NOTES ON TRANSNORMAL FUNCTIONS ON RIEMANNIAN MANIFOLDS

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Abstract. This article was part of author's 1993 master thesis at Fudan University. Since the results were cited in more recent literatures it perhaps is helpful to make it available on arXiv.

1. Introduction

Let M^n be a connected complete Riemannian manifold of dimension n, and let f be a smooth function defined on M. If f is not a constant function and there is a smooth (or even C^2) function $b : J := f(M) \to \mathbb{R}$ such that

$$\|\nabla f\|^2 = b(f), \tag{1.1}$$

then we call f a transnormal function. This equation was first studied by Elie Cartan in [4] where he began the project of classifying the isoparametric hypersufaces in the space forms. Later on, this equation appeared in the series of papers [8, 9, 6, 7, 10]. By studying the whole family of hypersurfaces defined by the level sets of the corresponding transnormal (isoparametric) functions, [8, 9] gave surprising restrictions on the isoparametric hypersurfaces in spheres. On general Riemannian manifolds this equation was first studied in [13]. The main result which was proved there is;

(1) There is no critical value in $\mathrm{int}(J)$*. So the focal varieties, i.e. the singular level sets of* f*, are only the level sets corresponding to the maximum or the minimum point of* J *(we denote them by* V_+ *and* V_-*).*

(2) (Theorem A of [13]*.) If* M *is a connected complete Riemannian manifold, and* f *is a transnormal function on* M*, then*

a) The focal varieties of f *are smooth submanifolds of* M*.*

b) Each regular level set of f *is a tube over either of the focal varieties.*

These results are generalizations of the geometry provided by the isoparametric family in [8] and [9]. In this paper, we will show that the existence of transnormal functions puts very strong restriction on the topology and geometry of the manifolds. In particular, we can prove that if a simply-connected compact three manifold supports a transnormal function then this manifold has to be a three-sphere. We can also show that the level hypersurfaces of transnormal function in $\mathbb{S}^n$ are all isoparametric hypersurfaces (i.e. All principle curvatures are constant on the hypersurface). The interesting point is the interaction between topology and geometry, i.e. that the geometry of transnormal functions restrict the topology of the manifold where they are defined and the topological structure, on the other hand, helps us to get more geometric information of the leaves of the foliation provided by the functions. On

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complete manifold, there are plenty of transnormal functions (See next section for examples). But one can give a complete classification of transnormal functions in $\mathbb{R}^n$ and $\mathbb{H}^n$.

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2. Transnormal foliations.

Examples. Before we study the general theory let's start with some concrete examples of transnormal functions.

Example 1. $M = \mathbb{R}^n$, $f = x_1^2 + x_2^2 + \cdots + x_k^2$.

Example 2. $M = \mathbb{S}^n, f = x_1^2 + x_2^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_{n+1}^2$.

Example 3. $M = \mathbb{S}^n$, all the polynomials (of degree 4) constructed using Clifford algebra representations in [7].

Example 4. M is a complete manifold with nonnegative Ricci curvature and a line, then the Busemann fuction is a transnormal function (See [12]). In this case, since the Busemann function satisfies $\|\nabla B\|^2 = 1$ we know that all the level sets are regular and we can have the splitting of the manifold by [13].

Definition 2.1. *(See [B]). A partition F of a complete Riemannian manifold M is called a transnormal foliation (or transnormal system) if every geodesic of M cuts the leaves (the connected elements) of F orthorgonally at none or all of its points. And a transnormal foliation is called regular if all leaves has same codimension. Otherwise it's called singular.*

Proposition 2.1. *Let M be a complete manifold, f be a transnormal function on M. Then the level sets of f yields a transnormal (might be singular) foliation of M.*

Proof. The proof follows directly from the geometry described in [13], mainly the Theorem A there. In fact, in our case we have codimension one foliation i.e. generic leaves are codimension 1. □

Lemma 2.1. *Let M be a compact manifold, F be a singular transnormal foliation of M with only one singular leave S. Then there exists a double cover $\pi : \widetilde{M} \to M$.*

Proof. By the Lemma 1 of [1], we know that the exponential map $\exp_S : N_S \to M$ preserves the leaves of the foliations (Over the N_S , the normal bundle of S in M, the foliation is given by the sections of constant length.) and it must have conjugate locus of S, otherwise, by the Theorem 2 [1] we know that M is diffeomorphic to a vector bundle over S, which is contradictory to the compactness of M. Let us denote the first conjugate locus of $\exp_S$ by $C(S)$. While S is the only singular leave of F we know $\exp_S(C(S)) = S$. And we can assume that $N_{2t}S = \{(s, y) \in N_S \,|\, \|y\| = 2t\}$ is the first conjugate locus. By the above, we know that the cut locus of S is N_tS and we denote $H_t = \exp_S(N_tS)$. Furthermore we have that $\exp_S(N_tS) \to H_t$ is a double cover. If we denote the deck transformation of this double cover by h we can construct $\widetilde{M}$ by gluing two copies of $N_{\le t}S$ along their boundary through h. From the construction it's quite clear that we have double cover $\pi : \widetilde{M} \to M$.□

Theorem 2.2. *If M is a simply-connected compact 3-manifold and with a Riemannian metric g and a smooth function f such that f is transnormal with respect to g. Then M is a three sphere.*

Proof. Let F be the transnormal foliation of M provided by f. By Lemma 2.1 and the simply-connectness of M we know that F has more than one singular leaves. By the Theorem 3 of [1] we know that F has exactly two singular leaves. Let S_1 and S_2 be the two singular leaves and $d_i (i = 1, 2)$ be the codimension of them in M. For (d_1, d_2) we have the following three cases;

i) $(d_1, d_2) = (3, 3)$. In this case, we know S_1 and S_2 are all points (denoted as p and q) then $M = N_{\leq t}\{p\} \cup N \leq t\{q\}$, is a three sphere by Brown's theorem ([2]).

ii) $(d_1, d_2) = (2, 2)$. In this case $S_1 = S_2 = \mathbb{S}^1$, and M is so-called lens space $L(p, q)$ (See, for example, [11] Pages 234-235) and we know that $L(1, q) = \mathbb{S}^3$ and $\pi_1((L(p, q)) = \mathbb{Z}_p$. By the simply-connectness assumption we also know that M is a three sphere.

iii) $(d_1, d_2) = (3, 2)$. But this case can't happen in our situation by some Mayer-Vietoris argument and it was essentially proved in Corollary 1 of [1]. For the sake of completeness we include a simple argument here. We know that $S_1 = p$ and $S_2 = \mathbb{S}^1$ and $M = N_{\leq t}\{p\} \cup N_{\leq t}\mathbb{S}^1$. In particular we have $H_t = \mathbb{S}^2$ and also $H_t = \exp_{\mathbb{S}^1}(N_t\mathbb{S}^1)$. But it's a contradiction since $\pi_1(\mathbb{S}^2) = 0$ and $\pi_1(N_t\mathbb{S}^1) = \mathbb{Z} \oplus \mathbb{Z}$. □

3. Geometric constrains of the transnormal foliation.

In this section we first show that if M happen to be space form with nonnegative curvature the level set of a transnormal function has some interesting geometric properties. Before we prove our results we need to set up some preliminary results. From (1.1) we know that $\frac{\text{grad} f}{\sqrt{b(f)}}$ (over where $b(f) \neq 0$) is a self-parallel vector field (cf.[5]), and the integral curve is geodesic perpendicular to $f^{-1}(\alpha)$. We give the following definition according to this observation.

Definition 3.1. *A geodesic segment $\sigma : (\alpha, \beta) \to M$ is called an f-segment, if $f(\sigma(t))$ is increasing and $\dot{\sigma}(t) = \frac{\text{grad} f}{\sqrt{b(f)}}(b(f) \neq 0)$. And $\sigma(t)$ is called an inverse f-segment if $\sigma(-t)$ is an f-segment(cf.* [13]*).*

To give the description which Wang provided in [13], we need to define the following map $\Phi(t, p)$. Suppose that $\alpha \in \text{int}(J)$, then $M_\alpha := f^{-1}(\alpha)$ is a hypersurface. Let $p \in M_\alpha$, and let $\xi(p)$ be the unit normal vector pointing to the f-increasing direction, then $\exp_p(t\xi(p))$ is the arc-length geodesic starting from p. This is a f-segment. We can define the smooth map $\Phi(t, p) := \exp_p(t\xi(p))$, and let $\phi_t(p) := \Phi(t, p)$. We know from lemmas of [13] that $\phi_t((M_\alpha)$ belongs to the level set of f. When $d\phi_t$ is non-degenerate for $0 \leq t \leq r$, it's a level hypersurface of f, and $d(M_\alpha, \phi_r(M_\alpha)) = r$. The f-segment $\phi_t(p)$ is the minimizing geodesic which joins M_α to $\phi_t(M_\alpha)$.

Remark 3.1. From [13] we know that $V_+ = \phi_r(M_\alpha)$, if r is the first degenerate point of $d\phi_t$.

Definition 3.2. *Let K be a submanifold of M, $p \in K, \sigma(t)$ be a geodesic starting from $p, \dot{\sigma}(0) \perp T_p(K)$, and $Y(t)$ be a Jacobi field along $\sigma(t)$. We call $Y(t)$ a K-Jacobi field provided that it satisfies: $Y(0) \in T_p(K)$, and $S_{\dot{\sigma}(0)}Y(0) + \dot{Y}(0) \in (T_p(K)^\perp$, where S is the second fundamental form of K (cf.* [3]*).*

The relation between the K-Jacobi fields and the function f is given in the following proposition.

Let M_α be a level set. We know from Theorem A of [13] that it's always a manifold, and we can assume that it is connected. Let q be a point of M, but $q \notin M_\alpha$, we can assume $\beta = f(q) > \alpha$. Consider $\sigma(t)$ a f-segment joining $p := \sigma(0)$ to $q = \sigma(\ell)$. We know from [3] that for any two vector $X, Y \in T_q(M_\beta)$, there exists two M_α-Jacobi fields J_i, $i = 1, 2$ such that $J_(\ell) = X, J_2(\ell) = Y$. We have the following proposition.

Proposition 3.1. *For M_β regular,*

$$D^2 f(X, Y) = \sqrt{b(f)}\langle \dot{J}_1(\ell), J_2(\ell)\rangle. \tag{3.1}$$

Proof. We divide the proof into two cases. Case 1). M_β is a hypersurface. In this case we can compute directly by using the concrete construction of J_i. Let $\gamma_i(s)$ be two curves in M_β such that $\gamma_i(0) = q$, $\dot{\gamma}_1(0) = X$ and $\dot{\gamma}_2(0) = Y$. Then by using the exponential map, we can get two families of geodesics $\gamma_i(t, s)$. These two families give two geodesic variations of $\sigma(t)$. Therefore, the variational vector fields are the M_β-Jacobi fields (they are $M\alpha$-Jacobi fields as well). From the construction we know $J_(\ell) = X; J_2(\ell) = Y$, Now we can do the following calculation:

$$\begin{aligned}
D^2 f(X, Y) &= D^2 f(Y, X) = XYf - D_X Y f \\
&= X\langle Y, \operatorname{grad} f\rangle - \langle D_X Y, \operatorname{grad} f\rangle \\
&= \langle Y, D_X \operatorname{grad} f\rangle = \langle Y, D_X(\sqrt{b(f)}\dot{\sigma}\rangle \\
&= \langle Y, \sqrt{b(f)} D_X \dot{\sigma}\rangle + \frac{1}{2} \cdot \frac{b'(f)}{\sqrt{b(f)}} \langle Y, \langle X, \operatorname{grad} f\rangle \dot{\sigma}\rangle \\
&= \sqrt{b(f)}\langle J_2, D_{J_1}\dot{\sigma}\rangle = \sqrt{b(f)}\langle \dot{J}_1, J_2\rangle.
\end{aligned}$$

Case 2). M_β is a focal submanifold. We can construct Jacobi fields as Case 1). From [13] we know that $\sigma(t)$ belongs to the level hypersurfaces, provided $\ell - \epsilon < t < \ell$. So Case 2) follows Case 1) by continuity. □

Remark 3.2. If $\sigma(t)$ is the inverse f-segment, we can get the similar result:

$$D^2 f(X, Y) = -\sqrt{b(f)}\langle \dot{J}_1, J_2\rangle. \tag{3.2}$$

The above proposition relates the Hession of f to the M_α-Jacobi fields. However, we have known the following equation on hypersurface M_α (See Theorem 5.1 on page 268 of [CR]):

$$\langle S_\xi X, Y >= -\frac{D^2 f(X, Y)}{\sqrt{b(f)}}, \tag{3.3}$$

where S is the second fundmental form of the level hypersurface, $\xi = \frac{\operatorname{grad} f}{\sqrt{b(f)}}$.

From the equation (3.1) and the Proposition 3.1 we can calculate the principal curvatures of the level set of f. In the case that M_α is a hypersurface, let $\sigma(t)$ be an f-segment joining M_α to another level hypersurface M_β. Then the M_α-Jacobi fields are the vector fields $J(t)$ satisfying

$$\ddot{J}(t) + R_t J = 0, \quad J(0) \in T_p(M_\alpha), \quad S_{\dot{\sigma}} J(0) = -\dot{J}(0), \tag{3.4}$$

where $R_t J = R(\dot{\sigma}(t), J(t))\dot{\sigma}(t)$. If we can solve (3.4) we can get some information about the principal curvatures of parallel hypersurfaces. Combining the global geometrical structure of transnormal function and the calculation given above, we can prove the following result.

Theorem 3.3. *Let $M = M(c)$ be a Riemannian manifold with nonnegative constant sectional curvature c and f be a transnormal function on M. Then all the regular leaves of the related transnormal foliation are isoparametric hypersurfaces, i.e. all the principal curvatures of the level hypersurfaces are constant on the hypersurface.*

Proof. When $c > 0$, we can assume $c = 1$, and we only need to give the proof for the case $c = 1$. We divide the proof into two cases.

Case 1). V_+ and V_- have codimension greater than 1. In this case we can apply the Theorem A in [13] to prove that the regular level set of f, say $M_\alpha = f^{-1}(\alpha), \alpha \in \text{int}(J)$, is an isoparametric hypersurface.

The map $\phi_r(p)$ defined as before is our starting point. We know from [3] that $d\phi_t(X)$ is the Jacobi field $J(t)$ along the f-segment $\phi_t(p)$, which have the initial value X. Whether $d\phi_r$ is degenerate is determined by if there is a Jacobi-field $J(t)$ with nonzero initial value, but vanishes at $\phi_r(p)$.

We denote $\ell_1 := d(M_\alpha, V_+), \ell_2 := d(V_+, V_-)$. From the facts described in the introduction and the assumption that V_+ and V_- have codimension greater than 1, we know that when $r \neq \ell_1 + k\ell_2$, $k = 0, 1, \cdots, \phi_r$ is a diffeomorphism from M_α to another hypersurface, and $d\phi_r$ degenerates if and only if $r = \ell_1 + k\ell_2$. However, we can solve the equation (3.4) explicitly. Let the principal curvatures at p be $\lambda_1 = \lambda_{1,1} = \lambda_{1,2} = \cdots = \lambda_{1,m_1} > \lambda_2 = \lambda_{2,1} = \lambda_{2,2} = \lambda_{2,m_2} > \cdots >= \lambda_g = \lambda_{g,1} = \lambda_{g,2} = \cdots = \lambda_{g,m_g}$, with corresponding principal vectors $X_{1,1}, X_{1,2}, \cdots, X_{1,m_1}, X_{2,1}, X_{2,2}, \cdots, X_{2,m_2}, \cdots, X_{g,1}, X_{g,2}, \cdots, X_{g,m_g}$, where m_i are the multiplicities of λ_i, g is the number of distinct principal curvatures. Solving the Jacobi equations:

$$\ddot{J}_{i,j}(t) + J_{i,j}(t) = 0, \quad J_{i,j} = X_{i,j}, \quad \dot{J}_{i,j}(0) = -\lambda_{i,j} X_{i,j}$$

we get the solutions

$$J_{i,j}(t) = (\cos t - \lambda_{i,j} \sin t)\widetilde{X}_{i,j}(t) \tag{3.5}$$

where $\widetilde{X}_{i,j}(t)$ is the parallel transport of $X_{i,j}$ along $\phi_t(p)$. So we conclude $J_{i,j}(t) = 0$ if and only if $t = \cot^{-1}(\lambda_{i,1})$. And ϕ_r is diffeomorphism if $0 < r < \ell_1$, hence we have: $\lambda_{1,j} = \cot(\ell_1)$ for $j = 1, \cdots, m_1$.

Similarly, because ϕ_r is a diffeomorphism if $\ell_1 < r < \ell_1 + \ell_2$ and degenerates at $r = \ell_1 + \ell_2$, we have that $\lambda_{2,j} = \cot(\ell_1 + \ell_2)$. Inductively, we can conclude that $\lambda_{i,j} = \cot(\ell_1 + (i-1)\ell_2)$. So $\{\lambda_j\}$ are independent of the point on M_α and we complete the proof in Case 1).

Case 2). Since ϕ_r must degenerate for some r (cf [3]), V_+ and V_- can not be both hypersurfaces. We might as well assume that V_+ has codimension greater than 1 and V_- is a hypersurface. In this case, $\phi_t(p)$ reaches V_+, at $t = \ell_1$, then $f(\phi_t(p))$ begins to decrease until $\phi_t(p)$ reaches V_-, but $\phi_r : M_\alpha \to V_-$ is a diffeomorphism, so $d\phi_{\ell_1+\ell_2}$ does not degenerate. When $t = \ell_1 + 2\ell_2, \phi_t(p)$ reaches V_+ again, and $d\phi_t$ degenerates. By the same way as Case 1) we can get $\lambda_{i,j} = \cot(\ell_1 + 2(i-1)\ell_2)$.

In both cases we all have M_α is isoparametric. For $c = 0$ one can do similarly. □

Remark 3.3. From the above proof we can conclude that $g\ell_2 = \pi$ for Case 1) and $2g\ell_2 = \pi$ for Case 2). This is just the geometry which Münzner described in [8]. And this result might be helpful to the classification of the isoparametric hypersurfaces in spheres.

Remark 3.4. In [13] it was claimed (in Theorem B) that the transnormal functions on $\mathbb{S}^n$ are all isoparametric functions. In fact the claim is not correct and Theorem 3.3 is the right version of the claim.

Theorem 3.4. *If $M = M(0)$, then any transnormal function on M is the function of distance to a totally geodesic submanfold.*

To prove this result we will use the following lemma which is a corollary of Cartan's fundamental equation (see [6], Proposition 4).

Lemma 3.1. *(See [F], Theorem 5.) If N is an isoparametric hypersurface in $M = M(c)$ of constant curvature $c \leq 0$, then the number of distinct principal curvatures g is ≤ 2. For $g = 1$, N is totally umbilic. For $g = 2$, the two distinct principal curvatures satisfy*

$$\lambda_1 \lambda_2 = c. \tag{3.6}$$

Proof of Theorem 3.4. From Theorem 3.3 we know that if f is a transnormal function on M, the level hypersurface $M_\alpha := f^{-1}(\alpha)$ is isoparametric. However, from Lemma 3.1. we know that g, the number of distinct principal curvature, ≤ 2, and at least one of the two principal curvatures is zero. If M is totally geodesic then the principal curvatures are all zero. From the proof of Theorem 3.3, we know that there is no level set of f with codimension greater than one. While $t = \int_\alpha^f \frac{1}{\sqrt{b(\eta)}}\, d\eta$, we can conclude $f = f(t)$, where t is the distance from M_α. In this case M has a topological type $M_\alpha \times \mathbb{R}$ or $M_\alpha \times \mathbb{S}^1$. If M_α is not totally geodesic, from the proof of Theorem 3.3 we know the focal manifold always exists. If λ_i are the principal curvatures of the focal manifold and M_α is the ℓ-tube of the focal manifold V (we can assume this focal manifold V is V_-, and $\dim(V) = m$). Then through the modified calculation of Lemma 3.1, we can get the principal curvatures of M_α are $\frac{\lambda_i}{\ell\lambda_i + 1}$, and $\frac{1}{\ell}$ with the multiplicity of $n - m - 1$. From Lemma 3.1, we know $\lambda_i = 0$, i.e. the focal manifold is totally geodesic.

Definition 3.5. *(See also* [13]*.) If f is a transnormal function on M and the second Beltrami differential of f is also a continuous function of itself, i.e. $\Delta f = a(f)$ for some continuous function a, we call f an isoparametric function. We call the corresponding foliation an isoparametric foliation.*

The further application of the sphere-bundle structure will give further properties on the singular leaves of the isoparametric foliation.

Theorem 3.6. *Let M be a connected Riemannian manifold, F be an isoparametric foliation on M given by an isoparametric function f. Then the singular leaves of F are minimal submanifolds in M.*

Proof. Before the proof we give the following convention of indices: $1 \leq i, j, k \leq m, 1 \leq A, B, C \leq n, m+1 \leq \alpha, \beta, \gamma \leq n$ where $\dim(M) = n + 1$, and m is the dimension of a fixed focal manifold.

We assume M_α is a singular focal manifold, and α is the maximum of f. Then we consider the M_α-Jacobi fields. Let $p \in M_\alpha$, $\xi(p)$ be a normal vector at p, and $e_1, e_2, \cdots, e_{n+1}$ be a local orthonormal frame field such that $e_1, \cdots, e_m \in T_p(M_\alpha)$, and $e_{n+1} = \xi(p)$, $S_{\xi(p)} e_i = -\lambda_i e_i$. Let $\sigma(t)$ be an inverse f-segment starting from p with $\dot{\sigma}(0) = e_{n+1}$. Let $\{e_A(t)\}$ be the parallel transport of $\{e_A\}$. By this convention $e_{n+1}(t) = \dot{\sigma}(t)$. Since $D^2 f(e_{n+1}, e_{n+1}) = \frac{1}{2}b'(f)$, at $\sigma(t)$ we have

$$\left(\Delta f - \frac{1}{2}b'(f)\right) = \sum_{A=1}^{n} D^2 f(e_A(t), e_A(t)). \tag{3.7}$$

Let $\{J_A(t)\}$ be the Jacobi fields which satisfy

$$\ddot{J}_A(t) + R_t J_A(t) = 0;$$

$$J_A(0) = \begin{cases} e_A, & 1 \le A \le m; \\ 0, & m+1 \le A \le n; \end{cases}$$

$$\dot{J}_A(0) = \begin{cases} \lambda_A e_A, & 1 \le A \le m; \\ e_A, & m+1 \le A \le n. \end{cases}$$

Then from (3.7), the tube lemma of [13], and the Remark 3.2, we have

$$\left(\Delta f - \frac{1}{2}b'(f)\right) = - \sum_{1 \le A,B,C \le n} g_{AB} g_{AC} \langle \dot{J}_B, J_C \rangle \sqrt{b(f)}.$$

Here $(g_{AB}(t))$ is given by the equations $e_A(t) = \sum_B g_{AB}(t) J_B(t)$. If we set $H = (h_{AB}) := (g_{AB})^{-1}$, then we have

$$\left(\Delta f - \frac{1}{2}b'(f)\right) = -\sqrt{b(f)}\,\mathrm{trace}(H^{-1} \cdot H_t). \tag{3.8}$$

In the following, we compute $\mathrm{trace}(H^{-1} \cdot H_t)$. Note that H(t) satisfies the matrix equation:

$$\ddot{H}(t) - R(t)H(t) = 0; \quad H(0) = \begin{pmatrix} I_{m\times m} & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$H_t(0) = \begin{pmatrix} \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \lambda_m & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Here $R(t) = (R_{AB}(t))$ with $R_{AB}(t) = -\langle R(\dot\sigma(t), e_A(t))\dot\sigma(t), e_B(t)\rangle$. Then we have the expansion:

$$H^{-1}(t) \quad = \quad \left(I - \frac{t^2}{2} \begin{pmatrix} \frac{R_{11}(0)}{1+t\lambda_1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \frac{R_{mm}(0)}{1+t\lambda_m} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} + o(t^2) \right) \cdot$$

$$\begin{pmatrix} \frac{1}{1+t\lambda_1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \frac{1}{1+t\lambda_m} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \frac{1}{t} \end{pmatrix}.$$

And

$$H_t(t) = \begin{pmatrix} \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \lambda_m & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix} + t \begin{pmatrix} R_{11}(0) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & R_{mm}(0) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} + O(t^2).$$

Hence we obtain

$$H^{-1}H_t = \begin{pmatrix} \frac{\lambda_1}{1+t\lambda_1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \frac{\lambda_m}{1+t\lambda_m} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \frac{1}{t} \end{pmatrix} + O(t). \tag{3.9}$$

From it we have that

$$\mathrm{trace}\left(H^{-1}H_t\right) = \sum_{i=1}^m \frac{\lambda_i}{1+t\lambda_i} + \frac{n-m}{t} + o(1). \tag{3.10}$$

Combining (3.8) and (3.10) we have that

$$-(a(f) - \frac{1}{2}b'(f)) = \sqrt{b(f)} \left(\sum_{i=1}^m \frac{\lambda_i}{1+t\lambda_i} + \frac{n-m}{t} \right) + \sqrt{b(f)} \cdot o(1).$$

From this we deduce

$$\sum_{i=1}^m \frac{\lambda_i}{1+t\lambda_i} = \frac{1}{\sqrt{b(f)}} \left(\frac{1}{2}b'(f) - a(f) - \sqrt{b(f)}\frac{n-m}{t} \right) + o(1).$$

Taking $t \to 0$ we have that

$$\begin{aligned}\sum_{i=1}^{m} \lambda_i &= \lim_{t \to 0} \frac{1}{\sqrt{b(f)}} \left(\frac{1}{2} b'(f) - a(f) - \sqrt{b(f)} \frac{n-m}{t} \right) \\ &= \lim_{f \to \alpha} \frac{1}{\sqrt{b(f)}} \left((m-n) \frac{\sqrt{b(f)}}{\int_f^{\alpha} \frac{1}{\sqrt{b(\eta)}} d\eta} - a(f) + \frac{1}{2} b'(f) \right).\end{aligned}$$

But the right hand side of the above equation is independent of the choice of $\xi(p)$. Hence $\sum_{i=1}^{m} \lambda_i = 0$, i.e. the mean curvature is zero. This proves the focal manifold is minimal. □

Corollary 3.7. *The same conclusion holds if $\Delta f = a(f)$ for $f \in (\alpha - \epsilon, \alpha)$. In particular, the result holds even when $a(\eta)$ is a multiple-valued function. If $\alpha = \max_M f$ we have*

$$\frac{n-m+1}{2} b'(\alpha) = a(\alpha).$$

The same equation holds at $\beta = \min_M f$.

Remark 3.5. Theorem 3.6 was claimed in Theorem D of [13]. But to the best knowledge of the author the proof here is the first one in the literature. The result generalizes the result of Münzner and Nomizu for the focal manifolds of the isoparametric family in $\mathbb{S}^n$, which in turn follows from a Cartan's fundamental identity.

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