JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
S 0894-0347(04)00465-5
Article electronically published on August 27, 2004

A MONOTONICITY FORMULA ON COMPLETE KÄHLER MANIFOLDS WITH NONNEGATIVE BISECTIONAL CURVATURE

LEI NI

0. Introduction

In [Y], Yau proposed to study the uniformization of complete Kähler manifolds with nonnegative curvature. In particular, one wishes to determine whether or not a complete Kähler manifold M with positive bisectional curvature is biholomorphic to $\mathbb{C}^m$. See also [GW], [Si]. For this sake, it was further asked in [Y] whether or not the ring of the holomorphic functions with polynomial growth, which we denote by $\mathcal{O}_P(M)$, is finitely generated, and whether or not the dimension of the spaces of holomorphic functions of polynomial growth is bounded from above by the dimension of the corresponding spaces of polynomials on $\mathbb{C}^m$. This paper addresses the latter questions. We denote by $\mathcal{O}_d(M)$ the space of holomorphic functions of polynomial growth with degree d. (See Section 3 for the precise definition.) Then $\mathcal{O}_P(M) = \bigcup_{d\geq 0}\mathcal{O}_d(M)$. In this paper, we show that

Theorem 0.1. *Let M^m be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Assume that M is of maximum volume growth.*[1]

Then

$$\dim_{\mathbb{C}}(\mathcal{O}_d(M)) \leq \dim_{\mathbb{C}}(\mathcal{O}_{[d]}(\mathbb{C}^m)). \tag{0.1}$$

Here $[d]$ is the greatest integer less than or equal to d. In the case that equality holds in (0.1), M is biholomorphic-isometric to $\mathbb{C}^m$.

Denote by $V_o(r)$ the volume of the ball of radius r centered at o. For manifolds with nonnegative Ricci curvature, $\frac{V_o(r)}{r^{2m}}$ is monotone decreasing by the Bishop volume comparison theorem. M is said to have the maximum volume growth if $\lim_{r\to\infty}\frac{V_o(r)}{r^{2m}} > 0$.

Although we did not prove the finite generation of the ring $\mathcal{O}_P(M)$, we can show that the quotient field generated by $\mathcal{O}_P(M)$ is finitely generated. In fact, this follows from the following rougher dimension estimate for the general case.

Received by the editors July 22, 2003.

2000 *Mathematics Subject Classification.* Primary 58J35, 53C55.

Key words and phrases. Monotonicity formula, holomorphic functions of polynomial growth, heat equation deformation of plurisubharmonic functions.

The author's research was partially supported by NSF grant DMS-0328624, USA.

[1]This sentence may be deleted, thanks to an improvement on Lemma 4.2 of this paper observed by Chen, Fu, Yin, and Zhu.

Theorem 0.2. *Let M^m be a complete Kähler manifold with nonnegative holomorphic bisectional curvature of complex dimension m. There exists a constant $C_1 = C_1(m)$ such that for every $d \geq 1$,*

$$\dim_{\mathbb{C}}(\mathcal{O}_d(M)) \leq C_1 d^m. \tag{0.2}$$

By an argument in [M], which is originally due to Poincaré and Siegel, the above result does imply that the rational functions field $\mathcal{M}(M)$ generated by $\mathcal{O}_P(M)$ is of transcendental degree at most m and is finitely generated. From this one can further construct a birational embedding of M into $\mathbb{C}^{m+2}$ in the case that M has positive bisectional curvature and admits nonconstant holomorphic functions of polynomial growth. In a future publication we shall study the finite generation of $\mathcal{O}_P(M)$ as well as the affine embedding of M, using the results and techniques developed here.

The new idea of this paper is a monotonicity formula for the plurisubharmonic functions, as well as positive currents. In order to illustrate our approach let us recall a classical result attributed to the Bishop-Lelong Lemma.

Let Θ be a closed (p,p) positive current in $\mathbb{C}^m$. Define

$$\nu(\Theta, x, r) = \frac{1}{r^{2m-2p}} \int_{B_x(r)} \Theta \wedge \left(\frac{1}{\pi}\omega_{\mathbb{C}^m}\right)^{m-p}. \tag{0.3}$$

Here $\omega_{\mathbb{C}^m}$ is the Kähler form of $\mathbb{C}^m$. Then

$$\frac{\partial}{\partial r}\nu(\Theta, x, r) \geq 0. \tag{0.4}$$

In particular, this monotonicity formula can be applied to the $(1,1)$ current $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|f|^2$, where f is a holomorphic function. Through the monotonicity (0.4), in [B] Bombieri derived a Schwarz's Lemma type inequality, out of which one can infer that the vanishing order of a polynomial is bounded by its degree.

However, this line of argument encountered difficulties when applied to the non-flat spaces. In [M] Mok made the first such attempt. The following result of Mok in [M] is particularly notable.

Theorem (Mok)**.** *Let M be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose that there exist positive constants C_2 and C_3 such that for some fixed point $o \in M$,*

$$V_o(r) \geq C_2 r^{2m} \tag{0.5}$$

and

$$0 < \mathcal{R}(x) < \frac{C_3}{(1 + r(x))^2}. \tag{0.6}$$

Here $V_o(r)$ is the volume of $B_o(r)$, the ball of radius r centered at o, $\mathcal{R}(x)$ is the scalar curvature function and $r(x)$ is the distance function to o. Then M is biholomorphic to an affine algebraic variety.

The key step of the proof to the above result is to obtain estimate (0.2) or a multiplicity estimate, from which (0.2) can be derived. (See Section 3 for the definition and derivation.) The extra assumptions (0.5) and (0.6) were needed to compensate the failure of (0.4) on curved manifolds.

The main contribution of this paper is to establish a new monotonicity formula on any complete Kähler manifold with nonnegative bisectional curvature.

The monotonicity formula was established through the heat equation deformation of the initial plurisubharmonic functions (or positive $(1,1)$-current). In the case of plurisubharmonic functions, the monotonicity formula has the following simple form. One can refer to Theorem 1.1 for the general case.

Let M be a complete Kähler manifold. Let $v(x,t)$ be a family of plurisubharmonic functions deformed by the heat equation $\left(\frac{\partial}{\partial t} - \Delta\right) v(x,t) = 0$ such that $w(x,t) = \frac{\partial}{\partial t} v(x,t)$ is continuous for each $t > 0$. Then

$$\frac{\partial}{\partial t}\left(tw(x,t)\right) \geq 0. \tag{0.7}$$

Here we assume that $v(x,t)$ is plurisubharmonic just for the sake of simplicity. This assumption in general can be ensured by a recent established maximum principle for tensors on complete manifolds (see [NT2, Theorem 2.1]) if the initial function $v(x,0)$ is plurisubharmonic and of reasonable growth. The monotonicity of $tw(x,t)$ replaces (0.4) in the non-flat case. The dimension estimates in Theorem 0.1 and Theorem 0.2 can be proved by comparing the value of $tw(x,t)$ at $t=0$ with its limit as $t \to \infty$. In the proof of Theorem 0.1, the sharp upper bound on the heat kernel by Li-Tam-Wang [LTW, Theorem 2.1][2] was used. In the proof of Theorem 0.2, we make use of the less precise 'moment' estimates proved in [N1, Theorem 3.1] by the author.

The estimate (0.7) follows from a gradient estimate of Li-Yau type, which resembles the trace form of Hamilton's Li-Yau-Hamilton differential inequality [H], originally also called the differential Harnack inequality, for the Ricci flow. See also [Co] for the Kähler version. Indeed, the derivation of (0.7) was motivated by the earlier work of Chow and Hamilton in [CH] on the linear trace Li-Yau-Hamilton inequality, as well as in [NT1] by Luen-Fai Tam and the author for the Kähler case. In fact, the author discovered (0.7) when trying to generalize Chow's interpolation [C] between Li-Yau's estimates and the linear trace Harnack (Li-Yau-Hamilton) estimates for the Ricci flow on Riemann surfaces to the high dimension.[3] The parabolic approach here is also influenced by a discussion held with G. Perelman, in which Perelman attributed the success of parabolic methods to an 'uncertainty principle'. This suggests that an elliptic method may only be possible after deeper understandings of the geometry of Kähler manifolds with nonnegative curvature, such as a total classification of such manifolds up to biholomorphisms. Since one can think of the heat equation deformation of a plurisubharmonic function as a parabolic deformation of related currents, the work here suggests that there exist strong connections between the Kähler-Ricci flow and the other curvature flows. The recent works of Perelman [P] and Huisken-Sinestrari [HS] also suggest some strong dualities between the Ricci flow and the mean curvature flow. It is not clear whether or not the parabolic deformation of the currents in this paper has any connection with the mean curvature flow. This certainly deserves further deeper investigations in future projects. The previous work [NT2] is also crucial to this paper, especially the tensor maximum principle on complete manifolds [NT2, Theorem 2.1].

[2] It turns out that this is not necessary. Li-Yau's well-known heat kernel estimates are enough. Please see Remark 4.2.

[3] This was successfully carried out in Theorem 1.3, with the generous help from Professor Ben Chow.

There have been many articles on estimating the dimension of the harmonic functions of polynomial growth in the last few years. See, for example, [CM, LT1-2], [LW]. One can refer to [L2] for an updated survey on the subject. The previous results on harmonic functions conclude that the dimension has upper bound of the form $C_5\, d^{2m-1}$, which is sharp in the power for the harmonic functions. Since the space of harmonic functions is far larger than the space of holomorphic functions, the estimate is neither sharp for the holomorphic functions, nor enough for the geometric applications such as the affine embedding considered in the above-mentioned work of Mok. On the other hand, estimate (0.2) is sharp in the power and strong enough to draw some complex geometric conclusions out of it.

In [LW], the problem of obtaining the sharp upper bound for the dimension of the space of harmonic functions of polynomial growth was studied for manifolds with nonnegative sectional curvature and maximum volume growth. An asymptotically sharp estimate was proved there. But the estimate as (0.1) is still missing. Due to the apparent difference in the nature of the two problems, the method in this paper is quite different from the previous papers on harmonic functions. The exceptional cases are either $m = 1$ or $d = 1$. For both cases, the sharp bounds have been proved by Li-Tam [LT1], [LT2].) The consequence on the equality of the estimate (0.1), when $m = 1$, namely the case of Riemann surfaces, was implicit in the work of [LT2], and was also considered in [L1] for $d = 1$.

Combining the estimate (0.3) with Hörmander's L^2-estimate of the $\bar{\partial}$-operator, we can obtain some topological consequences on the complete Kähler manifolds with nonnegative bisectional curvature. For example, we have the following result.

Corollary 0.1. *Let M^m be a complete Kähler manifold with nonnegative bisectional curvature. Assume that the transcendence degree of $\mathcal{M}(M)$, $deg_{tr}(\mathcal{M}(M))$ is equal to m. Then M has finite fundamental group.*

Since it is still unknown whether or not a complete Kähler manifold with positive bisectional curvature is simply-connected, the result above gives some information on this question as well as on the uniformization problem. The assumption $deg_{tr}(\mathcal{M}(M))$ can be replaced by the positivity of the Ricci curvature and some average curvature decay conditions.

The concept of the transcendence degree of $\mathcal{M}(M)$ has a geometric meaning as in classical algebraic geometry. It is the same as the so-called Kodaira dimension of M (denoted by $k(M)$). Please see Section 5 for the details of the definition. In the case that the transcendence degree of $\mathcal{M}(M)$, equivalently $k(M)$, is smaller than the dimension of M^m, we have the following improved version of Theorem 0.1.

Theorem 0.3. *Let M^m be a complete Kähler manifold with nonnegative bisectional curvature. If $k(M) = deg_{tr}(\mathcal{M}(M)) \leq m-1$, we have that*

$$\dim_{\mathbb{C}}(\mathcal{O}_d(M)) \leq \dim_{\mathbb{C}}(\mathcal{O}_{[d]}(\mathbb{C}^{k(M)})). \tag{0.8}$$

The case of equality implies the splitting $M = M_1 \times \mathbb{C}^{k(M)}$, with $\mathcal{O}_P(M_1) = \mathbb{C}$.

Theorems 0.1–0.3 can all be generalized to the spaces of polynomial growth holomorphic sections of holomorphic Hermitian line bundles. Please see Theorem 3.2, Corollary 4.2 and Theorem 4.3 for details.

We organize the paper as follows. In Section 1 we derive the gradient estimate of Li-Yau-Hamilton type for the symmetric tensors, from which the monotonicity formula (0.7) is derived in Section 2. Theorem 0.1 and Theorem 0.2 are proved in

Section 4 and Section 3 respectively. The more general estimates are also proved for the holomorphic sections of polynomial growth for line bundles with finite 'Lelong number at infinity'. (See Section 4 for the precise definition.) As another application, we also include unified treatments on the Liouville theorem for the plurisubharmonic functions on complete Kähler manifolds with nonnegative bisectional curvature (namely, any continuous plurisubharmonic function of $o(\log r)$ growth is a constant), as well as on the optimal gap theorem [NT2, Corollary 6.1]. They can all be phrased as the positivity of the 'Lelong number at infinity' for non-flat, nonnegative holomorphic line bundles. These results, which are presented in Section 2 as a warm-up to the later cases, were originally proved in [NT2] by Luen-Fai Tam and the author, using different methods. In fact, Corollary 6.1 in [NT2] is slightly more general than what was proved in Theorem 2.2 here. Theorem 0.2 is a weaker result than Theorem 0.1. However, it is enough for the geometric purpose in mind. Since its proof is painless once we establish the monotonicity, we treat it separately in Section 3 in order to illustrate the idea first and leave the more technical sharp estimates to Section 4. In Section 5 we prove Corollary 0.1 and Theorem 0.3.

As was pointed out in Remark 4.1, our argument proves Theorem 0.1 for the general case (without the maximum volume growth assumption) if one can prove Lemma 4.2 without assuming the maximum volume growth. Therefore, Theorem 0.1 would hold if one can prove Lemma 4.2 without assuming the maximum volume growth. Recently, B. Chen, X. Fu, L. Yin and X. Zhu proved [CFYZ] that Lemma 4.2 of the current paper indeed holds in general due to the surprising effectiveness of Li-Yau's heat kernel estimates. (The proof is even easier. Please see Remark 4.2 in Section 4.) Hence Theorem 0.1 is now true in general without assuming the maximum volume growth condition. They also proved that the equality case implies that the manifold is isometric to $\mathbb{C}^m$, following the line of arguments in [N2] (the result, in a sense, is a special case of [N2] since $\mathbb{C}^m$ can be viewed as an expanding Kähler-Ricci soliton).[4] The proof in [CFYZ] also needs to appeal a general splitting theorem proved by Luen-Fai Tam and the author in [NT2]. The proof for the equality case presented in the proof of Theorem 4.1 of this paper is simpler.

Acknowledgment

The author would like to thank Professors Peter Li, Luen-Fai Tam and Jiaping Wang for helpful discussions and their interest in this work. The author would also like to thank Professors Ben Chow, John Milnor and the Clay foundation for supporting his trip in April to Simons' lectures at SUNY, Stony Brook. Professor Ben Chow also helped with the proof of Theorem 1.3 by pointing out that certain error terms in the proof can be absorbed into the matrix differential Harnack (LYH) expression. Professor Jiaping Wang suggested to the author that he should look for a quantified version of the Liouville theorem proved earlier in [NT2]. This helped us to formulate results in Section 2. It is a pleasure to record our gratitude to them. Finally, we would like to thank the referee, whose comments were very helpful.

[4]The paper [N2] has been available on arXiv since November of 2002, and the author did send a copy of [N2] to X.-P. Zhu late in 2002. Unfortunately, [N2] was not mentioned in [CFYZ].

1. A new differential inequality of Li-Yau-Hamilton type

In this section we derive a new differential inequality of Li-Yau-Hamilton type for the symmetric (1,1) tensor $h_{\alpha\bar\beta}(x,t)$ satisfying the Lichnerowicz-Laplacian heat equation:

$$\left(\frac{\partial}{\partial t}-\Delta\right)h_{\gamma\bar\delta}=R_{\beta\bar\alpha\gamma\bar\delta}h_{\alpha\bar\beta}-\frac{1}{2}\left(R_{\gamma\bar s}h_{s\bar\delta}+R_{s\bar\delta}h_{\gamma\bar s}\right). \tag{1.1}$$

We assume that $h_{\alpha\bar\beta}(x,t)$ is semi-positive definite (denoted briefly as $h_{\alpha\bar\beta}(x,t)\geq 0$) and that M has nonnegative bisectional curvature. Applying the new estimate of this section to the complex Hessian of a plurisubharmonic function, we can obtain a parabolic version generalization of the classical three-circle theorem for the subharmonic functions on the complex plane to curved high-dimensional Kähler manifolds. The condition $h_{\alpha\bar\beta}(x,t)\geq 0$ can be ensured in most cases, provided that $h_{\alpha\bar\beta}(x,0)\geq 0$, by the maximum principle proved recently in [NT2]. Before we state the result, let us start with a definition.

For any (1, 0) vector field V we define

$$\begin{aligned}Z_h(x,t)=&\frac{1}{2}\left(g^{\alpha\bar\beta}\nabla_{\bar\beta}div(h)_\alpha+g^{\gamma\bar\delta}\nabla_\gamma div(h)_{\bar\delta}\right)+g^{\alpha\bar\beta}div(h)_\alpha V_{\bar\beta}+g^{\gamma\bar\delta}div(h)_{\bar\delta}V_\gamma\\&+g^{\alpha\bar\beta}g^{\gamma\bar\delta}h_{\alpha\bar\delta}V_{\bar\beta}V_\gamma+\frac{H}{t}.\end{aligned} \tag{1.2}$$

Here

$$div(h)_\alpha=g^{\gamma\bar\delta}\nabla_\gamma h_{\alpha\bar\delta},\quad div(h)_{\bar\delta}=g^{\alpha\bar\beta}\nabla_{\bar\beta}h_{\alpha\bar\delta} \tag{1.3}$$

and

$$H=g^{\alpha\bar\beta}h_{\alpha\bar\beta}. \tag{1.4}$$

In the context where the meaning is clear we drop the subscript h in Z_h.

Theorem 1.1. *Let M be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Let $h_{\alpha\bar\beta}(x,t)\geq 0$ be a symmetric $(1,1)$ tensor satisfying (1.1) on $M\times(0,T)$. Assume that for any $\epsilon'>0$,*

$$\int_{\epsilon'}^{T}\int_M e^{-ar^2(x)}\|h\|^2\,dv\,dt<\infty. \tag{1.5}$$

Then

$$Z(x,t)\geq 0, \tag{1.6}$$

for any $(1,0)$ vector V. If $Z(x_0,t_0)=0$ for some point (x_0,t_0) with $t_0>0$ and $h_{\alpha\bar\beta}(x,t)>0$, then M is flat.

The assumption (1.5) is to ensure the validity of the maximum principle on complete noncompact manifolds, which is false in general. In order to prove the theorem, let us start with several lemmas.

Lemma 1.1.

$$\left(\frac{\partial}{\partial t}-\Delta\right)div(h)_\alpha=-\frac{1}{2}R_{\alpha\bar t}div(h)_t,\quad\left(\frac{\partial}{\partial t}-\Delta\right)div(h)_{\bar\alpha}=-\frac{1}{2}R_{\bar\alpha t}\,div(h)_{\bar t}. \tag{1.7}$$

Proof. Direct calculation shows that

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(g^{\gamma\bar{\delta}}\nabla_\gamma h_{\alpha\bar{\delta}}\right) &= g^{\gamma\bar{\delta}}\frac{\partial}{\partial t}\left(\partial_\gamma h_{\alpha\bar{\delta}} - \Gamma^p_{\alpha\gamma}h_{p\bar{\delta}}\right)\\
&= \nabla_\gamma\left(\Delta h_{\alpha\bar{\gamma}} + R_{\alpha\bar{\gamma}s\bar{t}}h_{\bar{s}t} - \frac{1}{2}R_{\alpha\bar{t}}h_{t\bar{\gamma}} - \frac{1}{2}R_{t\bar{\gamma}}h_{\alpha\bar{t}}\right)\\
&= \nabla_\alpha R_{s\bar{t}}h_{\bar{s}t} + R_{\alpha\bar{\gamma}s\bar{t}}\nabla_\gamma h_{\bar{s}t} - \frac{1}{2}\nabla_\gamma R_{\alpha\bar{t}}h_{t\bar{\gamma}}\\
&\quad - \frac{1}{2}R_{\alpha\bar{t}}\nabla_\gamma h_{t\bar{\gamma}} - \frac{1}{2}\nabla_t Rh_{\alpha\bar{t}} - \frac{1}{2}R_{t\bar{\gamma}}\nabla_\gamma h_{\alpha\bar{t}} + \nabla_\gamma(\Delta h_{\alpha\bar{\gamma}}).
\end{aligned}
\tag{1.8}
$$

Now we calculate $\nabla_\gamma(\Delta h_{\alpha\bar{\gamma}})$. By definition,

$$
\nabla_\gamma(\Delta h_{\alpha\bar{\gamma}}) = \frac{1}{2}\nabla_\gamma\left(\nabla_s\nabla_{\bar{s}} + \nabla_{\bar{s}}\nabla_s\right)h_{\alpha\bar{\gamma}}.
$$

On the other hand,

$$
\begin{aligned}
\nabla_\gamma\nabla_s\nabla_{\bar{s}}h_{\alpha\bar{\gamma}} &= \nabla_s\nabla_\gamma\nabla_{\bar{s}}h_{\alpha\bar{\gamma}}\\
&= \nabla_s\left[\nabla_{\bar{s}}\nabla_\gamma h_{\alpha\bar{\gamma}} - R_{\alpha\bar{p}\gamma\bar{s}}h_{p\bar{\gamma}} + R_{p\bar{\gamma}\gamma\bar{s}}h_{\alpha\bar{p}}\right]\\
&= \nabla_s\nabla_{\bar{s}}\nabla_\gamma h_{\alpha\bar{\gamma}} - \nabla_\gamma R_{\alpha\bar{p}}h_{p\bar{\gamma}} - R_{\alpha\bar{p}\gamma\bar{s}}\nabla_s h_{p\bar{\gamma}} + \nabla_p Rh_{\alpha\bar{p}} + R_{p\bar{s}}\nabla_s h_{\alpha\bar{p}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\nabla_\gamma\nabla_{\bar{s}}\nabla_s h_{\alpha\bar{\gamma}} &= \nabla_{\bar{s}}\nabla_\gamma\nabla_s h_{\alpha\bar{\gamma}} + R_{p\bar{\gamma}\gamma\bar{s}}\nabla_s h_{\alpha\bar{p}} - R_{s\bar{p}\gamma\bar{s}}\nabla_p h_{\alpha\bar{\gamma}} - R_{\alpha\bar{p}\gamma\bar{s}}\nabla_s h_{p\bar{\gamma}}\\
&= \nabla_{\bar{s}}\nabla_s\nabla_\gamma h_{\alpha\bar{\gamma}} + R_{p\bar{s}}\nabla_s h_{\alpha\bar{p}} - R_{\gamma\bar{p}}\nabla_p h_{\alpha\bar{\gamma}} - R_{\alpha\bar{p}\gamma\bar{s}}\nabla_s h_{p\bar{\gamma}}.
\end{aligned}
$$

Combining the above three equations we have that

$$
\begin{aligned}
\nabla_\gamma(\Delta h_{\alpha\bar{\gamma}}) &= \Delta(\nabla_\gamma h_{\alpha\bar{\gamma}}) - \frac{1}{2}\nabla_\gamma R_{\alpha\bar{p}}h_{p\bar{\gamma}} - R_{\alpha\bar{p}\gamma\bar{s}}\nabla_s h_{p\bar{\gamma}}\\
&\quad + \frac{1}{2}\nabla_p Rh_{\alpha\bar{p}} + R_{p\bar{s}}\nabla_s h_{\alpha\bar{p}} - \frac{1}{2}R_{\gamma\bar{p}}\nabla_p h_{\alpha\bar{\gamma}}.
\end{aligned}
$$

Plugging the above equation into (1.8), the first equation in the lemma is proved. The second one is the conjugation of the first. □

Lemma 1.2.

$$
\left(\frac{\partial}{\partial t} - \Delta\right)\left(g^{\alpha\bar{\beta}}\nabla_{\bar{\beta}}div(h)_\alpha\right) = 0, \qquad \left(\frac{\partial}{\partial t} - \Delta\right)\left(g^{\alpha\bar{\beta}}\nabla_\beta div(h)_{\bar{\alpha}}\right) = 0.
\tag{1.9}
$$

Proof. This follows from Lemma 1.1 and routine calculations. Indeed,

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(g^{\alpha\bar{\beta}}\nabla_{\bar{\beta}}div(h)_\alpha\right) &= \nabla_{\bar{\alpha}}\left(\frac{\partial}{\partial t}div(h)_\alpha\right)\\
&= \nabla_{\bar{\alpha}}\left[\Delta div(h)_\alpha - \frac{1}{2}R_{\alpha\bar{t}}div(h)_t\right],
\end{aligned}
$$

by Lemma 1.1. Therefore we have that

$$
\frac{\partial}{\partial t}\left(g^{\alpha\bar{\beta}}\nabla_{\bar{\beta}}div(h)_\alpha\right) = \nabla_{\bar{\alpha}}\left(\Delta div(h)_\alpha\right) - \frac{1}{2}R_{s\bar{\alpha}}\nabla_{\bar{s}}\left(div(h)_\alpha\right) - \frac{1}{2}\nabla_{\bar{t}}R\left(div(h)_t\right).
\tag{1.10}
$$

Now we calculate $\nabla_{\bar{\alpha}}\left(\Delta div(h)_\alpha\right)$. By definition,

$$
\nabla_{\bar{\alpha}}\left(\Delta div(h)_\alpha\right) = \frac{1}{2}\nabla_{\bar{\alpha}}\nabla_s\nabla_{\bar{s}}div(h)_\alpha + \frac{1}{2}\nabla_{\bar{\alpha}}\nabla_{\bar{s}}\nabla_s div(h)_\alpha.
$$

On the other hand,

$$\begin{aligned}\nabla_{\bar\alpha}\nabla_{\bar s}\nabla_s div(h)_\alpha &= \nabla_{\bar s}\nabla_{\bar\alpha}\nabla_s div(h)_\alpha\\ &= \nabla_{\bar s}\left[\nabla_s\nabla_{\bar\alpha} div(h)_\alpha + R_{\alpha\bar p s\bar\alpha} div(h)_p\right]\\ &= \nabla_{\bar s}\nabla_s\nabla_{\bar\alpha} div(h)_\alpha + (\nabla_{\bar s}R)\,(div(h)_s) + R_{s\bar p}\nabla_{\bar s} div(h)_p\end{aligned}$$

and

$$\begin{aligned}\nabla_{\bar\alpha}\nabla_s\nabla_{\bar s} div(h)_\alpha &= \nabla_s\nabla_{\bar\alpha}\nabla_{\bar s} div(h)_\alpha + R_{\bar p s}\nabla_{\bar s} div(h)_p - R_{p\bar\alpha}\nabla_{\bar p} div(h)_\alpha\\ &= \nabla_s\nabla_{\bar s}\nabla_{\bar\alpha} div(h)_\alpha + R_{\bar p s}\nabla_{\bar s} div(h)_p - R_{p\bar\alpha}\nabla_{\bar p} div(h)_\alpha.\end{aligned}$$

Combining the above three equalities we have that

$$\nabla_{\bar\alpha}\left(\Delta div(h)_\alpha\right) = \Delta\left(\nabla_{\bar\alpha} div(h)_\alpha\right) + \frac{1}{2}\nabla_{\bar s}R(div(h)_s) + \frac{1}{2}R_{s\bar p}\nabla_{\bar s} div(h)_p.$$

Plugging into (1.10), this completes the proof of the first equation of Lemma 1.2. The second one is the conjugation of the first. ☐

Since $h_{\alpha\bar\beta} \geq 0$ only (not strictly positive), we need the perturbation trick of [NT1]. Namely, we consider

$$\begin{aligned}\widehat{Z} = \frac{1}{2}&\left(g^{\alpha\bar\beta}\nabla_{\bar\beta} div(h)_\alpha + g^{\gamma\bar\delta}\nabla_\gamma div(h)_{\bar\delta}\right) + g^{\alpha\bar\beta} div(h)_\alpha V_{\bar\beta} + g^{\gamma\bar\delta} div(h)_{\bar\delta} V_\gamma\\ &+ g^{\alpha\bar\beta}g^{\gamma\bar\delta}\left(h_{\alpha\bar\delta} + \epsilon g_{\alpha\bar\delta}\right)V_{\bar\beta}V_\gamma + \frac{H+\epsilon m}{t}.\end{aligned}\tag{1.11}$$

We can simply denote $\widetilde{h}_{\alpha\bar\beta} = h_{\alpha\bar\beta} + \epsilon g_{\alpha\bar\beta}$, which is strictly positive definite. Let V be the vector field which minimizes $\widehat{Z}$. Then the first variation formula gives

$$div(h)_\alpha + \widetilde{h}_{\alpha\bar\gamma}V_\gamma = 0 \quad\text{and}\quad div(h)_{\bar\alpha} + \widetilde{h}_{\gamma\bar\alpha}V_{\bar\gamma} = 0.\tag{1.12}$$

Differentiating (1.12) we have that

$$\begin{aligned}\nabla_s div(h)_\alpha + (\nabla_s h_{\alpha\bar\gamma})\,V_\gamma + \widetilde{h}_{\alpha\bar\gamma}\nabla_s V_\gamma &= 0,\\ \nabla_s div(h)_{\bar\alpha} + (\nabla_s h_{\gamma\bar\alpha})\,V_{\bar\gamma} + \widetilde{h}_{\gamma\bar\alpha}\nabla_s V_{\bar\gamma} &= 0,\\ \nabla_{\bar s} div(h)_\alpha + (\nabla_{\bar s} h_{\alpha\bar\gamma})\,V_\gamma + \widetilde{h}_{\alpha\bar\gamma}\nabla_{\bar s} V_\gamma &= 0,\\ \nabla_{\bar s} div(h)_{\bar\alpha} + (\nabla_{\bar s} h_{\gamma\bar\alpha})\,V_{\bar\gamma} + \widetilde{h}_{\gamma\bar\alpha}\nabla_{\bar s} V_{\bar\gamma} &= 0.\end{aligned}\tag{1.13}$$

From (1.12) we also have the following alternative form of $\widehat{Z}$:

$$\widehat{Z} = -\frac{1}{2}\widetilde{h}_{\alpha\bar\beta}\nabla_{\bar\alpha}V_\beta - \frac{1}{2}\widetilde{h}_{\beta\bar\alpha}\nabla_\alpha V_{\bar\beta} + \frac{H+\epsilon m}{t}.\tag{1.14}$$

On the other hand, by Lemmas 1.1 and 1.2 we have that

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)\widehat{Z} = & \, div(h)_\alpha \left(\left(\frac{\partial}{\partial t}-\Delta\right)V_{\bar\alpha}\right) + div(h)_{\bar\alpha}\left(\left(\frac{\partial}{\partial t}-\Delta\right)V_\alpha\right)\\
& - \nabla_s div(h)_\alpha \nabla_{\bar s}V_{\bar\alpha} - \nabla_{\bar s} div(h)_\alpha \nabla_s V_{\bar\alpha}\\
& - \nabla_s div(h)_{\bar\alpha} \nabla_{\bar s}V_\alpha - \nabla_{\bar s} div(h)_{\bar\alpha} \nabla_s V_\alpha\\
& - \frac{1}{2}R_{\alpha\bar t}div(h)_t V_{\bar\alpha} - \frac{1}{2}R_{t\bar\alpha}div(h)_{\bar t}V_\alpha\\
& + R_{\alpha\bar\beta s\bar t}h_{\bar s t}V_\beta V_{\bar\alpha} - \frac{1}{2}R_{\alpha\bar s}h_{s\bar\gamma}V_\gamma V_{\bar\alpha} - \frac{1}{2}h_{\alpha\bar s}R_{s\bar\gamma}V_\gamma V_{\bar\alpha}\\
& + \widetilde{h}_{\alpha\bar\gamma}\left(\left(\frac{\partial}{\partial t}-\Delta\right)V_\gamma\right)V_{\bar\alpha} + \widetilde{h}_{\alpha\bar\gamma}V_\gamma\left(\left(\frac{\partial}{\partial t}-\Delta\right)V_{\bar\alpha}\right)\\
& - \nabla_s h_{\alpha\bar\gamma}\nabla_{\bar s}\left(V_\gamma V_{\bar\alpha}\right) - \nabla_{\bar s}h_{\alpha\bar\gamma}\nabla_s\left(V_\gamma V_{\bar\alpha}\right)\\
& - \widetilde{h}_{\alpha\bar\gamma}\left[\nabla_s V_\gamma \nabla_{\bar s}V_{\bar\alpha} + \nabla_{\bar s}V_\gamma\nabla_s V_{\bar\alpha}\right] - \frac{H+\epsilon m}{t^2}.
\end{aligned} \tag{1.15}
$$

Lemma 1.3.

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)\widehat{Z} = & \, \widetilde{h}_{\gamma\bar\alpha}\left[\nabla_p V_{\bar\gamma} - \frac{1}{t}g_{p\bar\gamma}\right]\left[\nabla_{\bar p}V_\alpha - \frac{1}{t}g_{\bar p\alpha}\right] + \widetilde{h}_{\gamma\bar\alpha}\nabla_{\bar p}V_{\bar\gamma}\nabla_p V_\alpha\\
& + R_{\alpha\bar\beta s\bar t}h_{\bar s t}V_\beta V_{\bar\alpha} - \frac{2\widehat{Z}}{t}.
\end{aligned} \tag{1.16}
$$

Proof. Using (1.12), (1.13) we can simplify (1.15) to

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)\widehat{Z} = & \, R_{\alpha\bar\beta s\bar t}h_{\bar s t}V_\beta V_{\bar\alpha}\\
& + \widetilde{h}_{\gamma\bar\alpha}\nabla_s V_{\bar\gamma}\nabla_{\bar s}V_\alpha + \widetilde{h}_{\gamma\bar\alpha}\nabla_{\bar s}V_{\bar\gamma}\nabla_s V_\alpha - \frac{H+\epsilon m}{t^2}.
\end{aligned} \tag{1.17}
$$

Combining with (1.14) we have the lemma. □

In order to apply the maximum principle on complete manifolds and prove the theorem we need the following result.

Lemma 1.4. *Under the assumption of Theorem 1.1, for any $\eta > 0$,*

$$\int_\eta^T \int_M e^{-ar^2(x)}\|div(h)\|^2\, dv\, dt < \infty \tag{1.18}$$

and

$$\int_\eta^T \int_M e^{-ar^2(x)}\left(\|\nabla_s div(h)_\alpha\|^2 + \|\nabla_{\bar s}div(h)_\alpha\|^2\right) dv\, dt < \infty. \tag{1.19}$$

Proof. To simplify the notation we first define

$$
\begin{aligned}
\Phi &= \|h\|^2,\\
\Psi &= \|div(h)\|^2,\\
\Lambda &= \|\nabla_s div(h)_\alpha\|^2 + \|\nabla_{\bar s}div(h)_\alpha\|^2.
\end{aligned}
$$

From (1.1), we have that

$$\left(\Delta - \frac{\partial}{\partial t}\right)\Phi \geq \Psi. \tag{1.20}$$

Here we have used the fact that M has nonnegative holomorphic bisectional curvature. The reader can refer to [NT2, Lemma 2.2] for a detailed proof of (1.20). It follows from an argument which goes back to Bishop and Goldberg [BG]. (See also [MSY].)

Let ϕ be a cut-off function such that $\phi = 0$ for $r(x) \geq 2R$ or $t \leq \frac{\eta}{2}$ and $\phi = 1$ for $r(x) \leq R$ and $t \geq \eta$. Multiplying by ϕ^2 on both sides of (1.20), integration by parts gives that

$$\int_0^T \int_M \Psi \phi^2 \, dv \, dt \leq 2 \int_0^T \int_M \Phi \left(|(\phi^2)_t| + 4|\nabla \phi|^2 \right) \, dv \, dt.$$

Then (1.18) follows from the assumption (1.5).

To prove (1.19) we need to calculate $\left(\Delta - \frac{\partial}{\partial t}\right) \Psi$. From Lemma 1.1, it is easy to obtain

$$\begin{aligned} \left(\Delta - \frac{\partial}{\partial t}\right) \Psi &= \Lambda + R_{\alpha \bar{t}} div(h)_{\bar{\alpha}} div(h)_t \\ &\geq \Lambda \end{aligned} \tag{1.21}$$

since M has nonnegative Ricci curvature. Repeating the above argument in the proof of (1.18) we can obtain the integral estimate on Λ. Hence we complete the proof of the lemma. □

Proof of Theorem 1.1. By translating the time and the limiting argument we can assume that $h_{\alpha\bar{\beta}}$ is well-defined on $M \times [0, t]$. Since $\widetilde{h}_{\alpha\bar{\beta}} \geq \epsilon g_{\alpha\bar{\beta}}$ on $M \times [0, T]$, by (1.12) and (1.13), we have

$$||V|| \leq C_1 \Psi^{\frac{1}{2}}$$

and

$$||\nabla V|| \leq C_2 \left(\Lambda^{\frac{1}{2}} + \Psi \right),$$

for some constants C_1 and C_2. Combining this with (1.14), we have that

$$|t^2 \widehat{Z}|^2 \leq C_3 t \left(\Phi + \Phi(\Psi^2 + \Lambda) + 1 \right) \tag{1.22}$$

for some constant C_3. By (1.16), the corresponding $t^2 \widehat{Z}$ satisfies that

$$\left(\frac{\partial}{\partial t} - \Delta \right) (t^2 \widehat{Z}) \geq 0 \tag{1.23}$$

for the vector field which minimizes $\widehat{Z}$. By Lemma 1.4 and (1.22), we have that

$$\int_0^T \int_M \exp(-a r_0^2(x)) \left(t^2 \widehat{Z} \right)^2 \, dv \, dt < \infty$$

for any $a > 0$. By the maximum principle of Karp-Li [KL] (see also [NT1, Theorem 1.1]), we have $t^2 \widehat{Z} \geq 0$ because it is obvious that $t^2 \widehat{Z} = 0$ at $t = 0$. Since this is true for the vector field V minimizing $\widehat{Z}$, we have $\widehat{Z} \geq 0$ for any (1,0) vector field. Letting $\epsilon \to 0$, we complete the proof of the fact that $Z(z, t) \geq 0$. If the equality holds, as in [N2, page 16], by the strong maximum principle and the right-hand side of (1.16) one has that

$$\nabla_\alpha V_{\bar{\beta}} \equiv \frac{1}{t} g_{\alpha\bar{\beta}}$$

and

$$\nabla_\alpha V_\beta \equiv 0.$$

This implies that locally V is given by the gradient of a holomorphic function f. In particular, $f_{\alpha\bar\beta} = \frac{1}{t} g_{\alpha\bar\beta}$ and $f_{\alpha\beta} = 0$. The flatness follows from the curvature expression in terms of metrics/potential functions and taking derivatives on the above-mentioned two sets of equalities. Namely, from the well-known expression (see, for example, [KM, page 117])

$$R_{\alpha\bar\beta\gamma\bar\delta} = -\frac{\partial^4(tf)}{\partial z^\alpha \partial z^{\bar\beta} \partial z^\gamma \partial z^{\bar\delta}} + g^{p\bar q}\left(\frac{\partial^3(tf)}{\partial z^{\bar q}\partial z^\alpha \partial z^\gamma}\right)\left(\frac{\partial^3(tf)}{\partial z^p \partial z^{\bar\beta}\partial z^{\bar\delta}}\right)$$

one can conclude that M is flat due to $f_{\alpha\beta} \equiv 0$. □

Remark 1.1. Theorem 1.1 was motivated by the so-called linear trace Li-Yau-Hamilton inequality for Ricci flow. The linear trace Li-Yau-Hamilton inequality of the real case was first proved in [CH] by Chow and Hamilton. In [NT1], the authors proved the corresponding result for the Kähler-Ricci flow. In fact, we can state Theorem 1.2 in [NT1] in a slightly more general way such that it can be used in classifying the Kähler-Ricci solitons, which was done in [N2]. The proof of the following Theorem 1.2 remains unchanged as in [NT1].

Theorem 1.2. *Let $(M, g_{\alpha\bar\beta}(x,t))$ be a complete solution to the Kähler-Ricci flow on $M \times (0,T)$ with nonnegative bisectional curvature. Assume that the curvature is uniformly bounded on $M \times \{t\}$ for any $t > 0$. Let h be a symmetric $(1,1)$ tensor satisfying (1.1). Assume also that $h_{\alpha\bar\beta}(x,t) \ge 0$ and (1.5) holds. Then*

$$\widetilde{Z} \ge 0 \tag{1.24}$$

where

$$\begin{aligned}\widetilde{Z} &= \frac{1}{2}[g^{\alpha\bar\beta}\nabla_{\bar\beta} div(h)_\alpha + g^{\gamma\bar\delta}\nabla_\gamma div(h)_{\bar\delta}] \\ &\quad + g^{\alpha\bar\beta}g^{\gamma\bar\delta}[R_{\alpha\bar\delta}h_{\gamma\bar\beta} + \nabla_\gamma h_{\alpha\bar\delta}V_{\bar\beta} + \nabla_{\bar\beta}h_{\alpha\bar\delta}V_\gamma + h_{\alpha\bar\delta}V_{\bar\beta}V_\gamma] + \frac{H}{t}.\end{aligned} \tag{1.25}$$

The equality in (1.24) holding for some positive time implies that $(M, g(t))$ is an expanding gradient Kähler-Ricci soliton provided that $h_{\alpha\bar\beta}(x,t) > 0$ and M is simply-connected.

Besides relaxing the assumption on $h_{\alpha\bar\beta}(x,t)$, another main advantage of stating the result as above is that the form here can be applied to cases without considering the initial value problem. The form stated in [CH], as well as in [NT1], with the initial value prevents the application to the expanding solitons. It is also more clear to separate the issue of preserving the nonnegativity of $h_{\alpha\bar\beta}(x,t)$ from the nonnegativity of $\widetilde{Z}$.

In fact, by combining the proof of Theorem 1.2 of [NT1] and the proof of Theorem 1.1 we can obtain the high-dimensional generalization of Chow's interpolation between Li-Yau's gradient estimate and Chow-Hamilton's linear trace differential Harnack inequality. Here the role of Li-Yau's inequality in [C] is replaced by the inequality proved in Theorem 1.1, and Chow-Hamilton's inequality is replaced by its Kähler analogue proved by Tam and the author, namely the inequality in Theorem 1.2. More precisely, for $1 \ge \tau \ge 0$, consider the Kähler-Ricci flow with speed adjusted by τ:

$$\frac{\partial}{\partial t} g_{\alpha\bar\beta}(x,t) = -\tau R_{\alpha\bar\beta}(x,t). \tag{1.26}$$

Define

$$Z^{(\tau)}(x,t) = Z(x,t) + \tau\left(g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} R_{\alpha\bar{\delta}} h_{\gamma\bar{\beta}}\right)(x,t).$$

Then we have the following result, which interpolates between Theorem 1.1 and Theorem 1.2.

Theorem 1.3. *Let $(M, g(t))$ be a solution to (1.26) with bounded nonnegative bisectional curvature. Let $h_{\alpha\bar{\beta}}(x,t)$ be the symmetric tensor as in Theorem 1.2. Then $Z^{(\tau)}(x,t) \geq 0$. Moreover, the equality holding for some $t > 0$ implies that $(M, g(t))$ is an expanding gradient soliton, provided that $h_{\alpha\bar{\beta}}(x,t) > 0$ and M is simply-connected.*

Proof. Tracing the computation in the proof of Theorem 1.1 and Theorem 1.2 in [NT1] we have that

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) Z^{(\tau)} = {} & Y_1^{(\tau)} + \frac{\tau}{2}\left[R_{\alpha\bar{p}} \nabla_p div(h)_{\bar{\alpha}} + R_{p\bar{\alpha}} \nabla_{\bar{p}} div(h)_\alpha\right] \\
& + \frac{\tau}{2}\left[R_{s\bar{p}} \nabla_\alpha \nabla_{\bar{s}} h_{p\bar{\alpha}} + R_{s\bar{p}} \nabla_{\bar{\alpha}} \nabla_p h_{\alpha\bar{s}}\right] + \tau R_{\alpha\bar{\beta}s\bar{t}} R_{\bar{\alpha}\beta} h_{t\bar{s}} \\
& + \tau(\tau-1) R_{\alpha\bar{\beta}} R_{\bar{\alpha}s} h_{\beta\bar{s}} + \tau R_{\bar{\alpha}\beta} div(h)_\alpha V_{\bar{\beta}} + R_{\bar{\alpha}\beta} div(h)_{\bar{\beta}} V_\alpha \\
& + div(h)_\alpha \left(\left(\frac{\partial}{\partial t} - \Delta\right) V_{\bar{\alpha}}\right) + div(h)_{\bar{\alpha}} \left(\left(\frac{\partial}{\partial t} - \Delta\right) V_\alpha\right) \\
& - \nabla_s div(h)_\alpha \nabla_{\bar{s}} V_{\bar{\alpha}} - \nabla_{\bar{s}} div(h)_\alpha \nabla_s V_{\bar{\alpha}} \\
& - \nabla_s div(h)_{\bar{\alpha}} \nabla_{\bar{s}} V_\alpha - \nabla_{\bar{s}} div(h)_{\bar{\alpha}} \nabla_s V_\alpha \\
& + \left(\tau R_{s\bar{t}} \nabla_t h_{\alpha\bar{s}} V_{\bar{\alpha}} - \frac{1}{2} R_{\alpha\bar{t}} div(h)_t V_{\bar{\alpha}} + \tau R_{s\bar{t}} \nabla_{\bar{s}} h_{t\bar{\alpha}} V_\alpha \right. \\
& \qquad \left. - \frac{1}{2} R_{t\bar{\alpha}} div(h)_{\bar{t}} V_\alpha\right) \\
& + (\tau - \frac{1}{2})\left(R_{\alpha\bar{s}} h_{s\bar{\gamma}} V_\gamma V_{\bar{\alpha}} + h_{\alpha\bar{s}} R_{s\bar{\gamma}} V_\gamma V_{\bar{\alpha}}\right) \\
& + h_{\alpha\bar{\gamma}} \left(\left(\frac{\partial}{\partial t} - \Delta\right) V_\gamma\right) V_{\bar{\alpha}} + h_{\alpha\bar{\gamma}} V_\gamma \left(\left(\frac{\partial}{\partial t} - \Delta\right) V_{\bar{\alpha}}\right) \\
& - \nabla_s h_{\alpha\bar{\gamma}} \nabla_{\bar{s}} \left(V_\gamma V_{\bar{\alpha}}\right) - \nabla_{\bar{s}} h_{\alpha\bar{\gamma}} \nabla_s \left(V_\gamma V_{\bar{\alpha}}\right) \\
& - h_{\alpha\bar{\gamma}} \left[\nabla_s V_\gamma \nabla_{\bar{s}} V_{\bar{\alpha}} + \nabla_{\bar{s}} V_\gamma \nabla_s V_{\bar{\alpha}}\right] - \frac{H}{t^2},
\end{aligned} \tag{1.27}$$

where

$$\begin{aligned}
Y_1^{(\tau)} &= \left[\tau^2 \Delta R_{s\bar{t}} + \tau^2 R_{s\bar{t}\alpha\bar{\beta}} R_{\bar{\alpha}\beta} + \tau \nabla_\alpha R_{s\bar{t}} V_{\bar{\alpha}} + \tau \nabla_{\bar{\alpha}} R_{s\bar{t}} V_\alpha + R_{s\bar{t}\alpha\bar{\beta}} V_{\bar{\alpha}} V_\beta + \tau \frac{R_{s\bar{t}}}{t}\right] h_{\bar{s}t} \\
&= \tau^2 \left[\Delta R_{s\bar{t}} + R_{s\bar{t}\alpha\bar{\beta}} R_{\bar{\alpha}\beta} + \nabla_\alpha R_{s\bar{t}} \frac{V_{\bar{\alpha}}}{\tau} + \nabla_{\bar{\alpha}} R_{s\bar{t}} \frac{V_\alpha}{\tau} + R_{s\bar{t}\alpha\bar{\beta}} \frac{V_{\bar{\alpha}}}{\tau} \frac{V_\beta}{\tau} + \frac{R_{s\bar{t}}}{\tau t}\right] h_{\bar{s}t} \\
&\geq 0.
\end{aligned} \tag{1.28}$$

Here we have used Cao's result [Co], applied to the vector field $\frac{V}{\tau}$, for our speed adjusted Kähler-Ricci flow. A similar simplification, utilizing the equations satisfied by the minimizing vector fields as before, gives that

$$Z^{(\tau)} = \tau R_{\alpha\bar\beta} h_{\bar\alpha\beta} - \frac{1}{2} h_{\alpha\bar\beta} \nabla_{\bar\alpha} V_\beta - \frac{1}{2} h_{\alpha\bar\beta} \nabla_\beta V_{\bar\alpha} + \frac{H}{t}$$

and

$$\left(\frac{\partial}{\partial t} - \Delta\right) Z^{(\tau)} = Y_1^{(\tau)} + Y_2^{(\tau)} - 2\frac{Z^{(\tau)}}{t} \tag{1.29}$$

where
(1.30)

$$Y_2^{(\tau)} = h_{\gamma\bar\alpha}\left[\nabla_p V_{\bar\gamma} - \tau R_{p\bar\gamma} - \frac{1}{t} g_{p\bar\gamma}\right]\left[\nabla_{\bar p} V_\alpha - \tau R_{\alpha\bar p} - \frac{1}{t} g_{\bar p\alpha}\right] + h_{\gamma\bar\alpha}\nabla_{\bar p} V_{\bar\gamma}\nabla_p V_\alpha \geq 0.$$

The theorem follows, by an argument similar to before, from (1.28)–(1.30). ☐

Theorem 1.3 shows strong connections between the Ricci flow and the linear heat equation (in [N3], further evidence is provided by the strong similarity in the entropy formulae for both cases). One can even think that Theorem 1.1 is the limiting case of its corresponding result for Kähler-Ricci flow. It is tempting to speculate that Kähler geometry might be a special case of the Kähler-Ricci flow geometry.

Remark 1.2. In [CN] we prove another high-dimensional generalization of Chow's interpolation, which also has some interesting consequences for Kähler-Ricci flow/ geometry. The new inequality is also strong enough for the applications considered in the later sections of this paper.

2. Nonnegative holomorphic line bundles

In this section we shall apply results in Section 1 to study the holomorphic line bundles on Kähler manifolds with nonnegative holomorphic bisectional curvature. First we illustrate the cases to which Theorem 1.1 can be applied.

Theorem 2.1. *Let (E, H) be a holomorphic vector bundle on M. Consider the Hermitian metric $H(x,t)$ deformed by the Hermitian-Einstein flow:*

$$\frac{\partial H}{\partial t} H^{-1} = -\Lambda F_H + \lambda I. \tag{2.1}$$

Here Λ means the contraction by the Kähler form ω, λ is a constant, which is a holomorphic invariant in the case M is compact, and F_H is the curvature of the metric H, which locally can be written as $F^j_{i\alpha\bar\beta} dz^\alpha \wedge d\bar z^\beta e_i^* \otimes e_j$ with $\{e_i\}$ a local frame for E. The transition rule for H under the frame change is $H^U_{i\bar j} = f_i^k \overline{f_j^k} H^V_{k\bar l}$ with transition functions f_i^j satisfying $e_i^U = f_i^j e_j^V$. Let

$$\rho = \frac{\sqrt{-1}}{2\pi}\Omega_{\alpha\bar\beta} dz_\alpha \wedge d\bar z_\beta = \frac{\sqrt{-1}}{2\pi}\sum_i F^i_{i\alpha\bar\beta} dz^\alpha \wedge d\bar z^\beta.$$

Assume that $\Omega_{\alpha\bar\beta}$ is smooth on $M \times (0,T]$. Then $\Omega_{\alpha\bar\beta}(x,t)$ satisfies (1.1). Therefore, if $\Omega_{\alpha\bar\beta}(x,t) \geq 0$, $Z_\Omega(x,t) \geq 0$, provided that $\Omega_{\alpha\bar\beta}(x,t)$ satisfies the growth *assumption of* (1.5)*, when the manifold is noncompact. In particular, if* $\Omega(x,t) = g^{\alpha\bar\beta}(x)\Omega_{\alpha\bar\beta}(x,t) > 0$*, one has that*

$$\Omega_t - \frac{|\nabla \Omega|^2}{\Omega} + \frac{\Omega}{t} \geq 0. \tag{2.2}$$

Proof. The proof on pages 10–12 of [N2] can be applied to this case without modification. □

The following Harnack inequality follows from an argument of Li-Yau [LY] by applying (2.2) and integrating along a space-time path. Notice that there is a slight notational discrepancy. From now on we use $\Delta = 4g^{\alpha\bar\beta}\frac{\partial^2}{\partial z^\alpha \partial \bar z^\beta}$, while in the last section the Laplacian operator is one quarter of the current one. However, the conclusion of the result stays the same.

Corollary 2.1. *Let M and Ω be as above. Assume that $\Omega(x,t) > 0$. Then for any $t_2 > t_1$,*

$$\Omega(x,t_2) \geq \Omega(y,t_1)\left(\frac{t_1}{t_2}\right)\exp\left(-\frac{r^2(x,y)}{4(t_2-t_1)}\right). \tag{2.3}$$

In particular,

$$\frac{\partial}{\partial t}\left(t\,\Omega(x,t)\right) \geq 0. \tag{2.4}$$

The factor 4 in (2.3) is introduced due to the factor 4 in our definition of the Laplacian operator. We use this convention since we have to use Li-Yau's heat kernel estimates on the heat kernels, which follows from a Harnack inequality of the same form as (2.3), extensively afterwards.

The Hermitian-Einstein flow (2.1) was studied, for example, by Donaldson [Dn] to flow a metric into an equilibrium solution under some algebraic stability assumptions. In this section we focus on the following two cases when Theorem 2.1 applies.

Case 1. This is the special case when $(E,H) = (L,H)$, a line bundle and $\lambda = 0$. Now the metric change can be expressed by a single function $v(x,t)$ with $H(x,t) = H(x)\exp(-v(x,t))$, where $H(x,0) = H(x)$, equivalently $v(x,0) = 0$. Then (2.1) reduces to the simple equation:

$$\left(\frac{\partial}{\partial t} - \Delta\right) v(x,t) = \Omega(x) \tag{2.5}$$

where $\Omega_{\alpha\bar\beta}(x)$ is the curvature form of the initial metric $H(x)$, $\Omega(x) = g^{\alpha\bar\beta}(x)\Omega_{\alpha\bar\beta}(x)$ and $\Omega_{\alpha\bar\beta}(x,t) = \Omega_{\alpha\bar\beta}(x) + v_{\alpha\bar\beta}(x,t)$. It is easy to see that $w(x,t) := \frac{\partial}{\partial t}v(x,t) = \Delta v(x,t) + \Omega(x) = \Omega(x,t)$ satisfies the heat equation $\left(\Delta - \frac{\partial}{\partial t}\right) w(x,t) = 0$ with the initial data $w(x,0) = \Omega(x)$. In the following we will focus on the line bundle case.

We follow the convention of calling (L,H) nonnegative if the curvature of (L,H), $\rho = \frac{\sqrt{-1}}{2\pi}\Omega_{\alpha\bar\beta}\, dz_\alpha \wedge d\bar z_\beta = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log(H)$, is a nonnegative $(1,1)$ form.

In order to ensure that, for the deformed metric $H(x,t)$, $\Omega_{\alpha\bar\beta}(x,t) \geq 0$ when $\Omega_{\alpha\bar\beta}(x,0) = \Omega_{\alpha\bar\beta}(x) \geq 0$ one needs some constraints on $\Omega(x)$. First we assume that $\Omega(x)$ is continuous. Furthermore, we also require that

$$\sup_{r\geq 0}\left(\exp(-ar)\fint_{B_o(r)} \Omega(y)\, dy\right) < \infty \tag{2.6}$$

and

$$\sup_{r\geq 0}\left(\exp(-ar^2)\fint_{B_o(r)}\Omega^2(y)\,dy\right)<\infty \tag{2.7}$$

for some positive constant $a>0$. Here, for any continuous function $f(y)$,

$$\fint_{B_o(r)} f(y)\,dv_y=\frac{1}{V_o(r)}\int_{B_o(r)} f(y)\,dv_y\,.$$

Case 2. The is the case when $\Omega_{\alpha\bar\beta}(x,0)$ is given by the Hessian of continuous plurisubharmonic functions. It has special interest for later applications. This corresponds to the metrics $H(x)=\exp(-u(x))$ being singular, as considered in [D]. In [D], due to the compactness of the manifolds and the local nature of the questions considered there, locally integrable functions are allowed. However, since we are interested in global properties of $u(x)$, especially the behavior near infinity, and our argument is (only) a global one, we require the functions to be defined on the whole of M. Moreover, in order to apply the tensor maximum principle [NT2, Theorem 2.1] we also put growth constraints on the plurisubharmonic functions. These constraints are similar to (2.6) and (2.7), which are specified as follows.

Let u be a continuous function on M. We say that u is of exponential growth if there exists $a>0$ such that

$$|u|(x)\leq \exp(a(r^2(x)+1)). \tag{2.6$'$}$$

By Proposition 2.1 of [NT1] we know that if $u(x)$ is of exponential growth, the equation $\left(\Delta-\frac{\partial}{\partial t}\right)\tilde v=0$, with $\tilde v(x,0)=u(x)$, has a solution on $M\times[0,T]$ for any $T>0$. Furthermore, we know that there exists a constant b such that

$$|\tilde v|(x,t)\leq \exp(b(r^2(x)+1)). \tag{2.8}$$

In this case, it is easy to check that $H(x,t)=\exp(-\tilde v(x,t))$ gives the solution to (2.1) and $v(x,t)=\tilde v(x,t)-u(x)$ solves (2.5) with $\Omega_{\alpha\bar\beta}(x,t)=\tilde v_{\alpha\bar\beta}(x,t)$.

The following lemma ensures that $\Omega_{\alpha\bar\beta}(x,t)\geq 0$ for the above two cases, with the help of the general maximum principle proved in [NT2].

Lemma 2.1. *Let M be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Let (L,h) be a nonnegative holomorphic line bundle. We assume that either we are in case 1 with* (2.6) *and* (2.7)*, or in case 2 with* (2.6$'$)*. Then* (2.1) *has a long-time solution with $\Omega_{\alpha\bar\beta}(x,t)\geq 0$.*

Proof. Case 2 is easier. Since (2.1) amounts to solving $\left(\Delta-\frac{\partial}{\partial t}\right)\tilde v=0$ with $\tilde v(x,0)=u(x)$, the result follows from Theorem 3.1 of [NT2].

For Case 1, clearly,

$$v(x,t)=\int_0^t\int_M H(x,y,s)\Omega(y)\,dv_y\,ds,$$

where $H(x,y,s)$ is the heat kernel, gives the solution to (2.5). It exists for all time due to (2.6). In order to show $\Omega_{\alpha\bar\beta}(x,t)\geq 0$, since $\Omega_{\alpha\bar\beta}(x,t)$ satisfies (1.1) by Theorem 2.1, we only need to check that the maximum principle [NT2, Theorem 2.1] applies. Due to the assumption (2.7), we only need to check that $v_{\alpha\bar\beta}(x,t)=\Omega_{\alpha\bar\beta}(x,t)-\Omega_{\alpha\bar\beta}(x)$ satisfies the assumptions of Theorem 2.1 of [NT2]. Notice that $v(x,t)$ satisfies the nonhomogeneous heat equation (2.5). Therefore $v(x,t)$ has pointwise control through the representation formula above. The by-now standard integration by parts arguments give the wanted integral estimates for $\|v_{\alpha\bar\beta}\|^2(x,t)$. The interested reader can refer to the proof of Lemma 6.2 of [NT2] for details on checking the conditions for the maximum principle. (See also Lemma 1.4.) The extra terms caused by the nonhomogeneous term $\Omega(x)$ are taken care of by the assumption (2.7). ☐

Combining Theorem 2.1 and Lemma 2.1 we are in a position to apply the monotonicity formula (2.4). Next we prove the following gap theorem, which combines the Liouville theorem [NT2, Theorem 0.3] with a gap theorem, a slightly weaker version of [NT2, Corollary 6.1]. The proof here uses the result from Section 1 and fits the general duality principle in [N2].

Theorem 2.2. *Let M be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Let (L,H) be a nonnegative holomorphic line bundle on M with Hermitian metric H. We assume either in Case 1 that* (2.7) *holds and*

$$\int_0^r s\left(⨍_{B_o(s)}\Omega(y)\,dy\right)\,ds = o(\log r), \tag{2.9}$$

where $\Omega(y)=g^{\alpha\bar\beta}\Omega_{\alpha\bar\beta}(y)$; or in Case 2, with $H(x)=\exp(-u(x))$, that

$$\limsup_{r\to\infty}\frac{u(x)}{\log r}=0. \tag{2.10}$$

Then (L,H) is flat. Namely $\Omega_{\alpha\bar\beta}(x)\equiv 0$. In particular, if $L=K_M^{-1}$, the anti-canonical line bundle, this implies that M is flat. Moreover, in Case 2, this further implies that u is a constant.

Proof of Theorem 2.2. In Case 1, (2.9) implies (2.6) and in the second case, since we can replace u by u_+, the positive part of u, we can assume that u satisfies (2.6′). Therefore, by Lemma 2.1, we can apply Theorem 2.1, in particular (2.4), in both situations.

Assume that (L,H) is not flat. Then $\Omega(x)\geq 0$ and >0 somewhere. This implies that $\Omega(x,t)=\int_M H(x,y,t)\Omega(y)\,dv_y>0$ for $t>0$. By (2.4) we know that

$$\int_1^t \Omega(x,s)\,ds \geq C\log t \tag{2.11}$$

for $t\gg 1$ and some $C>0$ independent of t. On the other hand, by Theorem 3.1 of [N1], (2.9) implies that

$$\int_1^t \Omega(x,s)\,ds \leq \epsilon\log t \tag{2.12}$$

for $t\gg 1$ in Case 1. This proves that (L,h) is flat in the first case.

For Case 2, since $\tilde v_t(x,t)=\Omega(x,t)$, (2.11) implies that

$$\tilde v(x,t)\geq C\log t + C', \tag{2.13}$$

for $t\gg 1$ with positive constants C and C' independent of t. By the assumption (2.10) we know that for any $\epsilon>0$,

$$⨍_{B_o(r)} u\,dv \leq \epsilon\log r$$

for $r \gg 1$. Using Theorem 3.1 of [N1] we have that

$$\tilde{v}(x,t) \leq C(n)\epsilon \log t$$

for $t \gg 1$. This is a contradiction to (2.13). The contradictions show that (L, H) is flat in both cases.

For the last part of the theorem, the flatness of (L, H) implies that u is harmonic. But u also satisfies (2.10). Then u is a constant by a gradient estimate of Cheng and Yau [CY], or the result of Cheng stating that any sublinear growth harmonic function must be constant on any complete Riemannian manifold with nonnegative Ricci curvature. We should point out that there is another proof (see, for example, [N4, pages 338-339]) which is based on the mean value inequality of Li and Schoen. The advantage is that the mean value inequality can be proved via the De Giorgi/Moser iteration argument even in the case that the gradient estimate fails.

□

Remark 2.1. The results in Theorem 2.2 were proved earlier by Luen-Fain Tam and the author in [NT2] (cf. Theorem 0.3 and Corollary 6.1 therein). The proof in [NT2, Theorem 0.3], for the Liouville property on plurisubharmonic functions, uses the L^2-estimate of the $\bar{\partial}$-operator of Hörmander, as well as Hamilton's strong maximum principle for tensors satisfying (1.1). The proof of the gap result in [NT2, Corollary 6.1] uses the Liouville result above, along with some quite sophisticated techniques of solving the Poincaré-Lelong equation in [MSY] and [NST], as well as some new refinements through heat equation deformation (cf. Section 6 of [NT2]).

Corollary 2.2. *Let M be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Let $u(x)$ be a continuous plurisubharmonic function of exponential growth. Let $\tilde{v}(x,t)$ be the solution to $\left(\frac{\partial}{\partial t} - \Delta\right)\tilde{v}(x,t) = 0$. Then*

$$Z_w(x.t) := w_t + \nabla_\alpha w\, V_{\bar{\alpha}} + \nabla_{\bar{\alpha}} w\, V_\alpha + v_{\alpha\bar{\beta}} V_{\bar{\alpha}} V_\beta + \frac{w}{t} \geq 0 \tag{2.14}$$

on $M \times (0,\infty)$, for any $(1,0)$ vector field V. Here $w(x,t) = \Delta \tilde{v}(x,t)$ and $w_t = \frac{\partial w}{\partial t}$. In particular,

$$w_t - \frac{|\nabla w|^2}{w} + \frac{w}{t} \geq 0. \tag{2.15}$$

If (2.15) holds with equality for some (x_0, t_0) with $t_0 > 0$, this implies that M is flat, provided $v_{\alpha\bar{\beta}}(x,t)$ is positive definite. In particular, this is true if $(tw)_t(x,t) = 0$ for some (x_0, t_0).

Proof. This follows from Theorem 1.1, Theorem 2.1 and Lemma 2.1. □

Remark 2.2. The estimate of the form (2.14) was first proved for the plurisubharmonic functions deformed by the time-dependent (with metric evolved by Kähler-Ricci flow) heat equation in [NT1]. It was also used there to prove the Liouville theorem for the plurisubharmonic functions for the first time. However, due to complications caused by the Kähler-Ricci flow, the result requires various assumptions on the curvature of the initial metric on M. In particular, one has to assume that the curvature is bounded, which is rather artificial for the study of the function theory on M.

3. Dimension estimates I

Let M be a complete Kähler manifold with nonnegative bisectional curvature of complex dimension m. In this section, we shall further show applications of the gradient estimate (2.14) in the study of holomorphic functions of polynomial growth. In a sense, the results here and in the next section are quantified versions of the results in Section 2. Let us first fix some notation. We say that a holomorphic function f is of polynomial growth if there exists $d \geq 0$ and $C = C(d, f)$ such that

$$|f|(x) \leq C(r^d(x) + 1), \tag{3.1}$$

where $r(x)$ is the distance function to a fixed point $o \in M$. For any $d > 0$ we denote $\mathcal{O}_d(M) = \{f \in \mathcal{O}(M) | f(x) \text{ satisfies } (3.1)\}$. Let $\mathcal{O}_P(M)$ denote the space of holomorphic functions of polynomial growth. Since any sublinear growth holomorphic function is constant on a complete Kähler manifold with nonnegative Ricci curvature,

$$\mathcal{O}_P(M) = \mathbb{C} \cup \left(\bigcup_{d \geq 1} \mathcal{O}_d(M) \right).$$

We also define the order of f in the sense of Hadamard to be

$$\mathrm{Ord}_H(f) = \limsup_{r \to \infty} \frac{\log \log A(r)}{\log r}$$

where $A(r) = \sup_{B_o(r)} |f(x)|$. It is clear that if $f \in \mathcal{O}_P(M)$, then $Ord_H(f) = 0$. We say that f has finite order if $Ord_H(f) < \infty$. The first issue we are going to address is that of estimating the dimension of $\mathcal{O}_d(M)$. Let us summarize some simple observations in a lemma.

Lemma 3.1. *Let $f \in \mathcal{O}(M)$ be a nonconstant holomorphic function of order less than one, in the sense of Hadamard. Denote $u(x) = \log(|f|(x))$. Then there exists a solution $v(x,t)$ to the heat equation $\left(\Delta - \frac{\partial}{\partial t}\right) v(x,t) = 0$ such that $v(x,0) = u(x)$, where $v(x,t)$ is plurisubharmonic. Moreover, the function $w(x,t) := \Delta v(x,t) > 0$, for $t > 0$, and*

$$\frac{\partial}{\partial t} (t\, w(x,t)) \geq 0. \tag{3.2}$$

Proof. Let $u_j(x) = \log(|f|(x) + \frac{1}{j})$. Let $H(x,y,t)$ be the heat kernel. Then

$$v_j(x,t) = \int_M H(x,y,t) u_j(y)\, dy \tag{3.3}$$

gives the solution $v_j(x,t)$ such that $v_j(x,0) = u_j(x)$. Clearly $v_j(x,t)$ satisfies the assumptions of Lemma 2.1. Thus $v_i(x,t)$ are plurisubharmonic functions. Letting $j \to \infty$ in (3.3), we obtain

$$v(x,t) = \int_M H(x,y,t) u(y)\, dy,$$

a solution with $v(x,0) = u(x)$. Let $w(x,t) = v_t(x,t)$. Since $\{v_j\}$ is a decreasing sequence, $v(x,t)$ is also plurisubharmonic. To prove (3.2) we claim that $w_j(x,t) = (v_j)_t(x,t)$ satisfies (3.2). Since $w_j(x,t) \to w(x,t)$ uniformly on compact subsets of $M \times (0, \infty)$, the claim implies that $w(x,t)$ also satisfies (3.2). In order to prove (3.2) for w_j, we notice that $w_j(x,0) = \Delta u_j(x)$. By the strong maximum principle we have that $w_j(x,t) > 0$, otherwise u_j is harmonic, which implies that f is a constant by Cheng-Yau's gradient estimate [CY]. This proves (3.2). To show that $w(x,t) > 0$, observe that $\lim_{t\to 0} w(x,t) = \Delta u(x)$. We claim that f must vanish somewhere. Otherwise u is a harmonic function of sublinear growth, which implies u is a constant by Cheng-Yau's gradient estimate [CY] again. This then implies f is a constant, which contradicts the assumption. Therefore Δu must be a nonzero, nonnegative measure. This implies that $w(x,t)$ cannot be identically zero for $t > 0$. By the strong maximum principle we then have that $w(x,t) > 0$ for $t > 0$. □

Remark 3.1. 1) The monotonicity (3.2) can be justified for any holomorphic function. One only needs the order condition to ensure that $w(x,t) > 0$.

2) The proof shows that, on a complete Kähler manifold M with nonnegative Ricci curvature, any nonconstant holomorphic function f with $Ord_H(f) < 1$ must vanish somewhere. This in particular generalizes the fundamental theorem of algebra to complete Kähler manifolds with nonnegative Ricci curvature.

Recall that for any positive (p,p) current Θ one can define the Lelong number of Θ at x as

$$\nu(\Theta, x) = \lim_{r\to 0} \nu(\Theta, x, r) \tag{3.4}$$

where

$$\nu(\Theta, x, r) = \frac{1}{r^{2(m-p)}\pi^{m-p}} \int_{B_x(r)} \Theta \wedge \omega^{m-p}. \tag{3.5}$$

The existence of the limit in (3.4) is ensured by (0.4), the Bishop-Lelong Lemma. For $f(x) \in \mathcal{O}(M)$ we define Z_f to be the zero set of f. Z_f is a positive $(1,1)$ current. The Poincaré-Lelong Lemma (cf. [GH]) states that

$$\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log(|f|^2) = Z_f. \tag{3.6}$$

We define

$$ord_x(f) = \max\{m \in \mathbb{N} | D^\alpha f(x) = 0, |\alpha| < m\}.$$

It is well known that

$$ord_x(f) = \nu(Z_f, x). \tag{3.7}$$

One can refer to [D] or [GH] for details of the above cited results on the Lelong number and $ord_x(f)$. Using (3.4)–(3.7), some elementary computation shows that

$$ord_x(f) = \frac{1}{2m} \lim_{r\to 0} \left(\frac{r^2}{V_x(r)} \int_{B_x(r)} \Delta \log |f|\, dv \right). \tag{3.8}$$

Theorem 3.1. *Let M be a complete Kähler manifold with nonnegative holomorphic bisectional curvature of complex dimension m. Then there exists a constant $C_1 = C_1(m)$ such that for any $f \in \mathcal{O}_d(M)$,*

$$ord_x(f) \le C_1 d. \tag{3.9}$$

In particular, this implies that

$$\dim_{\mathbb{C}}(\mathcal{O}_d(M)) \le C_2 d^m \tag{3.10}$$

for some $C_2 = C_2(m)$.

Proof. Let $u(x)$, $v(x,t)$, and $w(x,t)$ be as in Lemma 3.1. From (3.2) we know that

$$(tw(x,t))_t \geq 0. \tag{3.11}$$

We are going to show that there exist positive constants $C_3 = C_3(m)$ and $C_4 = C_4(m)$ such that

$$\lim_{t\to 0}(tw(x,t)) \geq C_3\, ord_x(f) \tag{3.12}$$

and

$$tw(x,t) \leq C_4\, d \tag{3.13}$$

for $t \gg 1$.

We first show (3.12). The approximation argument in the proof of Lemma 3.1 shows that

$$w(x,t) = \int_M H(x,y,t)\Delta \log(|f|(y))\, dv_y.$$

As in the proof of Theorem 3.1 in [N1], using Li-Yau's [LY] lower bound for the heat kernel we have that

$$w(x,t) \geq C(m)\frac{1}{V_x(\sqrt{t})}\int_{B_x(\sqrt{t})} \Delta \log(|f|(y))\, dv_y.$$

Therefore

$$tw(x,t) \geq C(m)\frac{t}{V_x(\sqrt{t})}\int_{B_x(\sqrt{t})} \Delta \log(|f|(y))\, dv_y.$$

Now (3.12) follows easily from (3.8).

To prove (3.13), we first observe that, by Theorem 3.1 in [N1], for $t \gg 1$,

$$v(x,t) \leq C_5 d \log t \tag{3.14}$$

for some constant $C_5 = C_5(m)$, since from the assumption (3.1) one has $\log|f|(x) \leq d\log(r(x)+1) + C$. (Here one cannot apply Theorem 3.1 of [N1] directly since v is not always nonnegative. But we can use u_+ as the initial date to obtain a solution to the heat equation, which serves as a barrier from above for v.) We claim that this implies

$$tw(x,t) \leq C_5\, d$$

for $t \gg 1$. Otherwise, we have some $\epsilon > 0$ such that

$$tw(x,t) > (C_5 + \epsilon)d$$

for $t \gg 1$. Here we have used the monotonicity of $tw(x,t)$. Therefore

$$v(x,t) \geq (C_5 + \epsilon)d\log t - A$$

where A is independent of t. This contradicts (3.14). Since (3.9) follows from (3.11)–(3.13) and (3.10) follows from (3.9) by a simple dimension counting argument (cf. [M, page 221]) we complete the proof of the theorem. □

Remark 3.2. The dimension estimate as well as the multiplicity estimate (3.9) for holomorphic functions of polynomial growth was first considered in [M] by Mok. In [M], the estimate was obtained for manifolds with maximum volume growth as well as a pointwise quadratic decay assumption on the curvature (cf. (0.5) and (0.6)). Also, the constant in the estimate similar to (3.9), obtained in [M], depends on the local geometry of M. Here the constant depends only on the complex dimension. The estimate (3.10) is sharp in the power.

Denote by $\mathcal{M}(M)$ the meromorphic function field generated by $\mathcal{O}_P(M)$. Namely, any $F \in \mathcal{M}(M)$ can be written as $F = \frac{g}{h}$ with $g, h \in \mathcal{O}_P(M)$. A direct consequence of Theorem 3.1 is the following statement.

Corollary 3.1. *Let M be as in Theorem 3.1. Then the transcendence degree of $\mathcal{M}(M)$ over $\mathbb{C}$, $deg_{tr}(\mathcal{M}(M))$ satisfies*

$$deg_{tr}(\mathcal{M}(M)) \le m.$$

Moreover, in the equality case, $\mathcal{M}(M)$ is a finite algebraic extension over $\mathbb{C}(f_1, \cdots, f_m)$, where f_i are the transcendental elements in $\mathcal{M}(M)$. More precisely, there exist $g, h \in \mathcal{O}_P(M)$ and a polynomial P with coefficients in $\mathbb{C}(f_1, \cdots, f_m)$ such that $P(\frac{g}{h}) = 0$ and $\mathcal{M}(M) = \mathbb{C}(f_1, \cdots, f_m, \frac{g}{h})$.

Proof. This follows from the so-called Poincaré-Siegel arguments. See, for example, [M, pages 220-221] or [S, pages 176-178]. ☐

The dimension estimates for the holomorphic functions can be generalized for the holomorphic sections of polynomial growth of holomorphic line bundles with controlled positive part of the curvature. In particular, this applies to the nonpositive line bundles. (We call (L, H) nonpositive if the curvature form $\Omega_{\alpha\bar\beta}(x) \le 0$.) We treat the nonpositive line bundle with continuous curvature in this section first and leave the more complicated case when the curvature has partial positivity to the next section. Before we state the result let us define

$$\mathcal{O}_d(M, L) = \{s \in \mathcal{O}(M, L) \,|\, ||s||(x) \le C(r(x)+1)^d\}.$$

Here $r(x)$ is the distance function to a fixed point $o \in M$.

Theorem 3.2. *Let M be a complete Kähler manifold with nonnegative bisectional curvature. Let (L, H) be a Hermitian line bundle with nonpositive curvature. Then*

$$\dim(\mathcal{O}_d(M, L)) \le C_1 d^m. \tag{3.15}$$

Here $C_1 = C_1(m)$.

Proof. We assume that there exists $s \in \mathcal{O}_d(M, L)$. The well-known Poincaré-Lelong equation states that

$$\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log(||s||^2) = [s] - \rho, \tag{3.16}$$

which is semipositive definite by the assumption that (L, H) is nonpositive. Here $\rho = \frac{\sqrt{-1}}{2\pi}\Omega_{\alpha\bar\beta} dz_\alpha \wedge d\bar z_\beta$, and $[s]$ is the divisor defined by the zero locus of s. In particular, this implies that

$$\Delta \log ||s||^2(x) \ge -\Omega(x) \ge 0. \tag{3.17}$$

Now let $u(x) = \log(||s||)$ and solve the heat equation $\left(\Delta - \frac{\partial}{\partial t}\right) v(x,t) = 0$ with the initial data $v(x,0) = u(x)$. The solvability can be justified by the argument of Lemma 3.1. Similarly we have that $v(x,t)$ is plurisubharmonic and $w(x,t) = v_t(x,t)$ satisfies $\left(\Delta - \frac{\partial}{\partial t}\right) w(x,t) = 0$, $w(x,0) = \Delta \log ||s||$. Moreover, by Remark 3.1,

$$(tw(x,t))_t \ge 0.$$

The argument of Lemma 3.1 also implies that

$$w(x,t) = \int_M H(x,y,t)\,(\Delta \log ||s||(y))\, dv_y. \tag{3.18}$$

We denote by $mult_x([s])$ the multiplicity of the divisor $[s]$. It is from the definition that

$$mult_x([s]) = \frac{1}{2m} \lim_{r\to 0} \left(\frac{r^2}{V_x(r)} \int_{B_x(r)} \Delta \log \|s\|(y)\, dv_y \right).$$

Now the same argument as in the proof of (3.12) shows that

$$\lim_{t\to 0} tw(x,t) \geq C_2(m)\, mult_x([s]). \tag{3.19}$$

We claim that

$$\lim_{t\to \infty} tw(x,t) \leq C_3(m) d. \tag{3.20}$$

In fact, since $v(x,t) = \int_M H(x,y,t) \log \|s\|(y)\, dv_y$ we have that

$$v(x,t) \leq C_4(m) d \log t \tag{3.21}$$

for $t \gg 1$, by Theorem 3.1 of [N1], as in the proof of Theorem 3.1. Now a similar argument as in Theorem 3.1 shows (3.20). Therefore we have

$$mult_x([s]) \leq C_5(m) d, \tag{3.22}$$

from which (3.15) follows by the same dimension-counting argument as before. □

The proof of the above result as well as the proof of Theorem 3.1 gives the following improvement of an earlier result [NT2, Theorem 4.3].

Corollary 3.2. *Let M be a complete Kähler manifold with nonnegative bisectional curvature. Assume that M admits a nonconstant holomorphic function of polynomial growth and that the bisectional curvature is positive at some point. Then*

$$V_x(r) \geq C_1 r^{m+1} \tag{3.23}$$

and

$$⨍_{B_x(r)} \mathcal{R}(y)\, dv_y \leq \frac{C_2}{r^2} \tag{3.24}$$

for some positive constants C_1 and C_2 (which might depend on x and certainly on M), independent of r.

Proof. We only prove (3.24) here and leave (3.23) to the interested reader. By the assumption that M admits a nonconstant holomorphic function of polynomial growth and that M has quasi-positive bisectional curvature, the proof of [NT2, Theorem 4.3] implies that there exists a smooth strictly plurisubharmonic function $u(x)$ on M such that $u(x) \leq C(\log r(x) + 2)$. By the proof of Lemma 4.2 of [NT2] we can have a nontrivial $s \in \mathcal{O}_d(M, K_M)$ for some $d > 0$, only depending on M. Now we apply the argument of Theorem 3.2 to the case $L = K_M$. Notice that $-\Omega(y) = \mathcal{R}(y)$, the scalar curvature. Now combining (3.17) and (3.18), we have that

$$w(x,t) \geq \int_M H(x,y,t) \mathcal{R}(y)\, dv_y.$$

Applying Theorem 3.1 of [N1] we then have

$$w(x,t) \geq C_3(m) ⨍_{B_x(\sqrt{t})} \mathcal{R}(y)\, dv_y. \tag{3.25}$$

Now (3.24) follows from (3.20), (3.25) and the monotonicity of $tw(x,t)$. □

Note that the special case $m = 1$ of (3.23) recovers the earlier result of Wu [W].

Remark 3.3. Theorem 3.2 can be applied to the canonical line bundle to give the dimension estimates for the canonical sections of polynomial growth. In Corollary 3.2, (3.24) also holds for the case with general nonpositive line bundles by assuming that there exists a holomorphic section of polynomial growth.

The corollary and [NT2] also suggest the following conjecture.

Conjecture 3.1. *Under the assumptions of Corollary 3.2 one should be able to prove that M has maximum volume growth. Namely, $\mathcal{O}_P(M) \neq \mathbb{C}$, (3.24), and M being of maximum volume growth are all equivalent if M has quasi-positive bisectional curvature.*

The intuition for this is that every transcendental holomorphic function of polynomial growth seems to contribute to the volume by a factor of r^2. On the other hand, under the assumption of Corollary 3.2 one in fact has $(f_1, \cdots, f_m)$ to form a local coordinate near any given point, by Corollary 5.2 in Section 5. See also Corollary 6.2 of [NT2]. The conjecture has been partially verified. The details will appear in a forthcoming paper.

4. Dimension estimates–the sharp ones

Let M be a complete Kähler manifold of complex dimension m. Under the assumption that M has nonnegative Ricci curvature, the function $\frac{V_x(r)}{r^n}$ is monotone decreasing ($n = 2m$ is the real dimension). If it has a positive limit $\theta = \lim_{r\to\infty} \frac{V_x(r)}{r^n}$ we say that M is of maximum volume growth. In [LW] the authors proved some asymptotically sharp dimension estimates for harmonic functions of polynomial growth on a complete Riemannian manifold with nonnegative sectional curvature and maximum volume growth. Here we shall show the sharp dimension estimate for $\mathcal{O}_d(M)$ for complete Kähler manifolds with nonnegative bisectional curvature and maximum volume growth.

Theorem 4.1. *Let M^m be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Assume that M is of maximum volume growth. Then*

$$ord_x(f) \leq [d]. \tag{4.1}$$

In particular,

$$\dim_{\mathbb{C}}(\mathcal{O}_d(M)) \leq \dim_{\mathbb{C}}(\mathcal{O}_{[d]}(\mathbb{C}^m)). \tag{4.2}$$

Here $[d]$ is the greatest integer less than or equal to d. If the equality holds in (4.1) for some $f \in \mathcal{O}_P(M)$, the universal cover $\widetilde{M}$ of M splits as $\widetilde{M} = M_1^{m-l} \times \mathbb{C}^l$, with $l \geq 1$. If the equality holds in (4.2) for some $d \geq 1$, M is biholomorphic-isometric to $\mathbb{C}^m$.

We need several lemmas to prove the above results. The first one is the sharpened version of (3.12), which does not require the manifold being of maximum volume growth.

Lemma 4.1. *Let u, v, w be as in Lemma 3.1. Then*

$$\lim_{t\to 0} tw(x,t) = \frac{1}{2} ord_x(f). \tag{4.3}$$

Proof. Since $w(x,t)$ solves the heat equation $\left(\Delta - \frac{\partial}{\partial t}\right) w(x,t) = 0$ with the initial data being the positive measure $\Delta \log(|f|(y))$, we can apply Theorem 3.1 of [N1] to w. (It is easy to see that the result proved in [N1] can be generalized to the case of the initial data being the positive measure.) Therefore we have that

$$\fint_{B_x(r)} \Delta \log(|f|(y))\, dv_y \le C(m) w(x, r^2) \tag{4.4}$$

for some constant $C(m) > 0$. On the other hand, from the proof of Theorem 3.1 we know that

$$t w(x,t) \le C(m)\, d \tag{4.5}$$

for $t \gg 1$. (From now on, $C(m)$ denotes a positive constant depending only on the dimension, which may be different from line to line.) This then implies that

$$\fint_{B_x(r)} \Delta \log(|f|(y))\, dv_y \le \frac{C(m)\, d}{r^2} \tag{4.6}$$

for $r \gg 1$.

It is well known that

$$H(x,y,t) \sim \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{r^2(x,y)}{4t}\right) + \text{ lower order terms} \tag{4.7}$$

as $t \to 0$. By (3.8) we also know that for $\epsilon > 0$ there exists $\delta > 0$ such that

$$2m\, ord_x(f) - \epsilon \le \frac{r^2}{V_x(r)} \int_{B_x(r)} \Delta \log |f|\, dv \le 2m\, ord_x(f) + \epsilon \tag{4.8}$$

for $r \le \delta$. Write

$$\begin{aligned} t w(x,t) &= t \int_M H(x,y,t) \Delta \log(|f|(y))\, dv_y \\ &= t \int_{r(x,y) \ge \delta} H(x,y,t) \Delta \log(|f|(y))\, dv_y \\ &\quad + t \int_{r(x,y) \le \delta} H(x,y,t) \Delta \log(|f|(y))\, dv_y \\ &= I + II. \end{aligned}$$

Here I and II denote the first and the second term in the second line of the above exhibition, respectively. In the following we are going to show that I has limit 0, as well as

$$\frac{1}{2}\, ord_x(f) - 2\epsilon \le \liminf_{t \to 0} II \tag{4.9}$$

and

$$\limsup_{t \to 0} II \le \frac{1}{2}\, ord_x(f) + 2\epsilon. \tag{4.10}$$

Clearly, (4.3) is a consequence of these conclusions. Using Li-Yau's upper bound on the heat kernel estimate we have

$$\begin{aligned}
I &\le \frac{C(n)t}{V_x(\sqrt{t})}\int_\delta^\infty \exp(-\frac{s^2}{5t})\left(\int_{\partial B_x(s)} \Delta \log(|f|(y))\, dA\right)\, ds\\
&\le \frac{C(n)t}{V_x(\sqrt{t})}\exp(-\frac{\delta^2}{5t})\int_{B_x(\delta)} \Delta \log|f|(y)\, dv_y\\
&\quad + C(n)t\int_\delta^\infty \exp(-\frac{s^2}{5t})\left(\frac{s}{\sqrt{t}}\right)^n\left(\fint_{B_x(s)} \Delta \log(|f|(y))\, dv_y\right)\left(\frac{2s}{5t}\right)\, ds\\
&= III + IV.
\end{aligned}$$

Here we have used the volume comparison theorem and assumed that $\sqrt{t} \le \delta$. III and IV denote the term in the second and third line, respectively. Clearly $\lim_{t\to 0} III = 0$. On the other hand,

$$IV \le C(n)\int_{\frac{\delta^2}{5t}}^\infty \exp(-\tau)\tau^{\frac{n}{2}-1}t\tau\left(\fint_{B_x(\sqrt{5t\tau})} \Delta \log|f|(y)\, dv_y\right)\, d\tau.$$

Using the estimate (4.6) we have that $\lim_{t\to 0} IV = 0$. Therefore we have shown that

$$\lim_{t\to 0} I = 0. \tag{4.11}$$

Now we prove (4.10). Using (4.7), for $t \ll 1$,

$$\begin{aligned}
II &\le t\int_0^\delta \frac{1}{(4\pi t)^{\frac{n}{2}}}\exp(-\frac{s^2}{4t})\left(\int_{\partial B_x(s)} \Delta \log|f|(y)\, dy\right)\, ds + \frac{\epsilon}{2}\\
&= t\frac{1}{(4\pi t)^{\frac{n}{2}}}\exp(-\frac{\delta^2}{4t})\left(\int_{B_x(\delta)} \Delta \log|f|(y)\, dy\right)\\
&\quad + t\int_0^\delta \frac{1}{(4\pi t)^{\frac{n}{2}}}\exp(-\frac{s^2}{4t})\left(\int_{B_x(s)} \Delta \log|f|(y)\, dy\right)\left(\frac{s}{2t}\right)\, ds + \frac{\epsilon}{2}\\
&= V + VI + \frac{\epsilon}{2}.
\end{aligned}$$

Here V and VI denote the term in the second and the third line, respectively. The term V has limit 0 as in the estimate of I. To estimate the term VI we use (4.8) and the fact that

$$V_x(s) \sim \omega_n s^n$$

for $s \to 0$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Indeed,

$$
\begin{aligned}
VI &\le t\int_0^{\delta} \frac{\omega_n s^n}{(4\pi t)^{\frac{n}{2}}}\exp(-\frac{s^2}{4t})\left(\fint_{B_x(s)} \Delta \log|f|(y)\,dy\right)\left(\frac{s}{2t}\right)\,ds\\
&= \frac{\omega_n}{\pi^{\frac{n}{2}}}\int_0^{\frac{\delta^2}{4t}} \frac{1}{4}\exp(-\tau)\tau^{\frac{n}{2}-1}\left((4t\tau)\fint_{B_x(\sqrt{4t\tau})} \Delta\log|f|(y)\,dv_y\right)d\tau\\
&\le \frac{m}{2}\frac{\omega_n}{\pi^{\frac{n}{2}}}\,ord_x(f)\int_0^{\frac{\delta^2}{4t}} \exp(-\tau)\tau^{\frac{n}{2}-1}d\tau+\frac{\epsilon}{2}\\
&= \frac{m}{2}\frac{\omega_n}{\pi^{\frac{n}{2}}}\,ord_x(f)\Gamma(\frac{n}{2})+\frac{\epsilon}{2}\\
&= \frac{1}{2}\,ord_x(f)+\frac{\epsilon}{2}.
\end{aligned}
$$

Here $\Gamma(z)$ is the standard Gamma function. This proves (4.10). The proof for (4.9) is similar. ☐

Notice that we do not make use of the maximum volume growth in Lemma 4.1. The next lemma sharpens (3.13), which makes use of the maximum volume growth assumption.

Lemma 4.2. *Let M be as in Theorem 4.1 and let u, v, w be as in Lemma 3.1. Then*

$$
\limsup_{t\to\infty}\frac{v(x,t)}{\log t}\le\frac{1}{2}d, \tag{4.12}
$$

which, in particular, implies that

$$
\lim_{t\to\infty} tw(x,t)\le\frac{1}{2}d. \tag{4.13}
$$

In order to prove the above lemma we need a result of Li-Tam-Wang [LTW, Theorem 2.1] on the upper bound of the heat kernel under the maximum volume growth assumption.

Theorem 4.2 (Li-Tam-Wang)**.** *Let M be a complete Riemannian manifold with nonnegative Ricci curvature and maximum volume growth. For any $\delta > 0$, the heat kernel of M must satisfy the estimate*

$$
\begin{aligned}
&\frac{\omega_n}{\theta(\delta r(x,y))}(4\pi t)^{-\frac{n}{2}}\exp\left(-\frac{1+9\delta}{4t}r^2(x,y)\right)\le H(x,y,t)\\
&\qquad\le(1+C(n,\theta)(\delta+\beta))\,\frac{\omega_n}{\theta}(4\pi t)^{-\frac{n}{2}}\exp\left(-\frac{1-\delta}{4t}r^2(x,y)\right),
\end{aligned} \tag{4.14}
$$

where

$$
\beta=\delta^{-2n}\max_{r\ge(1-\delta)r(x,y)}\{1-\frac{\theta_x(r)}{\theta_x(\delta^{2n+1}r)}\}. \tag{4.15}
$$

Note that β is a function of $r(x,y)$ such that

$$
\lim_{r(x,y)\to\infty}\beta=0. \tag{4.16}
$$

Proof of Lemma 4.2. By (4.16) and the fact that

$$A_x(s) \sim n s^{n-1}\theta,$$

as $s \to \infty$, where $A_x(s)$ is the area of the sphere $\partial B_x(s)$, we know that for any $\epsilon > 0$, there exists a positive constant $A > 0$ such that for $s = r(x,y) \geq A$,

$$\beta \leq \epsilon \quad \text{and} \quad \frac{A_x(s)}{\theta} \leq ((1+\epsilon)n)\, s^{n-1}. \tag{4.17}$$

Now, we estimate $v(x,t) = \int_M H(x,y,t) \log |f|(y)\, dv_y$. Using the upper bound of Li-Yau we have that

$$\begin{aligned} v(x,t) &= \int_{r(x,y)\leq A} H(x,y,t) \log|f|(y)\, dv_y + \int_{r(x,y)\geq A} H(x,y,t)\log|f|(y)\, dv_y \\ &\leq \frac{C(n)}{V_x(\sqrt{t})} \int_0^A \exp(-\frac{s^2}{5t}) \left(\int_{\partial B_x(s)} \log|f|(y)\, dA_y \right) ds + II \\ &\leq \frac{C(n)}{V_x(\sqrt{t})} \int_{B_x(A)} \log|f|(y)\, dv_y + II \\ &= I + II. \end{aligned}$$

Here we use II to represent the second term of the first line and I to represent the first term of the third line. Clearly

$$\lim_{t\to\infty} \frac{I}{\log t} = 0. \tag{4.18}$$

We claim that

$$\limsup_{t\to\infty} \frac{II}{\log t} \leq \frac{1}{2}\, d. \tag{4.19}$$

The lemma follows easily from (4.18) and (4.19). To prove (4.19) we need the estimate (4.14) of Li, Tam, and Wang. Notice that this is the only place where the maximum volume growth condition is used. Indeed, by (4.14) and (4.17), for the given fixed δ and $\epsilon > 0$,

$$\begin{aligned} II &\leq (1 + C(n,\theta)(\delta+\epsilon))\, \frac{\omega_n}{(4\pi t)^{\frac{n}{2}}} \int_A^\infty \exp(-\frac{1-\delta}{4t}s^2) \frac{1}{\theta} \left(\int_{\partial B_x(s)} \log|f|(y)\, dA_y \right) ds \\ &\leq (1 + C(n,\theta)(\delta+\epsilon))\, \frac{\omega_n}{(4\pi t)^{\frac{n}{2}}} \int_A^\infty \exp(-\frac{1-\delta}{4t}s^2) \frac{A_x(s)}{\theta} \left(d\log s + \tilde{C} \right) ds \\ &\leq (1 + C(n,\theta)(\delta+\epsilon))\, \frac{\omega_n}{(4\pi t)^{\frac{n}{2}}} \int_A^\infty \exp(-\frac{1-\delta}{4t}s^2) n(1+\epsilon) s^{n-1} \left(d\log s + \tilde{C} \right) ds. \end{aligned}$$

Here $\tilde{C}$ is the constant in (3.1). Let

$$III = (1 + C(n,\theta)(\delta+\epsilon))\,(1+\epsilon) \frac{d\, n\, \omega_n}{(4\pi t)^{\frac{n}{2}}} \int_A^\infty \exp(-\frac{1-\delta}{4t}s^2) s^{n-1} \log s\, ds$$

and

$$IV = (1 + C(n,\theta)(\delta+\epsilon))\, \tilde{C}(1+\epsilon) \frac{n\omega_n}{(4\pi t)^{\frac{n}{2}}} \int_A^\infty \exp(-\frac{1-\delta}{4t}s^2) s^{n-1}\, ds.$$

Then we have $II \leq III + IV$. It is easy to check that

$$\lim_{t\to\infty} \frac{IV}{\log t} = 0.$$

For III we have that

$$III \leq (1 + C(n,\theta)(\delta+\epsilon)) \frac{(1+\epsilon) d\, n\, \omega_n}{4(\pi(1-\delta))^{\frac{n}{2}}} \times \int_{\frac{(1-\delta)A^2}{4t}}^{\infty} \exp(-\tau)\tau^{\frac{n}{2}-1} \left(\log t + \log(\frac{4\tau}{1-\delta}\right) d\tau.$$

Then we have that

$$\begin{aligned} \limsup_{t\to\infty} \frac{III}{\log t} &\leq (1 + C(n,\theta)(\delta+\epsilon)) \frac{(1+\epsilon) d\, n\, \omega_n}{4(\pi(1-\delta))^{\frac{n}{2}}} \int_0^{\infty} \exp(-\tau)\tau^{\frac{n}{2}-1}\, d\tau \\ &= (1 + C(n,\theta)(\delta+\epsilon)) \frac{(1+\epsilon) d\, n\, \omega_n}{4(\pi(1-\delta))^{\frac{n}{2}}} \Gamma(\frac{n}{2}) \\ &= (1 + C(n,\theta)(\delta+\epsilon)) \frac{(1+\epsilon) d}{2(1-\delta)^{\frac{n}{2}}}. \end{aligned}$$

Since δ and ϵ are arbitrary chosen positive constants, this proves (4.19). □

Proof of Theorem 4.1. By Lemmas 4.1 and 4.2 we deduce (4.1), from which the theorem follows by dimension counting. More precisely, for a fixed $o \in M$ and a local coordinates chart $(z_1, \cdots, z_m)$ with $z_i(o) = 0$, one can define the map

$$\Phi : \mathcal{O}_d(M) \to \mathbb{C}^{K_{[d]}}$$

by, for all $|\alpha| \leq [d]$,

$$\Phi(f) = (f(o), D_z f(o), \cdots, D_z^{\alpha} f(o), \cdots)$$

where

$$K_{[d]} = \dim(\mathcal{O}_d(M)) = 1 + \begin{pmatrix} m \\ m-1 \end{pmatrix} + \cdots + \begin{pmatrix} m+[d]-1 \\ m-1 \end{pmatrix} = \begin{pmatrix} m+[d] \\ m \end{pmatrix}.$$

Assume that the conclusion of the theorem is not true. Then there exists $f \neq 0$ such that $\Phi(f) = 0$. This implies that $ord_x(f) > d$, which is a contradiction to (4.1).

In the case that equality holds in (4.1), this implies that $(tw)_t(x,t) = 0$ for all $t > 0$. If $v_{\alpha\bar\beta} > 0$, Theorem 1.1, in particular Corollary 2.2, implies that M is flat. Since it is of maximum volume growth, M must be isometric to $\mathbb{C}^m$. Otherwise the universal cover of M splits by the general splitting result proved in Theorem 0.1 of [NT2]. The above argument shows that one of the factors, where $v_{\alpha\bar\beta}$ is positive, is $\mathbb{C}^l$, $l \geq 1$ since f is not a constant.

In the case that equality holds in (4.2), it is easy to see that the map Φ above is an isomorphism by the dimension consideration and (4.1). Hence there exist $f_i \in \mathcal{O}_{[d]}(M)$ such that

$$f_i = z_i^{[d]} + \text{ higher order terms.}$$

Consider $v(x,t)$ with the initial data $u(x) = \frac{1}{2}\log(\sum_{i=1}^m |f_i|^2)$. Clearly $u(x) \leq d\log(r + C)$. On the other hand, $\frac{\sqrt{-1}}{2\pi}\partial\bar\partial u$ has Lelong number d at o, by Lemma 4.4.14 in [Ho]. Therefore(4.1) (more precisely (2.15) or (3.2)) also holds with equality for such choice of v. On the other hand, we claim that $v_{\alpha\bar{\beta}}(x,t) > 0$. Otherwise, by applying Theorem 0.1 of [NT2] (passing to its universal cover, if needed) one splits the manifold M into a product of two factors, and on one (nontrivial, if $v_{\alpha\bar{\beta}}$ is not positive definite) of the two factors, $v(x,t)$, therefore $u(x)$, is harmonic, therefore constant, thanks to Cheng-Yau's gradient estimate. But this contradicts the choice of $u(x)$ since $u(x)$ has only an isolated singularity at o, and the splitting implies that the singularity of $u(x)$ cannot be zero dimensional. Therefore we have that $v_{\alpha\bar{\beta}}(x,t) > 0$. By the above argument for the equality case in (4.1) or (2.15), we have that M is flat, therefore isometric to $\mathbb{C}^m$. This completes the proof of the theorem.

Notice that the above argument also works for the case in which M is not assumed to be of maximum volume growth. One just needs to notice that $\mathcal{O}_d(M)$ can be viewed as a subspace of the corresponding function space of its universal cover $\tilde{M}$ and the above argument proves that $\tilde{M} = \mathbb{C}^m$. The equality also forces $\mathcal{O}_d(M) = \mathcal{O}_d(\tilde{M}) = \mathcal{O}_d(\mathbb{C}^m)$. This implies that every polynomial of degree less than d is equivariant, so are their common zeros. This particularly implies that the covering has only one sheet since every point of $\tilde{M}$ is a unique common zero of some linear holomorphic functions. Therefore its orbit under the deck transformation contains only itself. $\square$

Remark 4.1. Theorem 4.1 was conjectured by Yau [Y]. The only place where the maximum volume growth assumption was used is in Lemma 4.2, where the asymptotically sharp heat kernel upper bound estimate of Li, Tam, and Wang was applied. If the similar estimate as (4.12) holds for the general case without assuming maximum volume growth, then the argument given above provides the sharp dimension bound for the general case without any modifications.

A localized version of the estimate (4.1) was proved for polynomials on $\mathbb{C}^m$ by Bombieri [B] in the study of the algebraic values of meromorphic maps. Similar localization can be derived from the above proof of Theorem 4.1 for the holomorphic functions of polynomial growth on complete Kähler manifolds satisfying Theorem 4.1.

The following corollary is a simple consequence of the sharp estimate. The interested reader might want to compare the corollary with the example of [Do], for which there are more sub-quadratic harmonic functions than linear growth ones. The example of 'round-off' cones on pages 3–4 of [NT2] shows that one cannot expect that $\mathcal{O}_{1+\epsilon}(M) = \mathcal{O}_1(M)$. Namely, one cannot conclude that f is indeed linear. However, this is the case if one assumes a stronger 'closeness' assumption as in Theorem 0.3 of [NT2].

Corollary 4.1. *Let M be as in Theorem 4.1. Then, for any $\epsilon > 0$,*

$$\dim(\mathcal{O}_{2-\epsilon}(M)) \leq m+1.$$

Remark 4.2. As pointed out in the previous remark, the sharp estimate, Theorem 0.1, without assuming maximum volume growth, follows if one can prove Lemma 4.2 without assuming maximum volume growth. In fact, Lemma 4.2 follows from a nice observation by B. Chen, X. Fu, L. Yin and X. Zhu in a recent posting [CFYZ]. For the sake of the reader we include a simplified exposition here. Using the notation as in Lemma 4.2, by the estimates in the proof of (4.19), we have that

$$\begin{aligned}
v(x,t) &= \int_M H(x,y,t)\log|f(y)|\,dv_y \\
&= \int_{r(x,y)>\sqrt{t}} H(x,y,t)\left(\log|f(y)| - \frac{1}{2}d\log t\right) + \int_{r(x,y)\le\sqrt{t}} H(x,y,t)\log|f(y)| \\
&\quad + \int_{r(x,y)>\sqrt{t}} H(x,y,t)\frac{1}{2}d\log t\,dv_y \\
&\le \int_{r(x,y)>\sqrt{t}} H(x,y,t)\left(\log|f(y)| - \frac{1}{2}d\log t\right) + \frac{1}{2}d\log t + C(f).
\end{aligned}$$

On the other hand, the first term above, denoted by I, satisfies

$$I \le \int_{\sqrt{t}}^{\infty} \frac{C(n)d}{V_x(\sqrt{t})}\int_{\partial B_x(s)} \exp(-\frac{s^2}{5t})\log\frac{s}{\sqrt{t}}\,dA\,ds \le \tilde{C}(n)$$

by the heat kernel estimate of Li and Yau. Therefore, Lemma 4.2 holds for the general case. The equality case was also treated in [CFYZ]. The argument of using the equality in (1.15) and the strong maximum principle to get more information on M can be found in [N2, page 16], which was completed in September of 2002 and has been available on arXiv since November of 2002. The argument for the equality case (on (4.2)) in the proof of Theorem 4.1 also works assuming neither the maximum volume growth nor the simply-connectedness due to the fact that equality in (4.2) is stable under the lifting. The argument in the proof of Theorem 4.1 seems easier than the induction argument in [CFYZ], via the splitting theorem of [NT2].

Similarly, replacing the rough estimates (3.12) and (3.14) by the precise estimates in Lemma 4.1 and Lemma 4.2 we can have the corresponding result for the nonpositive line bundle case.

Corollary 4.2. *Let M be a complete Kähler manifold with nonnegative bisectional curvature. Let (L,H) be a Hermitian line bundle with nonpositive curvature. Then for any $s \in \mathcal{O}_d(M,L)$,*

$$mult_x([s]) \le d + \nu_\infty. \tag{4.20}$$

In particular,

$$\dim(\mathcal{O}_d(M,L)) \le \dim_{\mathbb{C}}(\mathcal{O}_{[d]}(\mathbb{C}^m)).$$

In the next result we combine the techniques in the proof of Theorem 4.1 and Theorem 3.2/Corollary 4.2 to obtain the dimension estimate for the holomorphic sections of polynomial growth on line bundles with controlled positive curvature. (We assume the curvature $\rho = \frac{\sqrt{-1}}{2\pi}\Omega_{\alpha\bar\beta}(x)dz^{\alpha}\wedge dz^{\bar\beta}$ to be a continuous current, for the sake of simplicity.)

Theorem 4.3. *Let M be a complete Kähler manifold with nonnegative bisectional curvature. Let (L,H) be a holomorphic line bundle over M such that $\|\Omega_{\alpha\bar\beta}\|(x)$ satisfies (2.6), (2.7). Assume that the curvature of (L,H), $\rho = \frac{\sqrt{-1}}{2\pi}\Omega_{\alpha\bar\beta}\,dz_\alpha\wedge d\bar z_\beta$ satisfies*

$$\nu(\Omega_+, x, \infty) = \limsup_{r\to\infty}\frac{1}{2m}\left(r^2 \fint_{B_x(r)} \Omega_+(y)\,dv_y\right) < \infty.$$

Then for any $s \in \mathcal{O}_d(M, L)$,

$$mult_x([s]) \le C(m)\ (d + \nu_\infty)\,. \tag{4.21}$$

Here we denote $\nu(\Omega_+, x, \infty)$ *by* ν_∞. *In particular, this implies that*

$$\dim(\mathcal{O}_d(M, L)) \le C(m)(d + \nu_\infty)^m. \tag{4.22}$$

If, furthermore, we assume that M *has maximum volume growth, then*

$$mult_x([s]) \le d + \nu_\infty. \tag{4.23}$$

In particular,

$$\dim(\mathcal{O}_d(M, L)) \le \dim_{\mathbb{C}}(\mathcal{O}_{[d+\nu_\infty]}(\mathbb{C}^m)). \tag{4.24}$$

Proof. The situation is more general than Case 1 considered in Section 2 since the bundle is not assumed to be nonnegative. The assumptions (2.6) and (2.7) on $\|\Omega_{\alpha\bar\beta}\|$ (in fact only $\Omega(x)$ satisfying (2.6) and (2.7) is needed) ensure that we have a global solution $v(x,t)$ to the Hermitian-Einstein heat equation (2.1) (equivalently, (2.5)). We also consider the equation $\left(\Delta - \frac{\partial}{\partial t}\right)\tilde v(x,t) = 0$ with initial data $\tilde v(x, 0) = \Delta \log \|s\|(x)$. By the approximation argument of Lemma 3.1, since both $\Omega_{\alpha\bar\beta}(x,t)$ and $\tilde v_{\alpha\bar\beta}(x,t)$ satisfy (1.1), and $\left(\Omega_{\alpha\bar\beta} + \tilde v_{\alpha\bar\beta}\right)(x, 0) \ge 0$, by the proof of Lemma 2.1, we know that $\left(\Omega_{\alpha\bar\beta} + \tilde v_{\alpha\bar\beta}\right)(x,t) \ge 0$. Here one needs the assumptions (2.6) and (2.7) on $\|\Omega_{\alpha\bar\beta}\|$. Now we can apply Theorem 1.1 to $\left(\Omega_{\alpha\bar\beta} + \tilde v_{\alpha\bar\beta}\right)(x,t)$. In particular we have that $(tw(x,t))_t \ge 0$. Here $w(x,t) = \Omega(x,t) + \tilde v_t(x,t)$. One can show that

$$\lim_{t\to 0} t\, w(x,t) = \frac{1}{2} mult_x([s]) \tag{4.25}$$

still holds since $\Omega_{\alpha\bar\beta}(x, 0)$ does not contribute to the Lelong number by the continuity assumption on Ω. Similar to the proof of (3.13) we have that

$$\limsup_{t\to\infty} t\tilde v_t(x,t) \le C(m)\, d. \tag{3.13$'$}$$

This can be justified as follows. Since $\Omega(x,t)$ satisfies the heat equation with $\Omega(x, 0) = \Omega(x)$, by Theorem 3.1 of [N1] we have that

$$\Omega(x,t) \le C_7(m) \sup_{r \ge \sqrt{t}} \left(\fint_{B_x(r)} \Omega_+(y)\, dv_y \right),$$

which implies that

$$\limsup_{t\to\infty} t\Omega(x,t) \le C(m)\nu_\infty. \tag{4.26}$$

Here we have used the definition of ν_∞. Combining (3.13$'$) and (4.26) we get (4.21). This finishes the proof for the general case.

For the case of maximum volume growth, we need to show that

$$\lim_{t\to\infty} t\, w(x,t) \le \frac{1}{2}\left(d + \nu_\infty\right). \tag{4.27}$$

Applying Lemma 4.2 to $\tilde v(x,t)$, we then have that

$$\limsup_{t\to\infty} t\tilde v_t(x,t) \le \frac{1}{2} d.$$

A similar argument as in the proof of Lemma 4.2 also shows that

$$\limsup_{t\to\infty} t\, \Omega(x,t) \le \frac{1}{2}\nu_\infty.$$

Combining them together we have (4.27). □

Remark 4.3. One can think of $\nu(\Omega, x, \infty)$ as the Lelong number of Ω at infinity. Similarly, one can define the Lelong number of a plurisubharmonic function at infinity. Then Theorem 2.2 can be simply rephrased as that any nonnegative line bundle has positive Lelong number at infinity, if it is not flat, and any nonconstant plurisubharmonic function has positive Lelong number at infinity, respectively.

5. Further applications

In this section we show some consequences of the multiplicity estimates proved in Sections 3 and 4. We first define the holomorphic maps Φ_j inductively, for $j \in \mathbb{N}$, as follows:

$$\Phi_1(x) = (f_1^1(x), \cdots, f_{k_1}^1(x)) \in \mathbb{C}^{k_1}$$

where f_i^1 form a basis for $\mathcal{O}_1(M)/\mathbb{C}$, and where $k_1 = \dim(\mathcal{O}_1(M)/\mathbb{C})$. Suppose that we have defined the map $\Phi_j : M \to \mathbb{C}^{k_j}$ as

$$\Phi_j(x) = (f_1^j(x), \cdots, f_{k_j}^j(x)),$$

where $k_j = \dim(\mathcal{O}_j(M)/\mathbb{C})$ and the f_i^j form a basis of $\mathcal{O}_j(M)/\mathbb{C}$. We define

$$\Phi_{j+1}(x) = (f_1^{j+1}(x), \cdots, f_{k_j}^{j+1}(x), f_{k_j+1}^{j+1}(x), \cdots, f_{k_{j+1}}^{j+1}(x))$$

such that $f_i^{j+1}(x) = f_i^j(x)$ for all $i \leq k_j$ and f_i^{j+1} with $i \geq k_j + 1$ form a basis of $\mathcal{O}_{j+1}(M)/\mathcal{O}_j(M)$. We define the Kodaira dimension $k(M)$ of M as

$$k(M) = \max_{j \in \mathbb{N}}\{\max_{x \in M}(rank(\Phi_j(x)))\}. \tag{5.1}$$

In the case $\mathcal{O}_P(M) = \mathbb{C}$ we define $k(M) = 0$.

The first result follows closely the consideration (in complex algebraic geometry) of the study of canonical embedding of a compact Kähler manifold through the sections of holomorphic line bundles [KM], [D].

Proposition 5.1. *Let M be a complete Kähler manifold with nonnegative bisectional curvature. Let $\mathcal{M}(M)$ be the quotient field generated by $\mathcal{O}_P(M)$. Then*

$$deg_{tr}(\mathcal{M}(M)) = k(M). \tag{5.2}$$

Proof. If $\max_{x \in M}(rank(\Phi_j)) \geq k$, it is easy to see that $deg_{tr}(\mathcal{M}(M)) \geq k$ since there are at least k holomorphic functions in $\mathcal{O}_j(M)$ which are transcendental. This implies that $deg_{tr}(\mathcal{M}(M)) \geq k(M)$.

Now we show that $deg_{tr}(\mathcal{M}(M)) \leq k(M)$ too. By Theorem 3.1 we have that $ord_x(f) \leq C_1(m)d$ for any $f \in \mathcal{O}_d(M)$. Assume that the conclusion is not true. Then we have $F_1, \cdots, F_{k(M)+1} \in \mathcal{M}(M)$ such that they are transcendental over $\mathbb{C}$. We can assume that they have the form $F_i = \frac{f_i}{f_0}$ with $f_j \in \mathcal{O}_{d_0}(M)$. By counting the monomials formed by F_i we conclude that

$$\dim(\mathcal{O}_{d_0\, p}(M)) \geq C_2(m)\,(p)^{k(M)+1} \tag{5.3}$$

for some positive constant $C_2(m)$.

On the other hand, by the definition of $k(M)$ we know that there exists j_0, sufficiently big, such that for $j \geq j_0$, $\max_{x \in M}(rank(\Phi_j)(x)) = k(M)$. By the definition of Φ_j, the basis elements $\{f_i^j(x)\}$, $i = 1, \cdots, k_j$, are constant on $\Phi_j^{-1}(y)$, which is of dimension $m - k(M)$. Namely $f_i^j(x)$ are functions of at most $k(M)$ free variables. Let y be a regular value of Φ_j. Pick $x_0 \in \Phi_j^{-1}(y)$. This shows that it only takes dimension of $C_3(m)\tilde{q}^{k(M)}$ to get a nontrivial function $f \in \mathcal{O}_j(M)$ such that $ord_{x_0}(f) \geq \tilde{q}$. Here $C_3(m)$ is a positive constant only depending on m.

We can choose $d \gg 1$ such that $d_0|d$ and

$$C_2(m)\frac{d^{k(M)+1}}{d_0^{k(M)+1}} \geq C_3(m)(C_1(m)d+1)^{k(M)}.$$

Now let $j = d, q = d/d_0, \tilde{q} = C_1(m)d + 1$. From (5.3) this implies that there exists an $f \in \mathcal{O}_d(M)$ such that $ord_{x_0}(f) \geq C_1(m)d + 1$. This is a contradiction.

□

Corollary 5.1. *Let $F_1, \cdots, F_{k(M)}$ be the transcendental elements in $\mathcal{M}(M)$. Then the quotient field $\mathcal{M}(M)$ is a finite algebraic extension of $\mathbb{C}(F_1, \cdots, F_{k(M)})$. In particular, $\mathcal{M}(M)$ is finitely generated.*

Proof. The finiteness of the extension follows a similar argument to Proposition 5.1. Once we know that the extension is finite, the Primitive Element Theorem implies the finite generation of $\mathcal{M}(M)$ (cf. [ZS, page 84]). □

Remark 5.1. A result similar to Proposition 5.1 for compact manifolds, where the holomorphic functions in $\mathcal{O}_d(M)$ are replaced by the holomorphic sections of the power L^d of a fixed line bundle L, was known as the Serre-Siegel lemma (cf. (6.5) in [D]). The dimension estimates as in Theorem 3.1 are much easier to prove in that case via either the standard Moser iteration or a covering argument together with Schwarz's lemma.

Proof of Theorem 0.3. By the proof of Proposition 5.1, for a regular value y of the map Φ_d, any $f \in \mathcal{O}_d(M)$ is a constant on $\Phi_d^{-1}(y)$. Pick x_0 as before. Therefore, f is only a function of at most $k(M)$ variables. In fact, one can choose local holomorphic coordinates $(z_1, \cdots, z_l, \cdots, z_m)$ near x_0 with $z_i(x_0) = 0$, $l \leq k(M)$, such that any $f \in \mathcal{O}_d(M)$ is a function of $z' = (z_1, \cdots, z_l)$, locally. Then we define a map

$$\Phi' : \mathcal{O}_g(M) \to C^{K'}$$

similarly as in the proof of Theorem 4.1, where $K' = \binom{l+[d]}{l}$. It is defined, for all l-multi-index α with $|\alpha| \leq [d]$, by assigning

$$f \to (f(x_0), D_{z'}f(x_0), \cdots, D^{\alpha}_{z'}f(x_0)).$$

Therefore, if $\dim(\mathcal{O}_d(M)) \geq \dim(\mathcal{O}_d(\mathbb{C}^{k(M)})) + 1$, one can find $f \in \mathcal{O}_d(M)$ such that $ord_{x_0}(f) > d$. This is a contradiction with Theorem 4.1.

Notice that there exist $\{f_j\}$, $j = 1, \cdots, l$, in $\mathcal{O}_d(M)$ such that near x_0, $z_j = f_j$. If the equality holds in the dimension estimate, we know that $l = k(M)$ in the above and the map Φ' is onto. Therefore, for each multiple index α with $|\alpha| \leq d$, we can find $g_\alpha \in \mathcal{O}_d(M)$ such that $g_\alpha = (z')^\alpha +$ higher order terms near x_0. Here we, in particular, let $g_i = z_i^d +$ higher order terms. By the unique continuation we have the formal power series P_α with vanishing order greater than d such that $g_\alpha = f_1^{\alpha_1} \cdots f_{k(M)}^{\alpha_{k(M)}} + P_\alpha(f_1, \cdots, f_{k(M)})$. For the sake of simplicity we consider the case $k(M) = m-1$. Now define $u_i(x) = \log|g_i|$ and its heat equation deformation $v_i(x,t)$ similarly as in the proof of Theorem 4.1. Let $u(x) = \frac{1}{2}\log(\sum_i^{k(M)}|g_i|^2)$. Denote its heat equation deformation by $v(x,t)$. Clearly the equality in (2.15) holds for $v_i(x,t)$ and $v(x,t)$. This implies that the universal cover $\tilde{M}$ of M splits as $\tilde{M} = \mathbb{C}^{k'} \times \tilde{M}_1$.

Here $k' \geq k(M)$ by the consideration on the dimension of singularity of $u(x)$ as in the proof of Theorem 4.1. We claim that $k' = m-1 = k(m)$. Otherwise, $v(x,t)$ will be positive definite everywhere. By the L^2 estimates (see the proof of Corollary 0.1 below), one will have that $k(M) = m$, which is a contradiction. The equality also implies that linear functions of $\mathbb{C}^{k(M)}$ are equivariant. This implies that the deck transformation action on $\mathbb{C}^{k(M)}$ is trivial. Therefore $M = \mathbb{C}^{k(M)} \times M_2$, where M_2 is a quotient of $\tilde{M}_2$. Now it is clear that M_2 does not contain any nonconstant holomorphic functions of polynomial growth by the definition of $k(M)$. (Note again, $\tilde{M}_1$ may contain factors of $\mathbb{C}$, and in general we do not have that $\dim(\mathcal{O}_d(M)) = \dim(\mathcal{O}_d(\tilde{M}))$.) $\square$

By Theorem 4.3 of [NT2] we know that if M is simply-connected, M splits as $M = M' \times M''$, where $\dim(M') = k(M) = k(M')$ and M'' does not support any nonconstant holomorphic functions of polynomial growth. In fact, the argument in [NT2] proves more. We state the result in a theorem below. Before doing so we introduce some notation.

We denote the spaces of holomorphic functions with order, in the sense of Hadamard, less than or equal to a by $\mathcal{O}^{(a)}(M)$. We denote a manifold by M_a if $\mathcal{O}^{(b)}(M) = \mathbb{C}$ for all $b < a$. Let $\mathcal{M}_a(M)$ be the meromorphic function field generated by $\mathcal{O}^{(a)}(M)$. For any $1 > a \geq 0$, we say that M has the curvature decay $\mathbf{(CD)}_a$ if for any $\epsilon > 0$ there exists a positive constant C_ϵ such that for any r,

$$\fint_{B_o(r)} \mathcal{R}(y)\, dy \leq \frac{C_\epsilon}{(1+r)^{2-(a+\epsilon)}}. \tag{5.4}$$

The following result can be proved using the induction and the arguments of Theorem 4.2 and Theorem 4.3 of [NT2].

Theorem 5.1 ([NT2])**.** *Let M be a simply-connected complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Then $\mathcal{O}^{(1)}(M) \neq \mathbb{C}$. Namely, M supports nonconstant holomorphic functions of order at most 1. Moreover, there exists the following isometric-holomorphic splitting:*

$$M = \mathbb{C}^{k_1} \times M_P \times M_{a_1} \times M_{a_2} \times \cdots \times M_{a_{k_2}} \times M_1 \times N$$

where N is a compact Kähler manifold with nonnegative bisectional curvature. Here $0 \leq a_1 \leq a_2 \leq \cdots \leq a_{k_2} < 1$. $\mathcal{M}_{a_i}(M_{a_i})$ is of transcendence degree at least $\dim(M_{a_i})$. The same conclusion holds for the transcendence degree of $\mathcal{M}(M_P)$. Furthermore, M_{a_i} has the curvature decay property $\mathbf{(CD)}_{a_i}$, for all $a_i < 1$. In particular, $M = M' \times M''$, where $k(M') = k(M) = \dim(M')$ and $\mathcal{O}_P(M'') = \mathbb{C}$. In addition, any automorphism γ of M decomposes as automorphisms of each factor. Namely, $\gamma(x_1, x_2, \cdots, x_l) = (f_1(x_1), f_2(x_2), \cdots, f_l(x_l))$ if we write $x = (x_1, x_2 \cdots, x_l)$ according to the splitting of M. Here f_i is an automorphism of the i-th factor.

Intuitively, the factor manifolds in the above splitting picture 'open smaller and smaller angles at infinity' towards the right and eventually close up in the last factor N.

If M is not simply-connected, let $\tilde{M}$ be the universal covering (or just a covering space) of M. In general, $k(\tilde{M})$ can be much larger than $k(M)$. For example, let $\tilde{M} = \mathbb{C} \times \mathbb{C}$. Let $\gamma(z, w) = (-z, w + 2\pi)$. Let $G = \{\gamma^k | k \in \mathbb{Z}\}$ and $M = \tilde{M}/G$. Then $k(M) = 1$ and $k(\tilde{M}) = 2$. The following example indicates the advantage of estimates via the Kodaira dimension. Let Σ denote the 2-dimensional cigar soliton. Consider the manifold $\mathbb{C}^k \times \Pi^{2l}\Sigma$. It has Kodaira dimension k. But the volume growth order is $2k+2l$. This shows that the estimate in Theorem 0.3 is sharper than the consideration via the volume growth order in [LT1].

Finally we prove Corollary 0.1. In order to do so we need the following well-known result on the L^2-estimate of $\bar\partial$.

Theorem 5.2 (cf. [AV], [D]). *Let (E,H) be a Hermitian line bundle with semi-positive curvature on the complete Kähler manifold (M,g) of dimension m. Suppose $\varphi : M \to [-\infty, 0]$ is a function of class C^∞ outside a discrete subset S of M and, near each point $p \in S$, $\varphi(z) = A_p \log|z|^2$ where A_p is a positive constant and $z = (z_1, z_2, \ldots, z_m)$ are local holomorphic coordinates centered at p. Assume that $\Theta(E, H\exp(-\varphi)) = \Theta(E,H) + \partial\bar\partial\varphi \geq 0$ on $M \setminus S$, and let $\epsilon : M \to [0,1]$ be a continuous function such that $\Theta(E,H) + \partial\bar\partial\varphi \geq \epsilon\omega_g$ on $M\setminus S$. Then, for every C^∞ form θ of type $(m,1)$ with values in L on M which satisfies*

$$\bar\partial\theta = 0 \text{ and } \int_M \epsilon^{-1}|\theta|^2 e^{-\varphi}\, dv_g < \infty,$$

there exists a C^∞ form η of type $(m,0)$ with values in L on M such that

$$\bar\partial\eta = \theta \text{ and } \int_M |\eta|^2 e^{-\varphi}\, dv_g \leq \int_M \epsilon^{-1}|\theta|^2 e^{-\varphi}\, dv_g < \infty.$$

Proof of Corollary 0.1. The proof uses a nice idea from [NR]. The estimate in Theorem 3.1 also holds the key to making their argument work in this case. Let $\tilde M$ be a covering space of M. Denote by π the covering map. By the assumption that $deg_{tr}(\mathcal{M}(M)) = m$ we know that $k(M) = m$, from Proposition 5.1. In particular, this implies that there exists a smooth plurisubharmonic function ϕ, which can be constructed using the holomorphic functions in $\mathcal{O}_P(M)$ such that ϕ is strictly plurisubharmonic at some point $p \in M$. Moreover, it satisfies

$$0 \leq \phi(x) \leq C_5(M)\log(r(x)+2). \tag{5.5}$$

Here $r(x)$ is the distance function to a fixed point $o \in M$ and $C_5(M)$ is a positive constant only depending on M. Now we choose a small coordinate neighborhood W near p such that it is evenly covered by π such that $\sqrt{-1}\partial\bar\partial\phi > 0$ in W. Choose $p \in U \subset W$. Let U_i (W_i), $i = 1, \cdots, l$, be the disjointed pre-images of U (W), and let p_i be the pre-images of p. Here l is the number of the covering sheets, which could be infinity a priori. We use the coordinates $(z_1, \cdots, z_m)$ in W, as well as in U_i such that p (p_i) is the origin. Clearly, it does not hurt to assume that W is inside the ball $\{z | |z| \leq 1\}$. We also define $\varphi_p(x) = \rho(x)\log|z|^2$, where $\rho(x)$ is a cut-off function with support inside W, equal to 1 in U. By changing the constant in (5.5), we can make sure that $\phi + \varphi_p$ is plurisubharmonic and

$$\sqrt{-1}\partial\bar\partial\,(\phi + \varphi_p) > 0 \tag{5.6}$$

inside U. Now we denote by $\tilde\phi$ the lift of ϕ to the cover $\tilde M$. Similarly $\tilde\varphi_p$ is the lift of φ_p. Clearly (5.5) holds for $\tilde\phi$ and (5.6) holds for $\tilde\phi + \tilde\varphi_p$. Now we use Theorem 5.1 to show that for any $d \in \mathbb{N}$, $1 \leq i \leq l$, and $\alpha = (\alpha_1, \cdots, \alpha_m)$ with $|\alpha| = d$, we can construct holomorphic functions $f^i_\alpha(x)$ on $\tilde M$ such that

$$D^\beta f^i_\alpha(p_j) = \begin{cases} 1, \beta = \alpha, j = i, \\ 0, \text{ otherwise.} \end{cases}$$

To achieve this we first give such a function locally in U_i, which is trivial to do, and then extend it by cutting-off to the whole of $\tilde{M}$. Let's call it ζ. Notice that we can arrange ζ to be holomorphic in U_i. Now we let $\theta = \bar{\partial}\zeta$. Apply Theorem 5.1 with $E = K_{\tilde{M}}^{-1}$ with the metric $|\cdot|\exp(-\mu(\tilde{\phi}+\tilde{\varphi}_p))$ (namely $\varphi = \tilde{\phi}+\tilde{\varphi}_p$) with $\mu = d+m+3$. Theorem 5.1 then provides η such that $\bar{\partial}\eta = \theta$. Now $\zeta - \eta$ gives the wanted holomorphic function, since by the choice of μ, the finiteness of

$$\int_M |\eta|^2 \exp(-\mu(\tilde{\phi}+\tilde{\varphi}_p))\,dv \tag{5.7}$$

implies that η vanishes at least up to order $d+1$ at p_j. The finiteness of (5.7) also implies that

$$\int_M |f^i_\alpha|^2 \exp(-\mu(\tilde{\phi}+\tilde{\varphi}_p))\,dv \le B < \infty \tag{5.8}$$

for some positive constant B. Observe that $\varphi_p \le 0$. Combining with (5.5) we further have

$$\int_{B_{\tilde{o}}(r)} |f^i_\alpha|^2\,dv \le B(r(x)+2)^{\mu C_5}. \tag{5.9}$$

Using the mean value inequality of Li and Schoen we can conclude that $f^i_\alpha \in \mathcal{O}_{\frac{\mu C_5}{2}}(\tilde{M})$. Noticing that the f^i_α are linearly independent, the dimension of the space spanned by them is bounded from below by $C_6(m)d^m l$. But from Theorem 3.1 we also know that

$$\dim(\mathcal{O}_{\frac{\mu C_5}{2}}(\tilde{M})) \le C_7(m)\left(\frac{\mu C_5}{2}\right)^m.$$

Plugging $\mu = d+m+3$ we have that

$$C_6(m)d^m l \le C_7(m)\left(\frac{C_5(d+m+3)}{2}\right)^m,$$

which implies that $l \le \frac{C_7(m)}{C_6(m)}\left(\frac{C_5}{2}\right)^m$, by letting $d \to \infty$.

□

Corollary 5.2. *Let M^m be a complete Kähler manifold with nonnegative bisectional curvature. Assume that the Ricci curvature is positive somewhere and the scalar curvature $\mathcal{R}(x)$ satisfies (2.7) and (3.24). Namely,*

$$\sup_{r\ge 0}\left(\exp(-ar^2)\fint_{B_o(r)} \mathcal{R}^2(y)\,dy\right) < \infty$$

and

$$\fint_{B_o(r)} \mathcal{R}(y)\,dv_y \le \frac{C}{r^2}$$

for some positive constants a and C. Then $k(M) = m$. In particular, $\pi_1(M)$ is finite.

Proof. The result follows from Corollary 0.1 and Corollary 6.2 of [NT2]. □

Remark 5.2. We believe that under the assumption of Corollary 0.1 or 5.2, M should be simply-connected. But we do not have a proof for it at this moment.

References

[AT] A. Andreotti and G. Tomassini, *Some remarks on pseudoconcave manifolds*, 1970 Essays on Topology and Related Topics (Memoires dédiés à Georges de Rham) pp. 85–104 Springer, New York. MR0265632 (42:541)

[AV] A. Andreotti and E. Vesentini, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 81–130. MR0175148 (30:5333)

[B] E. Bombieri, *Algebraic Values of Meromorphic Maps*, Invent. Math. **10** (1970), 267–287. MR0306201 (46:5328)

[BG] R. L. Bishop and S. I. Goldberg, *On the second cohomology group of a Kähler manifold of positive curvature*, Proceedings of AMS. **16** (1965), 119–122. MR0172221 (30:2441)

[Co] H.-D. Cao, *On Harnack inequalities for the Kähler-Ricci flow*, Invent. Math. **109** (1992), 247–263. MR1172691 (93f:58227)

[CFYZ] B. Chen, X. Fu, L. Yin and X-P. Zhu, *Sharp dimension estimates of holomorphic functions and rigidity, arXiv.math.DG/0311164.*

[CY] S. Y. Cheng and S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), 333–354. MR0385749 (52:6608)

[C] B. Chow, *Interpolating between Li-Yau's and Hamilton's Harnack inequalities on a surface*, J. Partial Differential Equations **11** (1998), no. 2, 137–140. MR1626999 (99h:58182)

[CH] B. Chow and R. Hamilton, *Constrained and linear Harnack inequalities for parabolic equations*, Invent. Math. **129** (1997), 213–238. MR1465325 (98i:53051)

[CN] B. Chow and L. Ni, *A new matrix LYH estimate for Kähler-Ricci flow and the interpolation with entropy monotonicity*, in preparation.

[CM] T. Colding and W. Minicozzi, *Weyl type bounds for harmonic functions*, Invent. Math. **131** (1998), 257–298. MR1608571 (99b:53052)

[D] J.-P. Demailly, L^2 *vanishing theorems for positive line bundles and adjunction theory*, Transcendental Methods in Algebraic Geometry, CIME, Cetrro, 1994, Lecture Notes in Math. 1646, Springer-Verlag, 1996. MR1603616 (99k:32051)

[Dn] S. Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. (3) **50** (1985), no. 1, 1–26. MR0765366 (86h:58038)

[Do] H. Donnelly, *Harmonic functions on manifolds of nonnegative Ricci curvature*, Internat. Math. Res. Notices **8** (2001), 429–434. MR1827086 (2002k:53062)

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley and Sons, 1978. MR0507725 (80b:14001)

[GW] R. E. Greene and H. Wu, *Analysis on noncompact Kähler manifolds*, Proc. Sympos. Pure Math. **30** (1977), 69–100. MR0460699 (57:692)

[H] R. Hamilton, *The Harnack estimate for the Ricci flow*, J. Differential Geom. **37** (1993), 225–243. MR1198607 (93k:58052)

[Ho] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, 3rd Edition, North Holland, 1990. MR1045639 (91a:32001)

[HS] G. Huisken and C. Sinestrari, *Surgeries on mean curvature flow of hypersurfaces*, work in progress.

[KL] L. Karp and P. Li, *The heat equation on complete Riemannian manifolds*, unpublished.

[KM] K. Kodaira and J. Morrow, *Complex Manifolds*, Holt, Rinehart and Winston, Inc. 1971. MR0302937 (46:2080)

[L1] P. Li, *Harmonic functions of linear growth on Kähler manifolds with nonnegative Ricci curvature*, Math. Res. Lett. **2** (1995), 79–94. MR1312979 (95m:53057)

[L2] P. Li, *Curvature and function theory on Riemannian manifolds*, Survey in Differential Geometry vol. VII, International Press, Cambridge, 2000, 71–111. MR1919432 (2003g:53047)

[LS] P. Li and R. Schoen, L^p *and mean value properties of subharmonic functions on Riemannian manifolds*, Acta Math. **153** (1984), 279–301. MR0766266 (86j:58147)

[LT1] P. Li and L.-F. Tam, *Linear growth harmonic functions on a complete manifold*, J. Differential Geom. **29** (1989), 421–425. MR0982183 (90a:58202)

[LT2] P. Li and L.-F. Tam, *Complete surfaces with finite total curvature*, J. Differential Geom. **33** (1991), 139–168. MR1085138 (92e:53051)

[LTW] P. Li, L.-F. Tam and J. Wang, *Sharp bounds for Green's functions and the heat kernel*, Math. Res. Letters **4** (1997), 589–602. MR1470428 (98j:58110)

[LW] P. Li and J. Wang, *Counting massive sets and dimensions of harmonic functions*, J. Differential Geom. **53** (1999), 237–278. MR1802723 (2001k:53063)

[LY] P. Li and S. T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), 139–168. MR0834612 (87f:58156)

[M] N. Mok, *An embedding theorem of complete Kähler manifolds of positive bisectional curvature onto affine algebraic varieties*, Bull. Soc. Math. France **112** (1984), 197–250. MR0788968 (87a:53103)

[MSY] N. Mok, Y.T. Siu and S. T. Yau, *The Poincaré-Lelong equation on complete Kähler manifolds*, Compositio Math. **44** (1981), 183–218. MR0662462 (84g:32011)

[N1] L. Ni, *The Poisson equation and Hermitian-Einstein metrics on complete Kähler manifolds*, Indiana Univ. Math. J. **51** (2002), 679–704. MR1911050 (2003m:32021)

[N2] L. Ni, *Monotonicity and Kähler-Ricci flow*, to appear in Contemp. Math., arXiv: math.DG/ 0211214.

[N3] L. Ni, *The entropy formula for linear heat equation*, Jour. Geom. Anal. **14** (2004), 85–98. MR2030576

[N4] L. Ni, *Hermitian harmonic maps from complete Hermitian manifolds to complete Riemannian manifolds*, Math. Z. **232** (1999), 331–355. MR1718630 (2001b:58031)

[NR] T. Napier and M. Ramanchandran, *The L^2 $\bar\partial$-method, weak Lefschetz theorems, and the topology of Kähler manifolds*, Jour. AMS. **11** (1998), 375–396. MR1477601 (99a:32008)

[NST] L. Ni, Y. Shi and L.-F. Tam, *Poisson equation, Poincaré-Lelong equation and curvature decay on complete Kähler manifolds*, J. Differential Geom. **57** (2001), 339–388. MR1879230 (2002j:53042)

[NT1] L. Ni and L.-F. Tam, *Plurisubharmonic functions and the Kähler-Ricci flow*, Amer. J. Math. **125** (2003), 623–654. MR1981036 (2004c:53101)

[NT2] L. Ni and L.-F.Tam, *Plurisubharmonic functions and the structure of complete Kähler manifolds with nonnegative curvature*, J. Differential Geom. **64** (2003), 457–524. MR2032112

[P] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv: math.DG/ 0211159.

[S] I. Shafarevich, *Basic Algebraic Geometry*, Vol II, Springer, 1994, Berlin. MR1328834 (95m:14002)

[Si] Y.-T. Siu, *Pseudoconvexity and the problem of Levi*, Bull. Amer. Math. Soc. **84** (1978), 481–512. MR0477104 (57:16648)

[W] H. Wu, *Polynomial functions on complete Kähler manifolds*, Several complex variables and complex geometry, Proc. Symp. Pure Math. **52** (1989), 601–610. MR1128575 (92g:32017)

[Y] S. T. Yau, *Open problems in geometry*, Lectures on Differential Geometry, by Schoen and Yau **1** (1994), 365–404.

[ZS] O. Zariski and P. Samuel, *Commutative Algebra I*, Graduate Texts in Math. Springer, 1958, New York. MR0384768 (52:5641)

Department of Mathematics, University of California, San Diego, La Jolla, Californiz 92093

E-mail address: `lni@math.ucsd.edu`