partial\bar{z}^\delta}\Gamma^\beta_{\alpha\gamma}, \quad {}^N R^j_{i\bar{l}k} = -\frac{\partial}{\partial\bar{w}^l}\Gamma^j_{ik}.$$

At the points p and $f(p)$, where the normal coordinates are centered, we have that

$$R^M_{\bar{\beta}\alpha\bar{\delta}\gamma} = -\frac{\partial^2 g_{\bar{\beta}\alpha}}{\partial z^\gamma\partial\bar{z}^\delta}, \quad R^N_{\bar{j}i\bar{l}k} = -\frac{\partial^2 h_{\bar{j}i}}{\partial w^k\partial\bar{w}^l}.$$

For a smooth map $df(\frac{\partial}{\partial z^\alpha})$ can be written as $\frac{\partial w^i}{\partial z^\alpha}\frac{\partial}{\partial w^i} + \frac{\partial\bar{w}^i}{\partial z^\alpha}\frac{\partial}{\partial\bar{w}^i}$. But for a holomorphic map $df(\frac{\partial}{\partial z^\alpha}) = \partial f(\frac{\partial}{\partial z^\alpha}) = \frac{\partial w^i}{\partial z^\alpha}\frac{\partial}{\partial w^i}$, which we also write as $\frac{\partial f^i}{\partial z^\alpha}\frac{\partial}{\partial w^i}$ or $f^i_\alpha\frac{\partial}{\partial w^i}$. Similarly, $df(\frac{\partial}{\partial\bar{z}^\alpha}) = \overline{\partial}f(\frac{\partial}{\partial\bar{z}^\alpha})$. Recall the Hessian of the map is $D\,df(X,Y) = D_Y(df(X)) - df(\nabla_Y X)$. For the holomorphic map f,

$$D\,df\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right) = \sum f^i_{\alpha,\beta}dz^\alpha \otimes dz^\beta \otimes \frac{\partial}{\partial w^i}$$

is the nonzero part of the Hessian of f. Sometimes we also denote as $D_X\, d_Y f$. In local coordinates

$$f^i_{\alpha,\beta} = \frac{\partial^2 f^i}{\partial z^\alpha \partial z^\beta} - \Gamma^\gamma_{\alpha\beta}\frac{\partial f^i}{\partial z^\gamma} + \Gamma^i_{jk}\frac{\partial f^k}{\partial z^\alpha}\frac{\partial f^l}{\partial z^\beta}.$$

LEMMA 2.1. *For a holomorphic map $f : M \to N$,*

$$\begin{aligned}
&\langle \partial\overline{\partial}\|\partial f\|^2, v \wedge \overline{v}\rangle \\
&\quad = \|D_v \partial_{(\cdot)} f\|^2 - \sum_{\alpha,\beta=1}^{m} g^{\alpha\overline{\beta}} R^N(\partial f_\alpha, \overline{\partial} f_{\overline{\beta}}, \partial f(v), \overline{\partial} f(\overline{v})) \qquad (2.1)\\
&\qquad + \sum_{\alpha\beta} g^{\alpha\overline{\beta}}\big\langle \partial f\big(R^M_{v\overline{v}}\big)_\alpha, \overline{\partial} f_{\overline{\beta}}\big\rangle.
\end{aligned}$$

Here $\partial f_\alpha = \partial f(\frac{\partial}{\partial z^\alpha})$, $\overline{\partial} f_{\overline{\beta}} = \overline{\partial} f(\frac{\partial}{\partial \overline{z}^\beta})$, $R_{v\overline{v}}$ is viewed as a transformation $T'M \to T'M$ defined as $R_{v\overline{v}}(\frac{\partial}{\partial z^\alpha}) = R_{v\overline{v}\alpha\overline{\beta}} g^{\gamma\overline{\beta}} \frac{\partial}{\partial z^\gamma}$, and $\partial f(R^M_{v\overline{v}})_\alpha = \partial f(R^M_{v\overline{v}}(\frac{\partial}{\partial z^\alpha}))$.

In this paper $\langle \cdot, \cdot \rangle$ is the bilinear pairing extended (bilinearly) from the Riemannian metric.

PROOF. The proof is via direct computations by choosing normal coordinates centered at p and $f(p)$. We can also derive this from the classical Kodaira-Bochner formula for $(1,1)$-forms. (See, for example, [21] as well as lemma 2.1 of [26].) This is based on the following observation: Let η denote the $(1,1)$-form $f^*\omega_h$ with ω_h being the Kähler form of N. Then $\|\partial f\|^2$ is nothing but $\Lambda\eta$ (following the notation of [26] with Λ being the contraction using the Kähler metric ω_g). Hence the left-hand side of the formula (2.1) amounts to computing $\partial\overline{\partial}\Lambda\eta$. On the other hand, lemma 2.1 of [29] asserts that it equals $\sqrt{-1}\,\Delta_{\overline{\partial}}\eta$ since $d\eta = 0$. Now the Kodaira-Bochner formula (cf. lemma 2.1 of [26]) can be applied to obtain the right-hand side, since the Kodaira-Bochner formula expresses $\Delta_{\overline{\partial}}\eta$ in terms of curvature of M together with $\frac{1}{2}\big(\nabla_\gamma\nabla_{\overline{\gamma}} + \nabla_{\overline{\gamma}}\nabla_\gamma\big)\eta$. Note that the first two terms in the right-hand side of (2.1) comes from $\frac{1}{2}\big(\nabla_\gamma\nabla_{\overline{\gamma}} + \nabla_{\overline{\gamma}}\nabla_\gamma\big)\eta$. There are also cancellations for terms involving Ric^M. $\square$

COROLLARY 2.2.

(a) *Let M^m and N^n be two complete Kähler manifolds. Assume that the bisectional curvature of M is nonnegative and the bisectional curvature of N is nonpositive. Let $f : M \to N$ be a holomorphic map. Then* $\log(1 + \|df\|^2(x))$ *is plurisubharmonic. Moreover, if* $\log(1 + \|df\|^2(x))$ *is pluriharmonic, then f is totally geodesic.*

(b) *If M^m is a Riemann surface with nonnegative curvature* (*whose universal cover is $\mathbb{C}$*)*, the same result holds if N has nonpositive holomorphic sectional curvature.*

PROOF. Direct calculation shows that

$$\left\langle \sqrt{-1}\,\partial\overline{\partial}\log\bigl(1+\|\partial f\|^2\bigr), \frac{1}{\sqrt{-1}} v\wedge\overline{v}\right\rangle = \frac{\partial_v\overline{\partial}_{\overline{v}}\|\partial f\|^2}{1+\|\partial f\|^2} - \frac{|\partial_v\|\partial f\||^2}{\bigl(1+\|\partial f\|^2\bigr)^2}.$$

Under the curvature assumption of (a) we have that, under the normal coordinates

$$\begin{aligned}
&\langle \partial\overline{\partial}\log\bigl(1+\|\partial f\|^2\bigr), v\wedge\overline{v}\rangle\\
&\geq \frac{\|D_v\partial_{(\cdot)}f\|^2}{1+\|\partial f\|^2} - \frac{|\partial_v\|\partial f\||^2}{\bigl(1+\|\partial f\|^2\bigr)^2}\\
&= \frac{\|D_v\partial_{(\cdot)}f\|^2}{(1+\|\partial f\|^2)^2} + \frac{D_v f^i_\alpha D_{\overline{v}} f^{\overline{i}}_{\overline{\alpha}}\bigl|f^k_\gamma\bigr|^2 - \bigl|D_v f^i_\alpha f^{\overline{i}}_{\overline{\alpha}}\bigr|^2}{(1+\|\partial f\|^2)^2}\\
&\geq \frac{\|D_v\partial_{(\cdot)}f\|^2}{(1+\|\partial f\|^2)^2}.
\end{aligned}$$

From this estimate the claims in (a) follow easily. If $m=1$, let $v = a\frac{\partial}{\partial z^1}$ for some $a\in\mathbb{C}$. Hence

$$R^N(\partial f_\alpha, \overline{\partial} f_{\overline{\beta}}, \partial f(v), \overline{\partial} f(\overline{v})) = |a|^2 R_{i\overline{i}j\overline{j}}|f^i_1|^2|f^j_1|^2 \leq 0$$

under the assumption that N has nonnegative holomorphic sectional curvature [44]. The rest of the proof is the same. □

Another quantity which enjoys a similar Bochner-type formula is $\frac{(f^*\omega_h)^m}{\omega_g^m}$ for the case $m\leq n$. Here $\omega_h = \sqrt{-1}\sum_{i,j=1}^n h_{i\overline{j}}\,dw^i\wedge d\overline{w}^j$ is the Kähler form of N^n and $\omega_g = \sqrt{-1}\sum_{\alpha,\beta=1}^m g_{\alpha\overline{\beta}}dz^\alpha\wedge d\overline{z}^\beta$ is the Kähler form of M^m. For the equal dimensional case, a Bochner formula for the Laplacian operator Δ (instead of $\partial\overline{\partial}$) was considered previously by various people, including Kobayashi, Yau, Mok-Yau, etc. We refer the reader to [15, 44] and references therein for details.

LEMMA 2.3. *For a holomorphic map $f: M^m\to N^n$ such that df has rank m in a neighborhood of p. Let $D(x) = \frac{(f^*\omega_h)^m(x)}{\omega_g^m(x)}$ (which is positive in a neighborhood of p). Then for normal coordinates centered at p and $f(p)$ such that at p, $df\bigl(\frac{\partial}{\partial z^\alpha}\bigr) = \lambda_\alpha\delta_{i\alpha}\frac{\partial}{\partial w^i}$ (namely, $\{|\lambda_\alpha|^2\}$ are singular values of $\partial f: T'_pM\to T'_{f(p)}N$), we have at p,*

$$\begin{aligned}
&\left\langle \sqrt{-1}\,\partial\overline{\partial}\log D, \frac{1}{\sqrt{-1}}v\wedge\overline{v}\right\rangle\\
&= \sum_{\alpha=1}^m\sum_{m+1\leq i\leq n}\frac{|f^i_{\alpha v}|^2}{|\lambda_\alpha|^2} - \sum_{\alpha=1}^m R^N(\alpha,\overline{\alpha},\partial f(v),\overline{\partial f(v)})\\
&\quad + \mathrm{Ric}^M(v,\overline{v}).
\end{aligned}\tag{2.2}$$

Here $R^N(\alpha,\overline{\alpha},\partial f(v),\overline{\partial f(v)}) = R^N(\frac{\partial}{\partial w^\alpha},\frac{\partial}{\partial\overline{w}^\alpha},\partial f(v),\overline{\partial f(v)})$.

PROOF. As stated in the lemma, after unitary changes of frames of T'_pM and $T'_{f(p)}N$, df or ∂f can be expressed as $df\big(\frac{\partial}{\partial z^\alpha}\big)h = \partial f\big(\frac{\partial}{\partial z^\alpha}\big) = \lambda_\alpha \delta_{i\alpha}\frac{\partial}{\partial w^i}$. We can perform the computation at p and $f(p)$, where

$$R^M_{\alpha\bar\beta\gamma\bar\delta} = -g_{\alpha\bar\beta,\gamma\bar\delta}, \quad R^N_{i\bar j k\bar l} = -h_{i\bar j,k\bar l}.$$

Here $g_{\alpha\bar\beta,\gamma\bar\delta} = \frac{\partial^2 g_{\alpha\bar\beta}}{\partial z^\gamma \partial \bar z^\delta}$. Moreover, at $f(p)$, $h_{i\bar j,k} = h_{i\bar j,\bar k} = 0$. To simplify notations we write $\frac{\partial f^i}{\partial z^\alpha}$ as f^i_α and $\frac{\partial^2 f^i}{\partial z^\alpha \partial z^\gamma}$ as $f^i_{\alpha\gamma}$. For $v = \sum_\gamma v^\gamma \frac{\partial}{\partial z^\gamma}$, $f^i_{\alpha v} = \sum_\gamma f^i_{\alpha\gamma} v^\gamma$. With respect to such coordinates, let $A = (A_{\alpha\bar\beta})$ be the Hermitian symmetric matrix with

$$A_{\alpha\bar\beta} = f^i_\alpha h_{i\bar j}\overline{f^j_\beta}.$$

Then $D = \frac{\det(A)}{\det(g_{\alpha\bar\beta})}$. We denote $(A^{\alpha\bar\beta})$ as the inverse of A. Hence

$$\frac{\partial^2}{\partial z^\gamma \partial \bar z^\delta}\log D = \frac{\partial^2}{\partial z^\gamma \partial \bar z^\delta}\log\det(A) + \mathrm{Ric}^M(\gamma,\bar\delta).$$

Direct calculation shows that

$$\begin{aligned}(\log\det(A))_{\bar\delta} &= A^{\alpha\bar\beta}\Big[f^i_\alpha h_{i\bar j}\overline{f^j_{\beta\delta}} + f^i_\alpha h_{i\bar j,\bar l}\overline{f^j_\beta}\,\overline{f^l_\delta}\Big]\\ &= \sum_\alpha \frac{\overline{f^\alpha_{\alpha\delta}}\cdot\lambda_\alpha}{|\lambda_\alpha|^2}.\end{aligned}$$

The last line only holds at the point p, while the first holds in the neighborhood. Similarly,

$$\begin{aligned}(\log\det(A))_\gamma &= A^{\alpha\bar\beta}\Big[f^i_{\alpha\gamma}h_{i\bar j}\overline{f^j_\beta} + f^i_\alpha h_{i\bar j,k}\overline{f^j_\beta}f^k_\gamma\Big]\\ &= \sum_\alpha \frac{f^\alpha_{\alpha\gamma}\cdot\overline{\lambda_\alpha}}{|\lambda_\alpha|^2}.\end{aligned}$$

Taking the second derivative and at the end restricting to p, we have

$$\begin{aligned}(\log\det(A))_{\gamma\bar\delta} &= -A^{\alpha\bar s}\frac{\partial A_{t\bar s}}{\partial z^\gamma}A^{t\bar\beta}\Big[f^i_\alpha h_{i\bar j}\overline{f^j_{\beta\delta}} + f^i_\alpha h_{i\bar j,\bar l}\overline{f^j_\beta}\,\overline{f^l_\delta}\Big]\\ &\quad + \sum_{1\le\alpha\le m,1\le i\le n}\frac{f^i_{\alpha\gamma}\overline{f^i_{\alpha\delta}}}{|\lambda_\alpha|^2} + \sum_\alpha\sum_{ijkl}\frac{-R^N_{i\bar jk\bar l}f^i_\alpha\overline{f^j_\alpha}f^k_\gamma\overline{f^l_\delta}}{|\lambda_\alpha|^2}\\ &= \sum_{\alpha=1}^m\sum_{m+1\le i\le n}\frac{f^i_{\alpha\gamma}\overline{f^i_{\alpha\delta}}}{|\lambda_\alpha|^2} - \sum_{\alpha=1}^m R^N_{\alpha\bar\alpha\gamma\bar\delta}\lambda_\gamma\overline{\lambda_\delta}.\end{aligned}$$

Here we have used that

$$A^{\alpha\bar{s}}\frac{\partial A_{t\bar{s}}}{\partial z^{\gamma}}A^{t\bar{\beta}}\Big[f^i_\alpha h_{i\bar{j}}\overline{f^j_{\beta\delta}}+f^i_\alpha h_{i\bar{j},\bar{l}}\overline{f^j_\beta}\,\overline{f^l_\delta}\Big]=\sum_{\alpha,\beta=1}^{m}\frac{f^\alpha_{\beta\gamma}\overline{f^\alpha_\alpha}f^\alpha_\alpha\overline{f^\alpha_{\beta\delta}}}{|\lambda_\alpha|^2|\lambda_\beta|^2}$$
$$=\sum_{\alpha,\beta}\frac{f^\alpha_{\beta\gamma}\overline{f^\alpha_{\beta\delta}}}{|\lambda_\beta|^2}.$$

Putting all the above together we have the Bochner formula claimed. □

COROLLARY 2.4. *If* $\mathrm{Ric}^N_m \le 0$ *and* $\mathrm{Ric}^M \ge 0$*,* $\log D(x)$ *is a plurisubharmonic function.*

Remark 2.5. With respect to the coordinates specified in Lemma 2.3, we also have

$$\begin{aligned}&\langle\partial\bar{\partial}\log\|\partial f\|^2, v\wedge\bar{v}\rangle\\ &\quad=\frac{\sum_{i=1}^n\sum_{\alpha=1}^m|f^i_{\alpha v}|^2\|\partial f\|^2-\left|\sum_{\alpha=1}^m f^\alpha_{\alpha v}\overline{\lambda_\alpha}\right|^2}{\|\partial f\|^4}\\ &\qquad-\sum_{\alpha=1}^m\frac{|\lambda_\alpha|^2R^N(\alpha,\bar{\alpha},\partial f(v),\bar{\partial}f(\bar{v}))}{\|\partial f\|^2}+\sum_{\alpha=1}^m\frac{R^M_{v\bar{v}\alpha\bar{\alpha}}|\lambda_\alpha|^2}{\|\partial f\|^2}.\end{aligned}\tag{2.3}$$

For $\alpha > n$, $\lambda_\alpha = 0$ is understood in the above formula.

3 Proof of Theorem 1.1 and Theorem 1.3

To prove part (i) of Theorem 1.1, note that $u(x) = \log(\|\partial f\|^2(x) + 1)$ is a plurisubharmonic function by part (a) of Corollary 2.2. The growth assumption of the gradient in the theorem implies that $u(x) = o(\log(r(x)))$. Hence theorem 0.2 of [28] implies that u is a constant. This together with the second part of (a) in Corollary 2.2 implies that f is totally geodesic.

To prove part (ii) of Theorem 1.1, we use part (b) of Corollary 2.2 instead.

Here we should remark that the argument of proving the three-circle theorem in [18] works without any changes. Namely, one can conclude the following corollary.

COROLLARY 3.1. *Let* M, N *be as in Theorem* 1.1*. Let* $f : M \to N$ *be a holomorphic map. Let* $M(r) = \sup_{x\in B_p(r)}\|\partial f\|(x)$*. Then for any* $r_1 < r_2 < r_3$*:*

$$\log M(r_2) \le \frac{1}{\log r_3 - \log r_1}\big((\log r_3 - \log r_2)\log M(r_1) + (\log r_2 - \log r_1)\log M(r_3)\big).\tag{3.1}$$

This together with the consequences of (3.1) derived in [18] implies that the boundedness of $\|\partial f\|$ follows from (1.1).

To prove part (i) of Theorem 1.3, we argue by contradiction. Assume that there exists a holomorphic map f such that $\partial f : T'_x M \to T'_{f(x)}N$ is of full rank for some x. Let $D(x)$ be the function defined in Lemma 2.3. Since M is compact, $D(x)$ attains its maximum at some point x_0. Then in a neighborhood of x_0, $D(x) \neq 0$. Now let $\{|\lambda_\gamma|^2(x)\}$ be the singular value of ∂f at x. Define the following second-order elliptic operator pointwise

$$\mathcal{L} = \sum_{\gamma} \frac{1}{2|\lambda_\gamma|^2}\left(\nabla_\gamma \nabla_{\bar{\gamma}} + \nabla_{\bar{\gamma}} \nabla_\gamma\right).$$

Applying (2.2), we obtain that at x_0, with respect to the normal coordinates specified in Lemma 2.3,

$$0 \geq \mathcal{L} \log D \geq -\sum_{\gamma=1}^{m}\sum_{\alpha=1}^{m} \frac{1}{|\lambda_\gamma|^2} R^N\left(\alpha, \bar{\alpha}, \partial f\left(\frac{\partial}{\partial z^\gamma}\right), \overline{\partial f\left(\frac{\partial}{\partial z^\gamma}\right)}\right).$$

Note that $\sum_{\gamma=1}^{m}\sum_{\alpha=1}^{m} \frac{1}{|\lambda_\gamma|^2} R^N(\alpha, \bar{\alpha}, \partial f(\frac{\partial}{\partial z^\gamma}), \overline{\partial f(\frac{\partial}{\partial z^\gamma})})$ is nothing but the m-scalar curvature of $\Sigma = \mathrm{Span}\{\partial f(\frac{\partial}{\partial z^\gamma})\}$, which is negative by the assumption of Theorem 1.3 (namely the right-hand above is positive). This is a contradiction to $\mathcal{L} \log D \leq 0$ at x_0. A similar argument proves the same result if $\mathrm{Ric}^M > 0$ and $S_m^N \leq 0$.

Note that the above argument implies a rigidity result when $S_m^N \leq 0$ is allowed (but with other assumptions).

COROLLARY 3.2. *Assume that* $\dim_{\mathbb{C}} M = m \leq n = \dim_{\mathbb{C}} N$. *Let* (M, g) *be a compact Kähler manifold such that* $\mathrm{Ric}^M \geq 0$. *Let* (N^n, h) *be a complete Kähler manifold such that* $S_m^N(y) \leq 0$. *Then for any nondegenerate holomorphic map* $f : M \to N$, $D(x)$ *must be a constant. Moreover,* $\mathrm{Ric}^M \equiv 0$ *and* $S_m^N = 0$ *at least along a* m*-dimensional submanifold. Furthermore, if* $\mathrm{Ric}_m^N \leq 0$ *and* (M^m, g) *has nonnegative bisectional curvature, then* f *must be totally geodesic. The same holds for* $f : M \to N$ *if* $H^N \leq 0$ *and* $\mathrm{Ric}^M \geq 0$. *The map is constant if any of two inequalities is strict in both cases.*

PROOF. Given that f is holomorphic, the locus where $D \neq 0$ is open and dense, with its complement being a closed subvariety. Over this open dense subset

$$\mathcal{L} \log D \geq 0.$$

Hence $\log D$ must be a constant since it attains an interior maximum. The Ricci flatness and S_m^N vanishing along a m-submanifold follow from Lemma 2.3.

Under the condition that (M, g) has nonnegative bisectional curvature and that $\mathrm{Ric}_m^N \leq 0$, since $D = \mathrm{const}$ on M, we apply the operator $\mathcal{L}$ (which is well-defined due to $D \neq 0$) to $\|\partial f\|^2$. Then Lemma 2.1 implies that

$$\begin{aligned}\mathcal{L}\|\partial f\|^2 &= \sum_{i=1}^{n}\sum_{\alpha,\gamma=1}^{m}\frac{|f^i_{\alpha\gamma}|^2}{|\lambda_\gamma|^2} - \sum_{\alpha}\mathrm{Ric}^N(\alpha,\bar\alpha)|\lambda_\alpha|^2 \\ &\quad + \sum_{\alpha,\gamma} R^M_{\gamma\bar\gamma\alpha\bar\alpha}\frac{|\lambda_\alpha|^2}{|\lambda_\gamma|^2} \geq 0.\end{aligned} \tag{3.2}$$

The maximum principle implies that $\mathcal{L}\|\partial f\|^2 = 0$ and $\|\partial f\|$ is a constant. The part f being total geodesic follows from $f^i_{\alpha\gamma} = 0$ for all $1 \leq i \leq n$, $1 \leq \alpha, \gamma \leq m$.

For the case $H^N \leq 0$ and $\mathrm{Ric}^M \geq 0$, Lemma 2.1 implies that

$$\Delta\|\partial f\|^2 = \sum_{i=1}^{n}\sum_{\alpha,\gamma=1}^{m}\left|f^i_{\alpha\gamma}\right|^2 - R^N_{i\bar i j\bar j}\left|f^i_\alpha\right|^2\left|f^j_\gamma\right|^2 + \mathrm{Ric}^M_{\alpha\bar\alpha}\left|f^i_\alpha\right|^2 \geq 0.$$

The argument in the Appendix implies that the right-hand side above is nonnegative. The claim then follows by the maximum principle. □

Remark 3.3. Applying equation (2.3) to the special case of

$$f = \mathrm{id} : (M, g) \to (N, g') \quad \text{with } N = M \text{ and } g'_{\alpha\bar\beta} = g_{\alpha\bar\beta} + \varphi_{\alpha\bar\beta},$$

and employing the operator $\mathcal{L}$ yield the essential calculation in the derivation of the crucial C^2-estimate [1] (see also [36]) in solving the complex Monge-Ampère equation related to Kähler-Einstein metrics. In fact, the operator $\mathcal{L}$ is the same as Δ' on page 151 of [1]. Specifically, (2.3) implies that

$$\mathcal{L}\log\|\partial f\|^2 \geq \frac{1}{\|\partial f\|^2}\left(\sum_{\alpha,i} -|\lambda_\alpha|^2 R^N_{\alpha\bar\alpha i\bar i} + R^M_{i\bar i\alpha\bar\alpha}\frac{|\lambda_\alpha|^2}{|\lambda_i|^2}\right).$$

Observe also in this case $\|\partial f\|^2 = m + \Delta\varphi$,

$$\sum_i |\lambda_\alpha|^2 R^N_{\alpha\bar\alpha i\bar i} = \mathrm{Ric}^{g'}\left(\frac{\partial}{\partial w^\alpha}, \frac{\partial}{\partial \bar w^\alpha}\right)|\lambda_\alpha|^2,$$

which equals $\mathrm{Ric}^{g'}(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar z^\alpha}) = \mathrm{Ric}^g_{\alpha\bar\alpha} + c\varphi_{\alpha\bar\alpha} - tF_{\alpha\bar\alpha}$ due to the equation $\frac{\omega^m_{g'}}{\omega^m} = e^{-c\varphi+tF}$, following the notations of pages 97–100 of [36]. From the above estimate one can obtain the C^2-estimate (proposition on p. 151 of [1]) for the Monge-Ampère equation related to the Kähler-Einstein metrics easily (however in terms of the C^0-estimate, which was first proved via Moser's iteration for the case of zero first Chern class and is a highly delicate issue for the case of positive first Chern class).

Recall from the introduction that we say N has k-dimensional Ricci curvature bounded from above by $-\kappa$ (denoted by $\mathrm{Ric}^N_k(v, \bar v) \leq -\kappa|v|^2$), if when restricted to any k-dimensional subspace $\Sigma \subset T'_y N$, the Ricci curvature of curvature tensor of $R^N|_\Sigma$,

$$\mathrm{Ric}_{y,\Sigma}(v, \bar{v}) \doteqdot \sum_{\gamma=1}^{k} R(E_\gamma, \bar{E}_\gamma, v, \bar{v})$$

is bounded from above by $-\kappa|v|^2$ for any $v \in \Sigma$. Here $\{E_\gamma\}$ is a unitary basis of $\Sigma \subset T'_y N$. Note that for $k = 1$, Ric_1 is the same as the holomorphic sectional curvature. However, Ric^N_k for $k \geq 2$ is independent of the holomorphic sectional curvature H^N in view of the examples in [12, 32]. The following Schwarz-type estimate generalizes the previous one proved for $m = n$ (cf. p. 190 of [44]).

COROLLARY 3.4. *Let $f : M^m \to N^n$ ($m \leq n$) be a holomorphic map with M being a complete manifold. Assume that* Ric^M *is bounded from below and the scalar curvature $S^M(x) \geq -K$. Assume further that* $\mathrm{Ric}^N_m(x) \leq -\kappa < 0$. *Then we have the estimate*

$$D(x) \leq \left(\frac{K}{m\kappa}\right)^m.$$

PROOF. Note that Lemma 2.3 implies that

$$\Delta \log D \geq \kappa \sum_\gamma |\lambda_\gamma|^2 - K \geq m D^{\frac{1}{m}} \kappa - K.$$

The claimed result follows from a similar argument as in the proof of the classical Schwarz lemma (see theorem 7.23 of [44]) by applying suitable cutoff techniques and the maximum principle as in [4] (see also the next section). The lower bound of the Ricci curvature is needed to apply the Laplacian comparison theorem on the distance function (later a stronger lower bound is needed to apply the Hessian comparison theorem). □

Part (ii) of Theorem 1.3 is an immediate consequence of the above estimate applyed to $K = 0$. Note that the negative upper bound $\mathrm{Ric}^N_m \leq -\kappa$ holds if N is compact with $\mathrm{Ric}^N_m < 0$. A similar argument to the proof of Corollary 3.2 implies the following result.

COROLLARY 3.5. *Assume that* $\dim_{\mathbb{C}} M = m \leq n = \dim_{\mathbb{C}} N$. *Let (M, g) be a compact Kähler manifold such that $S^M \geq 0$. Let (N^n, h) be a complete Kähler manifold such that* $\mathrm{Ric}^N_m(y) \leq 0$. *For any nondegenerate holomorphic map $f : M \to N$, $D(x)$ must be a constant. Moreover, $S^M \equiv 0$ and* $\mathrm{Ric}^N_m = 0$, *at least along a m-dimensional submanifold.*

By flipping the sign we have the following consequences.

COROLLARY 3.6 (A hoop lemma-volume version). *Let $f : M \to N$ be a holomorphic map with M being compact. Assume that* $\mathrm{Ric}^M \geq K > 0$. *Assume* *further that* $\mathrm{Ric}^N_m(x) \le \kappa$ *with* $\kappa > 0$. *Then we have the estimate*

$$\max_{x \in M} D^{1/m}(x) \ge \frac{K}{\kappa},$$

provided that f *is nondegenerate.*

PROOF. At the maximum point of $D(x)$, say x_0, apply Lemma 2.3 as before. Pick v to be the unit direction such that $|\partial f(v)|$ is the smallest. Then the maximum principle implies that $0 \ge -\kappa \inf_{v,|v|=1} |\partial f(v)|^2 + K$. The claimed result follows easily. □

We should remark that a similar result can be obtained (with the same argument) for the harmonic maps between two Riemannian manifolds, namely, if $u : M \to N$ is a harmonic map between two compact Riemannian manifolds. Assume that the sectional curvature of N is bounded from above by κ and $\mathrm{Ric}^M \ge K$ with $\kappa, K > 0$. Then for nonconstant map u

$$\max_{x \in M} \|du\|^2(x) \ge \frac{K}{\kappa}. \tag{3.3}$$

The corollary above has the advantage that when the volume (or the stretching of the volume forms) is concerned *only the Ricci curvatures* of both the target and domain manifolds are involved. The result for the harmonic maps is less satisfying since it involves the bound of two different types of curvatures.

To prove part (iii) of Theorem 1.3, we need the following result from [22]:

Let (M, g) *be a complete Kähler manifold with* $\mathrm{Ric}^M \ge 0$. *Let* $u(x)$ *be a plurisubharmonic function on* M *satisfying that*

$$\lim_{x \to \infty} u(x) = o\big(\log(r(x))\big).$$

Then $\big(\sqrt{-1}\,\partial\bar\partial u\big)^m \equiv 0$.

Now let $u(x) = \log D(x)$. Lemma 2.3 implies that $u(x)$ is a plurisubharmonic function, and at the point where $D > 0$, it is strictly plurisubharmonic due to $\mathrm{Ric}^N_m < 0$. On the other hand, the growth assumption in part (iii) of Theorem 1.3 implies that $u(x) = o(\log(r(x)))$. Hence $\big(\sqrt{-1}\,\partial\bar\partial u\big)^m \equiv 0$. This is a contradiction at the point x with $D(x) > 0$. The contradiction shows that $D(x) \equiv 0$; namely, f is degenerate.

4 Extensions

In this section we extend the proofs in the previous section to obtain the following result towards the question (Q) raised in the introduction.

THEOREM 4.1. *Assume that* $\dim_{\mathbb{C}} M = m \le n = \dim_{\mathbb{C}} N$.

(i) *Let* (M, g) *be a complete Kähler manifold such that the holomorphic bisectional curvature is bounded from below by* $-K_1$ *for some* $K_1 > 0$. *Let* (N^n, h) *be a compact Kähler manifold such that* $S_m(y) \le -\kappa < 0$.

Let $\{|\lambda_\gamma|^2(x)\}$ *be the singular values of* ∂f *at* $T'_x M$. *Assume further that* $\mathrm{Ric}^M \geq -K$ *and* $D(x)$ *is bounded from above. Then*

$$\limsup_{x\to\infty}(\min_{\gamma}|\lambda_\gamma|^2(x)) \leq \frac{mK}{\kappa}. \tag{4.1}$$

In particular, if additionally $\mathrm{Ric}^M \geq 0$ *then* $f : M \to N$ *must be asymptotically degenerate in the sense that*

$$\limsup_{x\to\infty}\Big(\min_{1\leq\gamma\leq m}|\lambda_\gamma|^2(x)\Big) = 0.$$

(ii) *Let* (M, g) *be a Kähler manifold. Assume that for* $R > 0$, *the holomorphic bisectional curvature of* M *is bounded from below by* $-K_1$ *for some* $K_1 > 0$ *in* $B_p(R)$. *Let* (N^n, h) *be a compact Kähler manifold such that* $S_m(y) < -\kappa < 0$. *Let* $f : M \to N$ *be a holomorphic map. Let* $\{|\lambda_\gamma|^2(x)\}$ *be the singular values of* ∂f *at* $T'_x M$. *Let* $\sigma_{m-1}(\lambda)$ *be the* $(m-1)^{th}$ *symmetric function of the singular values* $\{|\lambda_\gamma|^2\}$. *Assume further that* $\mathrm{Ric}^M \geq -K$. *Then we have*

$$\begin{aligned}\sup_{B_p(\frac{R}{2})} D(x) \leq \frac{mK}{\kappa}&\sup_{B_p(R)}\sigma_{m-1}\\ &+\left(\frac{C_1}{R^2}+\frac{C_1}{R}\left(C(m)\left(\frac{1}{R}+\sqrt{K_1}\right)\right)\right)\frac{\sup_{B_p(R)}\sigma_{m-1}}{\kappa}.\end{aligned} \tag{4.2}$$

Here $C_1 > 0$ *is an absolute constant. If furthermore* (M^m, g) *is complete and has nonnegative bisectional curvature and* σ_{m-1} *satisfies that*

$$\limsup_{x\to\infty}\frac{\sigma_{m-1}(x)}{r^2(x)} = 0,$$

then f *must be degenerate.*

PROOF. To prove part (i), we apply the maximum principle of [34] at infinity. By the virtue of [34] we have a sequence of points $x_k \to \infty$ such that $\lim_{k\to\infty} D(x_k) \to \sup_M D$, which we may assume without loss of generality is positive, and

$$\lim_{k\to\infty}\langle\sqrt{-1}\,\partial\bar\partial\log D, \frac{1}{\sqrt{-1}}v\wedge\bar v\rangle\Big|_{x_k} \leq 0.$$

Applying Lemma 2.3, if denoting the lower bound of the Ricci curvature (of M) by $-K$, we have that

$$\limsup_{k\to\infty}\left(\kappa - K\sum_{\gamma=1}^m\frac{1}{|\lambda_\gamma|^2(x_k)}\right) \leq 0.$$

This implies that

$$\limsup_{k\to\infty}(\min_{\gamma}|\lambda_\gamma|^2(x_k)) \leq \frac{mK}{\kappa}.$$

This proves (4.1), which implies the rest of part (i).

To prove part (ii), let $\eta(t) : [0, +\infty) \to [0, 1]$ be a function supported in $[0, 1]$ with $\eta' = 0$ on $[0, \frac{1}{2}]$, $\eta' \le 0$, $\frac{|\eta'|^2}{\eta} + (-\eta'') \le C_1$. The construction of such η is elementary. Let $\varphi_R(x) = \eta(\frac{r(x)}{R})$. When the meaning is clear, we omit subscript R in φ_R. Clearly $D \cdot \varphi$ attains a maximum somewhere at x_0 in $B_p(R)$. Now we apply $\mathcal{L}$ to $\log(D\varphi)$ at the maximum point x_0 (where $D \cdot \varphi$ also attains its maximum). The first derivatives vanish at x_0, which implies

$$\nabla_\gamma D(x_0) = -D(x_0)\,\frac{\eta'(\frac{r(x)}{R})}{R\eta(\frac{r(x)}{R})}\nabla_\gamma r(x)\bigg|_{x_0},$$

$$\nabla_{\bar\gamma} D(x_0) = -D(x_0)\,\frac{\eta'(\frac{r(x)}{R})}{R\eta(\frac{r(x)}{R})}\nabla_{\bar\gamma} r(x)\bigg|_{x_0}.$$

Applying Lemma 2.3, we have that at x_0 (where we may assume $D\varphi > 0$),

$$\begin{aligned}
0 \ge \mathcal{L}\log(D\varphi) &\ge \kappa - K\sum_\gamma \frac{1}{|\lambda_\gamma|^2} + \mathcal{L}\log\varphi\\
&= \kappa - K\sum_\gamma \frac{1}{|\lambda_\gamma|^2} + \frac{\eta''}{R^2\varphi}\sum_\gamma \frac{|\nabla_\gamma r(x)|^2}{|\lambda_\gamma|^2}\\
&\quad + \frac{\eta'}{2R\varphi}\sum_\gamma \frac{\nabla^2_{\gamma\bar\gamma} r(x) + \nabla^2_{\bar\gamma\gamma} r(x)}{|\lambda_\gamma|^2} - \frac{|\eta'|^2}{\varphi^2R^2}\sum_\gamma \frac{|\nabla_\gamma r(x)|^2}{|\lambda_\gamma|^2}\\
&\ge \kappa - K\sum_\gamma \frac{1}{|\lambda_\gamma|^2} - \frac{C_1}{\varphi R^2}\sum_\gamma \frac{1}{|\lambda_\gamma|^2} - \frac{C_1}{\varphi R}\sum_\gamma \frac{C(m)(\frac{1}{R} + \sqrt{K_1})}{|\lambda_\gamma|^2}\\
&\quad - \frac{|\eta'|^2}{R^2\eta\varphi}\sum_\gamma \frac{1}{|\lambda_\gamma|^2}.
\end{aligned}$$

In the last line above we have used the complex Hessian comparison theorem of [17]. Now multiplying $D\varphi$ on both sides of the estimate above, we have at x_0

$$0 \ge D \cdot \varphi\kappa - m\varphi K\sigma_{m-1} - \frac{C_1}{R^2}\sigma_{m-1} - \frac{C_1}{R}\left(C(m)\left(\frac{1}{R} + \sqrt{K_1}\right)\right)\sigma_{m-1}.$$

From this we have that

$$\sup_{B_p(\frac{R}{2})} D \le \frac{mK}{\kappa}\sup_{B_p(R)}\sigma_{m-1} + \left(\frac{C_1}{R^2} + \frac{C_1}{R}\left(C(m)\left(\frac{1}{R} + \sqrt{K_1}\right)\right)\right)\frac{\sup_{B_p(R)}\sigma_{m-1}}{\kappa}.$$

This proves (4.2). In the above estimate, letting $K = 0$ and letting $R \to \infty$, noting that $\lim_{R\to\infty}\frac{\sup_{B_p(R)}\sigma_{m-1}}{R^2} = 0$, we have the rest of the claim in part (ii). Here we have used the complex Hessian comparison result assuming the bisectional curvature lower bound [17]. $\square$

It is clear from the proof that part (i) of Theorem 4.1 holds if $\mathrm{Ric} \geq -K$ outside a compact domain, or even only

$$\liminf_{x\to\infty} \mathrm{Ric}^M(x) \geq -K.$$

In particular, if $\liminf_{x\to\infty} \mathrm{Ric}^M(x) \geq 0$, we have

$$\limsup_{x\to\infty}\bigl(\min_{\gamma} |\lambda_\gamma|^2(x)\bigr) = 0.$$

For part (ii), if we choose η carefully, the following estimate can be proved: If $\mathrm{Ric}^M \geq -K$, the bisectional curvature is bounded from below by $-K_1$ outside $B_p(R_0)$ for some $R_0 > 0$ then

$$\begin{aligned}
&\sup_{B_p(\frac{R}{2})\setminus B_p(R_0)} D(x) \\
&\leq \frac{mK}{\kappa} \sup_{B_p(R)} \sigma_{m-1} + \frac{C_2}{R} \sup_{B_p(R_0)} \sigma_{m-1} \\
&\quad + \left(\frac{C_1}{R^2} + \frac{C_1}{R}\left(C(m)\left(\frac{1}{R} + \sqrt{K_1}\right)\right)\right)\frac{\sup_{B_p(R)} \sigma_{m-1}}{\kappa}.
\end{aligned} \tag{4.3}$$

Here C_1 is an absolute constant, and $C_2 = C_2(R_0)$.

A similar localization procedure also implies the following estimate:

COROLLARY 4.2. *Let $R > 0$ be a constant. Assume that scalar curvature $S^M(x) \geq -K$ and $\mathrm{Ric}^M \geq -K_1$ in $B_p(R)$, and that the k-Ricci of N $\mathrm{Ric}^N_k(x) \leq -\kappa$. Then we have the estimate*

$$\sup_{B_p(\frac{R}{2})} mD^{1/m}(x) \leq \frac{K}{\kappa} + \frac{1}{\kappa}\left(\frac{C_1}{R^2} + \frac{C_1}{R}\left(C(m)\left(\frac{1}{R} + \sqrt{K_1}\right)\right)\right). \tag{4.4}$$

Here C_1 is an absolute constant.

For any p one may define the *lower Ricci curvature radius* (abbreviated by $r^l_{\mathrm{Ric}}(p)$) as the biggest R such that $\mathrm{Ric} \geq -\frac{1}{R^2}$ in $B_p(R)$. For $R = r^l_{\mathrm{Ric}}$ the estimate simplifies into the form

$$\sup_{B_p(\frac{R}{2})} mD^{1/m}(x) \leq \frac{K}{\kappa} + \frac{1}{\kappa}\frac{C_1}{R^2}. \tag{4.5}$$

5 A Schwarz Lemma

In this section we prove Theorem 1.4. We start with a linear algebra lemma.

LEMMA 5.1. *Let A be a Hermitian symmetric matric that is semipositive. Let G be a positive Hermitian symmetric matrix. We denote by $(G^{\alpha\bar\beta})$ the inverse of G. Then for any s*

$$\sup_{v\neq 0} \frac{\langle A(v), \overline{v}\rangle}{\langle G(v), \overline{v}\rangle} \geq \frac{G^{s\bar\beta}A_{\alpha\bar\beta}G^{\alpha\bar s}}{G^{s\bar s}} \geq \inf_{v\neq 0} \frac{\langle A(v), \overline{v}\rangle}{\langle G(v), \overline{v}\rangle}. \tag{5.1}$$

PROOF. By linear algebra, there exist matrices a and g such that $A = \overline{a}^t \cdot a$ and $G = \overline{g}^t \cdot g$. The positivity of G implies that g is nonsingular. Let $\{E_\gamma\}$ be $\{\frac{\partial}{\partial z^\gamma}\}$. Now the middle term can be expressed as

$$\frac{\langle G^{-1}AG^{-1}(E_s), \bar{E}_s\rangle}{\langle G^{-1}(E_s), \bar{E}_s\rangle} = \frac{\langle \overline{a}^t \cdot a \cdot g^{-1} \cdot (\overline{g}^t)^{-1}(E_s), \overline{g^{-1} \cdot (\overline{g}^t)^{-1}(E_s)}\rangle}{\langle (\overline{g}^t)^{-1}(E_s), \overline{(\overline{g}^t)^{-1}(E_s)}\rangle}.$$

Let $w = (\overline{g}^t)^{-1}(E_s)$ and $v = g^{-1}(w)$ (which is clearly nonzero). Then the right-hand side can be written as

$$\frac{\langle \overline{a}^t \cdot a \cdot g^{-1}(w), \overline{g^{-1}(w)}\rangle}{|w|^2} = \frac{\langle a(v), \overline{a(v)}\rangle}{|g(v)|^2} = \frac{\langle A(v), \overline{v}\rangle}{\langle G(v), \overline{v}\rangle}.$$

Now the claimed result becomes obvious. □

Now we prove Theorem 1.4.[1] Let η and φ be the cutoff functions as in the last section. We consider $\|\partial f\|_m^2 \varphi$. It must attain a maximum somewhere in $B_p(R)$, say at x_0. Now we pick normal coordinates $(z_1, \dots, z_m)$ centered at x_0, and $(w_1, \dots, w_n)$ centered at $f(x_0)$ as before. Let $A = (A_{\alpha\bar{\beta}})$ locally with

$$A_{\alpha\bar{\beta}}(x) = f_\alpha^i(x) h_{i\bar{j}}(f(x)) \overline{f_\beta^j}(x).$$

By unitary changes of frame of $T'_{x_0}M$ and $T'_{f(x_0)}N$, we can assume that $f_\alpha^i = \delta_{i\alpha}\lambda_\alpha$ at x_0. We may also assume that

$$\|\partial f\|_m^2(x_0) = |\lambda_1|^2 \geq |\lambda_2|^2 \geq \cdots \geq |\lambda_m|^2.$$

Now let

$$W(x) = \frac{g^{1\bar{\beta}}(x) A_{\alpha\bar{\beta}}(x) g^{\alpha\bar{1}}(x)}{g^{1\bar{1}}(x)}.$$

By the choice of the normal coordinates specified as above, we have that $W(x_0) = |\lambda_1|^2 = \|\partial f\|_m^2(x_0)$. The above lemma implies that $W(x) \leq \|\partial f\|_m^2(x)$ for x in the neighborhood of x_0. Hence $W(x) \cdot \varphi(x)$ still attains a local maximum at x_0, which is the same as $\|\partial f\|_m^2 \cdot \varphi$ at x_0. In the terminology of viscosity solutions, $W(x)$ serves a smooth barrier for $\|\partial f\|_m^2(x)$. We shall apply the maximum principle to $\log(W(x) \cdot \varphi(x))$. For that we need another $\partial\bar{\partial}$-lemma.

LEMMA 5.2. *Under the above notations, at x_0, or at any point with the normal coordinates specified above,*

$$\left\langle \sqrt{-1}\,\partial\bar{\partial}\log W, \frac{1}{\sqrt{-1}} v \wedge \bar{v} \right\rangle = R^M_{1\bar{1}v\bar{v}} - R^N_{1\bar{1}\partial f(v)\overline{\partial f(v)}} + \frac{\sum_{i\neq 1}|f^i_{1v}|^2}{W}. \tag{5.2}$$

[1] Very recently we were informed by X. D. Wang that Theorem 1.4 was obtained by Chen-Cheng-Lu under the assumption that the sectional curvature of the domain manifold is bounded from below.

PROOF. We shall compute $\frac{\partial^2}{\partial z^\gamma \partial z^{\bar\gamma}} \log W$. Then the claimed result follows from this by linear combinations. Under the normal coordinates specified above, we have that

$$\frac{\partial g^{\alpha\bar\beta}}{\partial z^\gamma} = -g^{\alpha\bar\delta}\frac{\partial g_{s\bar\delta}}{\partial z^\gamma} g^{s\bar\beta}, \quad \frac{\partial^2 g^{\alpha\bar\beta}}{\partial z^\gamma \partial z^{\bar\gamma}} = -g^{\alpha\bar\delta}\frac{\partial^2 g_{s\bar\delta}}{\partial z^\gamma \partial z^{\bar\gamma}} g^{s\bar\beta} = R_{\alpha\bar\beta\gamma\bar\gamma}.$$

Hence we have at x_0, noting that $\frac{\partial g_{\alpha\bar\beta}}{\partial z^\gamma} = 0$ and $\frac{\partial h_{i\bar j}}{\partial w^k} = 0$,

$$\frac{\partial \log W}{\partial z^\gamma} = \frac{g^{1\bar\beta} f^i_{\alpha\gamma} h_{i\bar j} f^{\bar j}_{\bar\beta} g^{\alpha\bar 1}}{W} = \frac{f^1_{1\gamma} f^{\bar 1}_{\bar 1}}{W},$$

$$\frac{\partial \log W}{\partial z^{\bar\gamma}} = \frac{f^{\bar 1}_{\bar 1\bar\gamma} f^1_1}{W},$$

$$\begin{aligned}\frac{\partial^2 \log W}{\partial z^\gamma \partial z^{\bar\gamma}} &\\ = \frac{2\frac{\partial^2 g^{1\bar\beta}}{\partial z^\gamma \partial z^{\bar\gamma}} f^i_\alpha h_{i\bar j} f^{\bar j}_{\bar\beta} g^{\alpha\bar 1} + g^{1\bar\beta} f^i_\alpha f^{\bar j}_{\bar\beta}\frac{\partial^2 h_{i\bar j}}{\partial w^k \partial w^{\bar l}} f^k_\gamma f^{\bar l}_{\bar\gamma} g^{\alpha\bar 1} + g^{1\bar\beta} f^i_{\alpha\gamma} h_{i\bar j} f^{\bar j}_{\bar\beta\bar\gamma} g^{\alpha\bar 1}}{W} &\\ - \frac{|f^1_{1\gamma}|^2 |f^1_1|^2}{W^2} - R^M_{1\bar 1\gamma\bar\gamma}.&\end{aligned}$$

The claimed result follows by observing that $W = |f_1^1|^2$, and putting the above computations together. $\square$

Remark 5.3. The argument in the above proof also shows that $\|\partial f\|^2_m$ is a viscosity subsolution of (5.2), and the minimal eigenvalue of A, on the other hand, is a viscosity supersolution of (5.2). In fact, one can prove something slightly better. Let μ be the multiplicity of λ_1 (namely for $\gamma > \mu$, $\lambda_\gamma < \lambda_1$). If at point x_0, φ is a smooth barrier from above, namely $\varphi(x_0) = \lambda_1(x_0)$ and in a neighborhood of x_0, $\varphi(x) \geq \lambda_1(x)$. Then at x_0, it holds that

$$\begin{aligned}&\left\langle \sqrt{-1}\,\partial\bar\partial \log\varphi, \frac{1}{\sqrt{-1}} v \wedge \bar v\right\rangle \\ &\quad \geq R^M_{1\bar 1 v\bar v} - R^N_{1\bar 1 \partial f(v)\overline{\partial f(v)}} + \sum_{\gamma>\mu} \frac{1}{(\varphi - \lambda_\gamma)}\big(|f^1_{\gamma v}|^2 + |f^\gamma_{1v}|^2\big).\end{aligned} \tag{5.3}$$

Now with the above lemma we continue along the same line of argument for the proof of Theorem 4.1 and obtain at x_0, where $W \cdot \varphi$ attains its maximum, that

$$0 \geq \frac{\partial^2}{\partial z^1 \partial z^{\bar 1}} (\log(W\varphi)) \geq R^M_{1\bar 1 1\bar 1} - R^N_{1\bar 1 1\bar 1}|f_1^1|^2 + \frac{\partial^2 \log\varphi}{\partial z^1 \partial z^{\bar 1}} \geq$$

$$
\begin{aligned}
&\geq -K + \kappa |f_1^1|^2 + \frac{\eta''}{R^2\varphi}|\nabla_1 r(x)|^2 + \frac{\eta'}{2R\varphi}\Big(\nabla^2_{1\bar{1}} r(x) + \nabla^2_{\bar{1}1} r(x)\Big)\\
&\quad - \frac{|\eta'|^2}{\varphi^2 R^2}\cdot |\nabla_1 r(x)|^2\\
&\geq -K + \kappa |f_1^1|^2 - \frac{C_1}{\varphi R^2} - \frac{C_1}{\varphi R}\cdot C(m)\left(\frac{1}{R} + \sqrt{K_2}\right) - \frac{C_1|\eta'|^2}{R^2\eta\varphi}.
\end{aligned}
$$

Here we applied the complex Hessian comparison theorem of [17] with K_2 being the lower bound of the bisectional curvature in $B_p(R)$. Multiplying φ on both sides, we will have at x_0 the estimate

$$
\kappa |f_1^1|^2\varphi \leq K\varphi + \frac{C_1}{R^2} + \frac{C_1}{R}\cdot C(m)\left(\frac{1}{R} + \sqrt{K_2}\right) + \frac{C_1|\eta'|^2}{R^2\eta}.
$$

Hence we arrive at the estimate

$$
\sup_{B_p(\frac{R}{2})} \|\partial f\|^2_m(x) \leq \frac{1}{\kappa}\left(K + \frac{C_1}{R^2} + \frac{C_1}{R}\cdot C(m)\left(\frac{1}{R} + \sqrt{K_2}\right)\right). \tag{5.4}
$$

The claimed estimate in Theorem 1.4 follows by letting $R \to \infty$. The last statement on the holomorphic map being a constant map follows easily by applying the estimate to the case $K = 0$.

COROLLARY 5.4. *Let $R > 0$ be a constant such that the bisectional curvature of M is bounded from below on $B_p(R)$ by $-K_2$. Assume that $H^M(X) \geq -K|X|^4$ and that $H^N(Y) \leq -\kappa |Y|^4$. Then we have the estimate*

$$
\sup_{B_p(\frac{R}{2})} \|\partial f\|^2_m(x) \leq \frac{1}{\kappa}\left(K + \frac{C_1}{R^2} + \frac{C_1}{R}\cdot C(m)\left(\frac{1}{R} + \sqrt{K_2}\right)\right). \tag{5.5}
$$

Here C_1 is an absolute constant.

For any point p, we can similarly define the *lower bisectional curvature radius* being the biggest R such that the bisectional curvature is bounded by $-\frac{1}{R^2}$ on $B_p(R)$. Such a radius is denoted by $r^l_B(p)$. Clearly, if the bisectional curvature is nonnegative, $r^l_B(p) = \infty$. For $R = r^l_B(p)$ the above estimate has the simple form

$$
\sup_{B_p(\frac{R}{2})} \|\partial f\|^2_m(x) \leq \frac{1}{\kappa}\left(K + \frac{C_1}{R^2}\right). \tag{5.6}
$$

A consequence from the proof also implies the following result, which can be interesting in the study of holomorphic (even memomorphic) maps between compact Kähler manifolds.

COROLLARY 5.5 (A hoop lemma-length version).

(i) *Assume that M is compact, $H^M(X) \geq K|X|^4$, and $H^N(Y) \leq \kappa|Y|^4$, with $K, \kappa > 0$. Then for any nonconstant $f : M \to N$,*

$$\max_x \|\partial f\|_m^2(x) \geq \frac{K}{\kappa}.$$

(ii) *Assume that M is compact, $\mathrm{Ric}^M \geq K_1$, and that $H^N(Y) \leq \kappa|Y|^4$, with $K_1, \kappa > 0$. Then for any nonconstant holomorphic map $f : M \to N$,*

$$\max_x \|\partial f\|^2(x) \geq \frac{K_1}{\kappa}.$$

The proof of the second statement uses an estimate modifying (A.3) in the Appendix. Part (i) of the result is more satisfying since it only involves the holomorphic sectional curvature of both the target and domain manifolds. The proof of the Schwarz lemma implies the following result.

COROLLARY 5.6. *Let (M^m, g) be a compact Kähler manifold, and (N^n, h) another Kähler manifold. Assume either that $H^M(X) > 0$ and $H^N(Y) \leq 0$, or $H^M(X) \geq 0$ and $H^N(Y) < 0$. Then any holomorphic map $f : M \to N$ must be a constant.*

PROOF. Assume otherwise. Then $\|\partial f\|_m^2(x)$ attains a nonzero maximum somewhere, say at x_0. Applying the above proof of the Schwarz lemma at x_0, we have

$$0 \geq \frac{\partial^2 \log W}{\partial z^1 \partial z^{\bar{1}}} \geq R^M_{1\bar{1}1\bar{1}} - R^N_{1\bar{1}1\bar{1}}\|\partial f\|_m^2 > 0.$$

This contradiction proves the result. □

Note that under the assumptions $H^M(X) > 0$ and $H^N(Y) \leq 0$, the result also follows from the above Hoop lemma part (i) by taking $\kappa \to 0$. This part was also proved independently in [40], using a different method. Similarly, f must be constant if $\mathrm{Ric}^M > 0$ and $H^N \leq 0$.

Appendix

First we include an alternate algebraic part of the proof, by Royden, of the "classical" Schwarz lemma [35]:

> *Let $f : M^m \to N^n$ be a holomorphic map. Assume that the holomorphic sectional curvature of N, $H(Y) \leq -\kappa|Y|^4$, and the Ricci curvature of M, $\mathrm{Ric}(X, \bar{X}) \geq -K|X|^2$, with $\kappa, K > 0$. Let $d = \mathrm{rank}(f)$. Then*

$$\|\partial f\|^2(x) \leq \frac{2d}{d+1}\frac{K}{\kappa}. \tag{A.1}$$

The estimate $\|\partial f\|^2 \leq \frac{K}{\kappa}$ was proved (by S.-T. Yau [42]) either for M being a Riemann surface or for the case $m \geq 2$ assuming that the bisectional curvature of N is bounded from above by $-\kappa$ (cf. [42]). The above result of Royden covers

Yau's estimate for the Riemann surfaces case while allowing a weaker holomorphic sectional curvature upper bound on the target manifolds for any dimension of the domain manifolds. The algebraic ingredient is needed in showing, under the assumption that $H^N(Y) \le -\kappa |Y|^4$,

$$\Delta \|\partial f\|^2 \ge \frac{d+1}{2d}\kappa \|\partial f\|^4 - K\|\partial f\|^2. \tag{A.2}$$

By taking the trace in (2.1) of Lemma 2.1, we have that

$$\Delta \|\partial f\|^2 \ge -g^{\alpha\bar\beta} g^{\gamma\bar\delta} R^N\big(\partial f_\alpha, \overline{\partial} f_{\bar\beta}, \partial f_\gamma, \overline{\partial} f_{\bar\delta}\big) + g^{\alpha\bar\beta} g^{\gamma\bar\delta} \langle \partial f (R^M_{\gamma\bar\delta})_\alpha, \overline{\partial} f_{\bar\beta}\rangle.$$

With respect to normal coordinates chosen before (so that $f^i_\alpha = \delta_{i\alpha}\lambda_i$) the above can be written as

$$\Delta \|\partial f\|^2 \ge -R^N_{i\bar i j \bar j} |f^i_\alpha|^2 |f^j_\gamma|^2 + \mathrm{Ric}^M_{\alpha\bar\alpha} |f^i_\alpha|^2 \ge -R^N_{i\bar i j\bar j}|f^i_\alpha|^2|f^j_\gamma|^2 - K\|\partial f\|^2.$$

Thus the result follows easily after the pointwise estimate (under the assumption $H(X) \le -\kappa |X|^4$):

$$R^N_{i\bar i j\bar j}|f^i_\alpha|^2 |f^j_\gamma|^2 \le -\frac{d+1}{2d}\kappa\|\partial f\|^4. \tag{A.3}$$

The argument below, which is due to F. Zheng, proves (A.3), which is a lemma of Royden. To prove this, consider the vector $Y = \sum_{\lambda_i \ne 0} w^i \lambda_i \frac{\partial}{\partial w^i}$ (if $m \le n$, $Y \in \partial f(T'_x M)$). Direct calculation shows that

$$\begin{aligned}
\fint_{\mathbb{S}^{2d-1}} R^N(Y, \bar Y, Y, \bar Y) d\theta(w) &= \frac{2}{d(d+1)} R^N_{i\bar i j\bar j} |\lambda_i|^2 |\lambda_j|^2 \\
&= \frac{2}{d(d+1)} R^N_{i\bar i j\bar j} |f^i_\alpha|^2 |f^j_\gamma|^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\fint_{\mathbb{S}^{2d-1}} R^N(Y, \bar Y, Y, \bar Y)\, d\theta(w) &\le -\kappa \fint_{\mathbb{S}^{2d-1}} |Y|^4\, d\theta(w) \\
&= \frac{-\kappa}{d(d+1)}\Big(2\sum |\lambda_i|^4 + \sum_{i\ne j} |\lambda_i|^2\, |\lambda_j|^2\Big) \\
&= -\frac{\kappa}{d(d+1)}\Big(\|\partial f\|^4 + \sum |\lambda_i|^4\Big) \\
&\le -\frac{\kappa}{d(d+1)} \frac{d+1}{d}\|\partial f\|^4.
\end{aligned}$$

Putting the above together we have (A.3).

From the proof it is easy to see that *if the equality holds in* (A.1)*, f is totally geodesic*. This is due to the formula in Lemma 2.1. Moreover, *if f is an immersion at some point, then f is an isometric immersion*. We should remark that in a more recent paper [24] the estimate (A.1) has been generalized to a family of interpolating estimates that connects Theorem 1.4 with Yau-Royden's result. The rigidity for the equality case also holds for the estimates there. An easy consequence of this is that *any distance nondecreasing holomorphic map between quotients of complex hyperbolic spaces must be totally geodesic*. The general case without the distance nondecreasing assumption is only known for holomorphic immersions between compact hyperbolic quotients [2] under a restriction of the dimension of the target manifold in terms of the dimension of the domain manifold.

Now we prove that the k-hyperbolicity defined in the introduction is the same as: For any $x \in N$, $v_1 \wedge \cdots \wedge v_k \neq 0$ with $v_i \in T'_x N$, the pseudonorm

$$\|v_1 \wedge \cdots \wedge v_k\|_k \doteqdot \inf_{df(\widetilde{v}_1 \wedge \cdots \wedge \widetilde{v}_k) = v_1 \wedge \cdots \wedge v_k} \|\widetilde{v}_1 \wedge \cdots \wedge \widetilde{v}_k\|_p,$$

with $f : \mathbb{D}^k(1) \to N$ being holomorphic and $f(0) = x$, is a norm. Here $\|\cdot\|_p$ denotes the Poincaré metric on $\mathbb{D}^k$ (naturally extended to $\wedge_k T'N$). The proof is parallel to the $k = 1$ case. It is easy to show that if there exists a nondegenerate $f : \mathbb{C}^k \to N$, then the above pseudonorm has to vanish for any $v_1 \wedge \cdots v_k \neq 0$ in $\wedge_k T'_x N$ with $df(\widetilde{v}_1 \wedge \cdots \wedge \widetilde{v}_k) = v_1 \wedge \cdots \wedge v_k$, since if we let $\rho_r(z) = \frac{z}{r}$ and define $f_\ell = f(\rho_{\frac{1}{\ell}}(z))$, it is clear $df_\ell(\frac{\widetilde{v}_1}{\ell} \wedge \cdots \wedge \frac{\widetilde{v}_k}{\ell}) = v_1 \wedge \cdots \wedge v_k$. But $\|\frac{\widetilde{v}_1}{\ell} \wedge \cdots \wedge \frac{\widetilde{v}_k}{\ell}\|_p \to 0$ as $\ell \to \infty$. The other direction utilizes that N is compact. It suffices to show that if the pseudonorm is not a norm at x, then one can construct a nondegenerate holomorphic $f : \mathbb{C}^k \to N$. First, equip N with a Hermitian metric h, and consider $\mathcal{F} = \{\text{holomorphic } g : \mathbb{D}^k(1) \to N, g(0) = x\}$. We claim that there exists g_i such that $D(g_i)(0) \to \infty$. Otherwise there will be a uniform upper bound A for $D(g)$ for any $g \in \mathcal{F}$. This would imply that

$$0 < \|v_1 \wedge \cdots \wedge v_k\|_h \leq \sqrt{A}\|\widetilde{v}_1 \wedge \cdots \wedge \widetilde{v}_k\|_p.$$

This is a contradiction, since the right-hand side can be arbitrarily small! Now for the g_i with $D(g_i)(0) \to \infty$, we let $\ell_i = D(g_i)(0)$ and consider $f_i = g_i(\rho_{\ell_i}(z))$, which shall be defined on a sequence of balls whose union exhausts $\mathbb{C}^k$. Clearly $D(f_i)(0) = 1$. Restricted to any compact subset $K \subset \mathbb{C}^k$, by the compactness of N and passing to a subsequence (still denoted as $\{f_i\}$), we assume $f_i \to f_\infty$ for some $f_\infty : \mathbb{C}^k \to N$. Clearly $D(f_\infty)(0) = 1$, and hence is nondegenerate.

It seems that our definition of k-hyperbolicity is different from Eisenman's [6] for $k < n$ (cf. [5]), since his definition concerns k-dimensional measure for all real k-dimensional submanifolds, while we are only concerned with the metric/norm on elements in $\wedge^k T'N$, namely only k-dimensional "holomorphic" parallelepipeds in the holomorphic tangent space.

Acknowledgment. We would like to thank Min Ru, Bernard Shiffman,[2] and Fangyang Zheng for their interest in the results related to the k-hyperbolicity. We are grateful to Fangyang Zheng for pointing out a discrepancy in (2.2) in an earlier version, his contribution in the Appendix related to Royden's lemma, and many

[2] Particularly for the reference [6].

discussions. We also thank the referee for corrections/suggestions that improved the presentation of the paper.

Bibliography

[1] Aubin, T. *Nonlinear analysis on manifolds. Monge-Ampère equations.* Grundlehren der mathematischen Wissenschaften, 252. Springer, New York, 1982. doi:10.1007/978-1-4612-5734-9

[2] Cao, H.-D.; Mok, N. Holomorphic immersions between compact hyperbolic space forms. *Invent. Math.* **100** (1990), no. 1, 49–61. doi:10.1007/BF01231180

[3] Cheng, S. Y. Liouville theorem for harmonic maps. *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*. Proc. Sympos. Pure Math., XXXVI, pp. 147–151. Providence, R.I.: American Mathematical Society, 1980.

[4] Cheng, S. Y.; Yau, S.-T. Differential equations on Riemannian manifolds and their geometric applications. *Comm. Pure Appl. Math.* **28** (1975), no. 3, 333–354. doi:10.1002/cpa.3160280303

[5] Demailly, J.-P. Recent results on the Kobayashi and Green-Griffith-Lang conjectures. *Jpn. J. Math.* **15** (2020), no. 1, 1—120. doi:10.1007/s11537-019-1566-3

[6] Eisenman, D. *Intrinsic measures on complex manifolds and holomorphic mappings*. Memoirs of the American Mathematical Society, 96. American Mathematical Society, Providence, R.I., 1970 i+80 pp,

[7] Goldberg, S. *Curvature and homology.* Revised reprint of the 1970 edition. Dover, Mineola, N.Y., 1998.

[8] Gromov, M. Manifolds of negative curvature. *J. Differential Geom.* **13** (1978), no. 2, 223–230.

[9] Gromov, M; Thurston, W. Pinching constants for hyperbolic manifolds. *Invent. Math.* **89** (1987), no. 1, 1–12. doi:10.1007/BF01404671

[10] Gutiérrez, C. E. *The Monge-Ampère equation.* Second edition. Progress in Nonlinear Differential Equations and Their Applications, 89. Birkhäuser/Springer, Cham, 2016. doi:10.1007/978-3-319-43374-5

[11] Hernández, L. Kähler manifolds and 1/4-pinching. *Duke Math. J.* **62** (1991), 601–611. doi:10.1215/S0012-7094-91-06226-5

[12] Hitchin, N. On the curvature of rational surfaces. *Differential geometry (Proc. Sympos. Pure Math., Vol XXVII, Part 2, Stanford Univ., Stanford, Calif., 1973*), 65–80. Amer. Math. Soc., Providence, R.I., 1975.

[13] Jost, J.; Yau, S.-T. Harmonic maps and superrigidity. *Tsing Hua lectures on geometry & analysis*, 213–246. International Press, Cambridge, Mass., 1997.

[14] Kobayashi, S. *Hyperbolic manifolds and holomorphic mappings.* Pure and Applied Mathematics, 2. Marcel Dekker, New York, 1970.

[15] Kobayashi, S. *Hyperbolic complex spaces.* Springer, New York, 1998. doi:10.1007/978-3-662-03582-5

[16] Li, C.; Ni, L.; Zhu, X.-H. An application of a C^2-estimate for a complex Monge-Ampère equation. *International J. Math.* (special volume in honor of S.-S. Chern's 110th anniversary), forthcoming.

[17] Li, P.; Wang, J.-P. Comparison theorem for Kähler manifolds and positivity of spectrum. *J. Differential Geom.* **69** (2005), no. 1, 43–74. doi:10.4310/jdg/1121540339

[18] Liu, G. Three-circle theorem and dimension estimate for holomorphic functions on Kähler manifolds. *Duke Math. J.* **165** (2016), no. 15, 2899–2919. doi:10.1215/00127094-3645009

[19] Mok, N.; Siu, Y.-T.; Yeung, S-K. Geometric superrigidity. *Invent. Math.* **113** (1993), no. 1, 57–83. doi:10.1007/BF01244302

[20] Mori, S. Projective manifolds with ample tangent bundles. *Ann. of Math. (2)* **110** (1979), no. 3, 593–606. doi:10.2307/1971241

[21] Morrow, J.; Kodaira, K. *Complex manifolds.* Holt, Rinehart and Winston, New York–Montreal–London, 1971.

[22] Ni, L. Vanishing theorems on complete Kähler manifolds and their applications. *J. Differential Geom.* **50** (1998), no. 1, 89–122.

[23] Ni, L. Hermitian harmonic maps from complete Hermitian manifolds to complete Riemannian manifolds. *Math. Z.* **232** (1999), no. 2, 331–355. doi:10.1007/s002090050518

[24] Ni, L. General Schwarz lemmata and their applications. *Internat. J. Math.* **30** (2019), no. 13, 1940007, 17 pp. doi:10.1142/S0129167X1940007X

[25] Ni, L. The fundamental group, rational connectedness and the positivity of Kähler manifolds. *J. Reine Angew. Math. (Crelle)*, forthcoming. doi:10.1515/crelle-2020-0040

[26] Ni, L.; Niu, Y. Y. Sharp differential estimates of Li-Yau-Hamilton type for positive (p,p)-forms on Kähler manifolds. *Comm. Pure Appl. Math.* **64** (2011), no. 7, 920–974. doi:10.1002/cpa.20363

[27] Ni, L.; Niu, Y. Y. A gap theorem on Kähler manifold with nonnegative orthogonal bisectional curvature. *J. Reine Angew. Math.* **763** (2020), 111–127. doi:10.1515/crelle-2019-0002

[28] Ni, L.; Tam, L.-F. Plurisbuharmonic functions and the structure of complete Kähler manifolds with nonnegative curvature. *J. Differential Geom.* **64** (2003), no. 3, 457–524.

[29] Ni, L.; Tam, L.-F. Poincaré-Lelong equation via the Hodge-Laplace heat equation. *Compos. Math.* **149** (2013), no. 11, 1856–1870. doi:10.1112/S0010437X12000322

[30] Ni, L.; Wang, K. Isoperimetric comparison via viscosity. *J. Geom. Anal.* **26** (2016), no. 4, 2831–2841. doi:10.1007/s12220-015-9650-2

[31] Ni, L.; Wilking, B. Manifolds with $\frac{1}{4}$-pinched flag curvature. *Geom. Funct. Anal.* **20** (2010), no. 2, 571–591. doi:10.1007/s00039-010-0068-5

[32] Ni, L.; Zheng, F. Comparison and vanishing theorems for Kähler manifolds. *Calc. Var. Partial Differential Equations* **57** (2018), no. 6, Paper No. 151, 31 pp. doi:10.1007/s00526-018-1431-x

[33] Ni, L.; Zheng, F. Positivity and Kodaira embedding theorem. Preprint, 2018. arXiv:1804.09696 [math.DG]

[34] Omori, H. Isometric immersions of Riemannian manifolds. *J. Math. Soc. Japan* **19** (1967), 205–214. doi:10.2969/jmsj/01920205

[35] Royden, H. L. The Ahlfors-Schwarz lemma in several complex variables. *Comment. Math. Helv.* **55** (1980), no. 4, 547–558. doi:10.1007/BF02566705

[36] Siu, Y.-T. *Lecture on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics.* DMV Seminar, 8. Birkhäuser, Basel, 1987. doi:10.1007/978-3-0348-7486-1

[37] Siu, Y.-T.; Yau, S.-T. Compact Kähler manifolds of positive bisectional curvature. *Invent. Math.* **59** (1980), no. 2, 189–204. doi:10.1007/BF01390043

[38] Tosatti, V.; Yang, X. An extension of a theorem of Wu-Yau. *J. Differential Geom.* **107** (2017), no. 3, 573–579. doi:10.4310/jdg/1508551226

[39] Wu, D.; Yau, S.-T. Negative holomorphic curvature and positive canonical bundle. *Invent. Math.* **204** (2016), no. 2, 595–604. doi:10.1007/s00222-015-0621-9

[40] Yang, X. K. RC-positivity, vanishing theorem and rigidity of holomorphic maps. *Cambridge J. Math.* **6** (2018), no. 2, 183–212. doi:10.4310/CJM.2018.v6.n2.a2

[41] Yau, S. T. Intrinsic measures of compact complex manifolds. *Math. Ann.* **212** (1975), 317–329. doi:10.1007/BF01344467

[42] Yau, S.-T. A general Schwarz lemma for Kähler manifolds. *Amer. J. Math.* **100** (1978), no. 1, 197–203. doi:10.2307/2373880

[43] Yau, S.-T.; Zheng, F. Negatively $\frac{1}{4}$-pinched Riemannian metric on a compact Kähler manifold. *Invent. Math.* **103** (1991), no. 3, 527–535. doi:10.1007/BF01239525

[44] Zheng, F. *Complex differential geometry.* AMS/IP, Studies in Advanced Math, 18. American Mathematical Society, Providence, R.I.; International Press, Boston, Mass., 2000.

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Received November 2019.