Local Aronson–Bénilan estimates and entropy formulae for porous medium and fast diffusion equations on manifolds

Peng Lu

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA

Lei Ni

Department of Mathematics, University of California at San Diego, La Jolla, CA 92093, USA

Juan-Luis Vázquez

Departamento de Matemáticas, Universidad Autónoma de Madrid, Campus de Cantoblanco, 28049 Madrid, Spain

Cédric Villani

Institut Universitaire de France and Unité de Mathématiques Pures et Appliquées, École Normale Supérieure de Lyon, 46 allée d'Italie, F-69364 Lyon Cedex 07, France

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Abstract

In this work we derive local gradient and Laplacian estimates of the Aronson–Bénilan and Li–Yau type for positive solutions of porous medium equations posed on Riemannian manifolds with a lower Ricci curvature bound. We also prove similar results for some fast diffusion equations. Inspired by Perelman's work we discover some new entropy formulae for these equations.

Dans cet article nous établissons des bornes locales à la Aronson–Bénilan sur le gradient et le laplacien de la pression, pour des solutions positives d'équations des milieux poreux sur des variétés riemanniennes à courbure de Ricci minorée. Nous obtenons des résultats similaires pour certaines équations de diffusion rapide. Inspirés par le travail de Perelman, nous mettons en évidence de nouvelles formules d'entropie pour ces équations.

Key words: Porous medium equation, fast diffusion equation, Aronson–Bénilan estimate, Li–Yau type estimate, local gradient bound, flow on manifold, entropy formula

1 Introduction

The porous medium equation (PME for short)

$$(1.1) \qquad\qquad \partial_t u = \Delta u^m,$$

where $m > 1$, is a nonlinear version of the classical heat equation (case $m = 1$). For various values of $m > 1$ it has arisen in different applications to model diffusive phenomena like groundwater infiltration (Boussinesq's model, 1903, with $m = 2$), flow of gas in porous media (Leibenzon–Muskat model, $m \geq 2$), heat radiation in plasmas ($m > 4$), liquid thin films moving under gravity ($m = 4$), crowd-avoiding population diffusion ($m = 2$), and others. The mathematical theory started in the 1950's and got momentum in recent decades as a nonlinear diffusion problem with interesting geometrical aspects (free boundaries) and peculiar functional analysis (like generating a contraction semigroup in L^1 and in Wasserstein metrics). We refer to the monograph [V4] for an account of the rather complete theory concerning existence, uniqueness, regularity and asymptotic behavior of PME, mostly in the setting of the Euclidean space and on open subsets of it, as well as the different applications.

The mathematical treatment of PME can be done in a more or less unified way for all parameters $m > 1$. Our main estimates below are only valid for nonnegative solutions, hence we will keep the restriction $u \geq 0$. This is reasonable from physical grounds since u represents a density, a concentration, a temperature or a height in the usual applications. However, re-writing (1.1) in the more general form $\partial_t u = \Delta(|u|^{m-1}u)$, solutions with changing sign can also be considered, but the theory is less advanced. It has been proved that for given initial data $u_0 \in L^1(\mathbb{R}^n)$ with $u_0 \geq 0$, there exists a unique continuous weak solution $u(x,t) \geq 0$ of the initial value problem of (1.1), with a number of properties.

Some of the existence, uniqueness and regularity properties hold true for the so-called fast diffusion equation (FDE), which is equation (1.1) with $m \in (0, 1)$. FDE appears in plasma physics and in geometric flows such as the Ricci flow on surfaces and the Yamabe flow. However, there are marked differences between PME and FDE that justify a separate treatment of FDE, cf. [DK],

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Email addresses: `penglu@uoegon.edu` (Peng Lu), `lni@math.ucsd.edu` (Lei Ni), `juanluis.vazquez@uam.es` (Juan-Luis Vázquez), `cvillani@umpa.ens-lyon.fr` (Cédric Villani).

[V3]. In particular, the qualitative properties of FDE become increasingly complex for small m, far away from $m = 1$ (very fast diffusion).

As is typical of nonlinear problems, the mathematical theory of PME and FDE is based on a priori estimates. In 1979, Aronson and Bénilan obtained a celebrated second-order differential inequality of the form ([AB])

$$(1.2) \qquad \sum_i \frac{\partial}{\partial x^i} \left(m u^{m-2} \frac{\partial u}{\partial x^i} \right) \geq -\frac{\kappa}{t}, \qquad \kappa := \frac{n}{n(m-1)+2}$$

which applies to all positive smooth solutions of (1.1) defined on the whole Euclidean space[1] on the condition that $m > m_c := 1 - 2/n$. Note that $\sum \frac{\partial}{\partial x^i} \left(m u^{m-2} \frac{\partial u}{\partial x^i} \right) = \Delta \left(\frac{m}{m-1} u^{m-1} \right)$ when $m \neq 1$. Precisely for the heat equation formula (1.2) takes the form

$$(1.3) \qquad \Delta \log u + \frac{n}{2t} \geq 0.$$

obtained by setting $m = 1$. Estimate (1.2) has turned out to be a key estimate in the development of the theory of the PME and FDE posed in the whole Euclidean space. In [AB] the estimate was used to prove the existence of initial value problem for PME and FDE. However, it has turned out difficult to find variants of (1.2) that hold for flows posed on open domains, unless in dimension $n = 1$.

In 1986 Li and Yau studied a heat type flow on manifolds ([LY]). Among other things, they proved the following Li–Yau differential Harnack inequality. If (M^n, g) is a complete Riemannian manifold with nonnegative Ricci curvature and $u : M \times [0, \infty) \to \mathbb{R}$ is a positive solution to the heat equation $\partial_t u = \Delta u$, then there is a lower bound for $\Delta_g \log u$ that has the precise sharp form (1.3). This extends to the manifold setting the Euclidean case mentioned above. Li and Yau also proved a local result, which implies (1.3) when u is a global positive solution and M is a complete Riemannian manifold with nonnegative Ricci curvature. More precisely, they proved the following theorem.

Theorem 1.1. *Let (M^n, g) be a complete Riemannian manifold satisfying Ricci curvature $\mathrm{Ric}(M) \geq -K^2$ for some $K \geq 0$. Let $B(\mathcal{O}, 2R)$ be a ball of radius $2R$ centered at $\mathcal{O}$. Assume that $u(x, t)$ is a positive smooth solution to the heat equation on $B(\mathcal{O}, 2R) \times [0, \infty)$. Then for any $\alpha > 1$ the following estimate holds on $B(\mathcal{O}, R)$*

$$\sup_{B(\mathcal{O},R)} \left(\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \right) \leq \frac{C\alpha^2}{R^2} \left(\frac{\alpha^2}{\alpha^2 - 1} + KR \right) + \frac{n\alpha^2 K}{2(\alpha - 1)} + \frac{n\alpha^2}{2t}.$$

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[1] In dimension $n = 1$ the restriction is $m > 0$ for general solutions, but we may keep $m > -1$ for so-called maximal solutions ([ERV]).

Here C is a constant only depending on n.

The Li–Yau estimate implies in particular the following classical Aronson-type upper bound on the heat kernel $p(t, x, y)$: *For any $\epsilon > 0$,*

$$p(t, x, y) \leq C(\epsilon, n) \frac{e^{-\frac{r^2(x,y)}{(4+\epsilon)t} + C_1 \epsilon K^2 t}}{V^{\frac{1}{2}}(x, \sqrt{t}) V^{\frac{1}{2}}(y, \sqrt{t})},$$

where $r(x, y)$ is the distance between x and y, $C_1 = C_1(n) > 0$, $V(x, r)$ is the volume of the ball $B(x, r)$.

The theory of PME and FDE on manifolds has not been considered until recently. Demange studied these equations in relation to Sobolev inequalities [D1,D2,D3,D4]. The extension of the Aronson–Bénilan estimate (1.2) to the PME on a complete noncompact Riemannian manifold with nonnegative Ricci curvature was done in the book [V4, Chapter 10].

In this paper we prove an extension of the Aronson–Bénilan estimate to the PME flow for all $m > 1$ (Theorem 3.3) and the FDE flow for $m \in (m_c, 1)$ (Theorem 4.1) on complete Riemannian manifolds with Ricci curvature bounded below. The estimates are of local type, hence even on Euclidean space, they give more information. The estimates look much better when the Ricci curvature is nonnegative.

Recall that for the positive solution $u := \frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}$, such that $u^{\frac{1}{2}} \in W^{1,2}(M)$, to the heat equation, it was shown in [N1] that

$$(1.4) \qquad \frac{d\mathcal{W}}{dt} = -\int_M 2t \left(\left| \nabla_i \nabla_j f - \frac{1}{2t} g_{ij} \right|^2 + R_{ij} f_i f_j \right) u \, d\mu$$

where

$$\mathcal{W}(t) = \int_M \left(t |\nabla f|^2 + f - n \right) u \, d\mu.$$

Using a basic identity involved in the proof of Theorem 3.3, we also obtain entropy formulae in the style of Perelman ([P]). This new entropy formula is the PME/FDE analogue of (1.4).

ORGANIZATION. In §2 we introduce the main ideas of the regularity question for the PME in the Euclidean setting. We then pose the problems that have to be addressed. §3 contains the new local estimate for positive solutions of the PME on Riemannian manifolds with Ricci curvature bounded below. The estimate admits a version valid for the FDE if $m \in (m_c, 1)$, which is developed in §4. Consequences in the form of Harnack inequalities for PME and FDE are derived in Sections 3 and 4 respectively. §5 introduces and studies the entropies.

2 Regularity of solutions of PME and Aronson–Bénilan estimate

A key idea in the PME theory comes from the observation that we can write the equation as a diffusion equation for a substance with density $u(x,t) \geq 0$:

$$(2.1) \qquad \partial_t u = \nabla \cdot (c(u)\nabla u),$$

We find a case of density-dependent diffusivity, i.e., $c(u) = mu^{m-1}$, so that c vanishes at $u = 0$; this makes the equation degenerate parabolic. It also implies the property of finite propagation, appearance of free boundaries, and limited regularity. Typical solutions with free boundaries are only Hölder continuous in space and time.

The second key idea in the study of the PME is to write the equation as a law of mass conservation

$$(2.2) \qquad \partial_t u = -\nabla \cdot (u\mathbf{V}),$$

which identifies the speed as $\mathbf{V} = -mu^{m-2}\nabla u$, and this in turn allows to write $\mathbf{V}$ as a potential flow, $\mathbf{V} = -\nabla p$. This gives for the potential the expression $p = mu^{m-1}/(m-1)$. In the application to gases in porous media the potential is just the pressure and the linear speed-pressure relation is known as Darcy's law. Historically, the letter v has been used for the pressure instead of p, and we will keep that tradition. This variable v is crucial in the study of free boundaries and regularity. Note that the *pressure* $v := mu^{m-1}/(m-1)$ satisfies

$$(2.3) \qquad \partial_t v = (m-1)v\Delta v + |\nabla v|^2.$$

Thus, near the level $u = 0$ we have the formal approximation $\partial_t v \sim |\nabla v|^2$, that can be easily identified as movement of the front with speed $-\nabla v$. It also means that the equation is approximately first-order so that we expect the Lipschitz continuity of v near the free boundary.

Now we turn to regularity estimates for PME. The question of Lipschitz regularity of the pressure was solved in one space dimension, $n = 1$, by Aronson who proved a local estimate for v_x using the Bernstein technique ([Ar]): a bounded solution defined in a cylinder in space-time $Q = [a,b] \times [0,T]$ has a uniform bound for $|v_x|$ inside the domain, i.e., in $Q' = [a',b'] \times [T',T]$, with $a < a' < b' < b$, $0 < T' < T$. Bénilan proved that in that situation v_t is locally bounded in a similar way ([Be]).

The extension of such results to dimension $n > 1$ fails, even for globally defined solutions. Indeed, it was shown in 1993 that the so-called focusing solutions are not Lipschitz continuous at the focusing point ([AG]), though wide classes of solutions can be Lipschitz continuous under special conditions ([CVW]). And other kinds of pointwise gradient estimates also failed. The problem of

minimal regularity was reduced to proving Hölder regularity, and this was done around 1980 by Caffarelli and Friedman ([CF1], [CF2]). The proof of these results and the whole theory of the porous medium equation in several space dimensions was greatly affected by the existence of special one-sided estimates that we discuss next.

The Aronson–Bénilan estimate (1.2) can be written as

$$(2.4) \qquad \Delta v \geq -\frac{\kappa}{t}, \quad \text{when } m > m_c,$$

where we define $v = \log u$ for $m = 1$. Note that with this definition $v \leq 0$ for $0 < m < 1$ so care must be taken in manipulating inequalities when dealing with fast diffusion. Using the pressure equation (2.3), it immediately implies that for PME with $m > 1$

$$(2.5) \qquad v_t \geq |\nabla v|^2 - \frac{(m-1)\kappa}{t} v.$$

so in particular

$$(2.6) \qquad v_t \geq -(m-1)\kappa v/t, \quad \text{and} \quad u_t \geq -\kappa u/t.$$

Other forms of parabolic Harnack inequalities follow from such estimates, and lead to Hölder regularity statements easily. These estimates have been used for all kinds of purposes in the theory, like existence of solutions in optimal classes of data, or asymptotic behavior, cf. [V4].

A striking property of the Aronson–Bénilan estimate is the fact that the constant κ is optimal when $m > m_c$. Indeed, the Barenblatt (or Barenblatt–Pattle) solutions, in terms of the pressure, are given by $v = V_C$, where

$$(2.7) \qquad V_C(x,t) = \frac{(C\, t^{2\kappa/n} - \kappa x^2)_+}{2n\, t}, \quad C > 0.$$

Equality holds in (2.4) for $v = V_C$ on the set $\{V_C > 0\}$. When $m = 1$ the estimates (1.2) is optimal since equality holds in (1.3) for the Gaussian kernel. In some sense the Barenblatt solutions play for the PME a role that the Gaussian kernel plays for the heat equation.

Many attempts have been made to obtain an extension of the estimate or a suitable variant for problems where the PME or the FDE are not posed on the whole Euclidean space: this can take the form of boundary value problems in bounded domains of $\mathbb{R}^d$, the PME posed on a Riemannian manifold, or even better, a local estimate, valid in any one of the above two settings. A straight extension of the global estimate to boundary value problems in several dimensions has not been done. (In the case of homogeneous Dirichlet problems a literal extension is even false, in view of explicit solutions.) But there is a

hope for local estimates. In one space dimension, the local estimate of v_{xx}, hence of v_t, from below was obtained in [V1], and the bound has a correction term involving the distance to the boundary. But the method fails for $n > 1$ because it uses the previous knowledge of the local bound of v_x. A modified version of the local estimate will be the first objective of the present paper.

Another research direction concerns the extension of Aronson–Bénilan estimate to other equations, like the p-Laplacian equation or reaction-diffusion equations. Some work have been done, for example, in [EV] for the p-Laplacian heat equation on Euclidean space, and recently in [KN] for the p-Lapacian heat equation and (local and global) doubly nonlinear equation on manifolds.

3   Local Aronson–Bénilan estimates for the PME

We proceed now with the new estimates. Let $u \geq 0$ be a solution to the Porous Medium Equation (1.1), $m > 1$, posed on an n-dimensional complete Riemannian manifold (M^n, g). We will assume at least a local bound from below for the Ricci tensor. The initial and boundary-value problems for this equation are usually formulated in terms of weak solutions, or better continuous weak solutions [V4]. Our local estimate is more closely related to the result of Li–Yau mentioned in the introduction, than that of say [SZ].

We will work with the pressure v, which satisfies equation (2.3). We see that $\nabla v = m u^{m-2} \nabla u$, and in the case $m < 2$ this equation only makes sense over $u > 0$. In order to avoid this and other regularity difficulties in our computations we will assume that the solutions are positive and smooth everywhere. The smoothness property comes from local boundedness and positivity of u in view of standard non-degenerate parabolic theory. Application of our results for general weak solutions proceeds in a standard way by approximating, and using the maximum principle and the local compactness of the classes of solutions involved. We refrain from more details on this issue, cf. [D1,D4,V4].

3.1 Assuming that $u > 0$ we introduce the quantities $y = |\nabla v|^2/v$, and $z = v_t/v$ and the differential operator

$$\mathcal{L} := \frac{\partial}{\partial t} - (m-1)v\Delta.$$

We also introduce the differential expression $\ F_\alpha := \alpha z - y$. Using equation (2.3) we can write the equivalent formulae

$$(3.1) \qquad F_\alpha = (m-1)\Delta v + (\alpha - 1)\frac{v_t}{v} = \alpha(m-1)\Delta v + (\alpha - 1)\frac{|\nabla v|^2}{v}.$$

In particular, $F_1 = (m-1)\Delta v$. Though our main goal is to estimate F_1, we will use the localization technique of Li and Yau to estimate F_α for $\alpha > 1$. One reason is that sometimes the estimate of F_1 is not feasible, for instance, when we want to obtain local estimates.

The goal of this subsection is to calculate a formula for $\mathcal{L}(F_\alpha)$. The following formula is helpful in the calculation:

$$(3.2) \qquad \mathcal{L}\left(\frac{f}{g}\right) = \frac{1}{g}\mathcal{L}(f) - \frac{f}{g^2}\mathcal{L}(g) + 2(m-1)v\left\langle \nabla\left(\frac{f}{g}\right), \nabla \log g\right\rangle.$$

The Bochner-type formulae in Lemma 3.1 below is established by direct calculation. We shall write v_i (instead of $v_{,i}$) to denote partial derivatives, and v_{ij} to denote the Hessian tensor $H(v)_{ij}$ of v, while v_{ij}^2 denotes the standard squared norm of the Hessian (with implicit summation over indices); R_{ij} is the Ricci tensor and $R_{ij}v_iv_j = \mathrm{Ric}(\nabla v, \nabla v)$.

Lemma 3.1. *Let u be a positive smooth solution to (1.1) on manifold (M^n, g) for some $m > 0$, and let $v := \frac{m}{m-1}u^{m-1}$ be the pressure.[2] Then we have*

$$(3.3) \qquad \mathcal{L}(v_t) = 2\langle \nabla v, \nabla v_t\rangle + F_1 v_t,$$
$$\mathcal{L}(|\nabla v|^2) = 2|\nabla v|^2 F_1 + 2\langle \nabla(|\nabla v|^2), \nabla v\rangle$$
$$(3.4) \qquad\qquad -2(m-1)vv_{ij}^2 - 2(m-1)vR_{ij}v_iv_j.$$

The following proposition is a generalization of the computation carried out in Proposition 11.12 of [V4].

Proposition 3.2. *Let u and v be as in Lemma 3.1. Then*

$$\mathcal{L}(F_\alpha) = 2(m-1)v_{ij}^2 + 2(m-1)R_{ij}v_iv_j + 2m\langle \nabla F_\alpha, \nabla v\rangle$$
$$(3.5) \qquad\qquad +(\alpha-1)\left(\frac{v_t}{v}\right)^2 + F_1^2.$$

Proof. Using (3.2) and Lemma 3.1 we have

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[2] Recall that when $m = 1$, we interpret $v = \log u$.

$$\mathcal{L}\left(\frac{|\nabla v|^2}{v}\right) = \frac{1}{v}\left(2|\nabla v|^2 F_1 + 2\langle\nabla(|\nabla v|^2), \nabla v\rangle\right)$$

$$-2(m-1)v_{ij}^2 - 2(m-1)R_{ij}v_i v_j - \frac{|\nabla v|^4}{v^2}$$

$$+2(m-1)v\left\langle\nabla\left(\frac{|\nabla v|^2}{v}\right), \nabla\log v\right\rangle;$$

$$\mathcal{L}\left(\frac{v_t}{v}\right) = \frac{1}{v}\left(2\langle\nabla v, \nabla v_t\rangle + F_1 v_t\right)$$

$$-\frac{v_t}{v}\frac{|\nabla v|^2}{v} + 2(m-1)v\left\langle\nabla\left(\frac{v_t}{v}\right), \nabla\log v\right\rangle.$$

Putting together gives

$$\mathcal{L}(F_\alpha) = 2(m-1)v\langle\nabla F_\alpha, \nabla\log v\rangle + 2(m-1)v_{ij}^2 + 2(m-1)R_{ij}v_i v_j$$

$$+\alpha\frac{v_t}{v}F_1 - 2\frac{|\nabla v|^2}{v}F_1 - \alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v} + \frac{|\nabla v|^4}{v^2}$$

$$+\frac{2}{v}\langle\nabla v, \alpha\nabla v_t - \nabla(|\nabla v|^2)\rangle.$$

Using
$$\langle\nabla v, \nabla(vF_\alpha)\rangle = v\langle\nabla v, \nabla F_\alpha\rangle + F_\alpha|\nabla v|^2,$$
we can rewrite the last term in the above formula for $\mathcal{L}(F_\alpha)$ as

$$\frac{2}{v}\langle\nabla v, \alpha\nabla v_t - \nabla(|\nabla v|^2)\rangle = 2\langle\nabla v, \nabla F_\alpha\rangle + 2F_\alpha\frac{|\nabla v|^2}{v}.$$

Hence we get

$$\mathcal{L}(F_\alpha) = 2mv\langle\nabla F_\alpha, \nabla\log v\rangle + 2(m-1)v_{ij}^2 + 2(m-1)R_{ij}v_i v_j$$

$$+\alpha\frac{v_t}{v}F_1 - 2\frac{|\nabla v|^2}{v}F_1 - \alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v} + \frac{|\nabla v|^4}{v^2} + 2F_\alpha\frac{|\nabla v|^2}{v}.$$

Note that the last five terms simplify, as

$$\alpha z(z - y) - 2y(z - y) - \alpha zy + y^2 + 2(\alpha z - y)y = (\alpha - 1)z^2 + (z - y)^2.$$

This completes the proof of the proposition. $\qquad\square$

When $\alpha = 1$, (3.5) becomes the following formula in [V4]:

(3.6) $\qquad \mathcal{L}F_1 = 2(m-1)v_{ij}^2 + 2(m-1)R_{ij}v_i v_j + 2m\langle\nabla F_1, \nabla v\rangle + F_1^2.$

From this, for positive smooth solution u to (1.1) with $m > 1$ on a closed Riemannian manifold of dimension n with nonnegative Ricci curvature, the

following estimate follows easily from maximum principle (see Proposition 11.12 of [V4]):

$$(3.7) \qquad F_1 \geq -\frac{(m-1)\kappa}{t}, \quad \kappa := \frac{n}{n(m-1)+2}.$$

3.2 Now we prove a new local estimate for PME on complete manifolds. We employ the localization technique of Li and Yau ([LY], see also [KN]). Denote by $B(\mathcal{O}, R)$ the ball of radius $R > 0$ and centered $\mathcal{O}$ in (M^n, g), and denote by $r(x)$ the distance function from $\mathcal{O}$ to x. The following constant will appear repeatedly in our estimates:

$$(3.8) \qquad a = \frac{n(m-1)}{n(m-1)+2} = (m-1)\kappa.$$

Theorem 3.3. *Let u be a positive smooth solution to PME (1.1), $m > 1$, on the cylinder $Q := B(\mathcal{O}, R) \times [0, T]$. Let v be the pressure and let $v_{\max}^{R,T} := \max_{B(\mathcal{O},R)\times[0,T]} v$.*

(1) Assume that Ricci curvature $\mathrm{Ric} \geq 0$ on $B(\mathcal{O}, R)$. Then, for any $\alpha > 1$ we have

$$(3.9) \qquad \frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v} \leq a\alpha^2 \left(\frac{1}{t} + \frac{v_{\max}^{R,T}}{R^2}(C_1 + C_2(\alpha)) \right)$$

on $Q' := B(\mathcal{O}, R/2) \times [0, T]$. Here a is defined by (3.8) and the positive constants C_1 and $C_2(\alpha)$ depend also on m and n.

(2) Assume that $\mathrm{Ric} \geq -(n-1)K^2$ on $B(\mathcal{O}, R)$ for some $K \geq 0$. Then, for any $\alpha > 1$, we have, on Q',
(3.10)
$$\frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v} \leq a\alpha^2 \left(\frac{1}{t} + C_3(\alpha)K^2 v_{\max}^{R,T} \right) + a\alpha^2 \frac{v_{\max}^{R,T}}{R^2}(C_2(\alpha) + C_1'(KR)) \ .$$

Here, a and $C_2(\alpha)$ are as before and the positive constants $C_3(\alpha)$ and $C_1'(KR)$ depend also on m and n.

Acceptable values of the constants are:

$$C_1 := 40(m-1)(n+2), \qquad C_2(\alpha) := \frac{200a\alpha^2 m^2}{\alpha - 1}$$

$$C_3(\alpha) := \frac{(m-1)(n-1)}{\alpha - 1}, \qquad C_1'(KR) := 40(m-1)[3 + (n-1)(1 + KR)].$$

Note that $C_1'(0) = C_1$.

Proof. (i) We start with an auxiliary calculation about suitable cutoff functions. We take a cut-off function $\eta(x)$ of the form $\eta(x) := \theta\left(r(x)/R\right)$, where $\theta(t)$ is a smooth monotone function satisfying the following conditions $\theta(t) \equiv 1$ for $0 \leq t \leq \frac{1}{2}$, $\theta(t) \equiv 0$ for $t \geq 1$, $(\theta')^2/\theta \leq 40$, and $\theta'' \geq -40\theta \geq -40$. (40 is just a convenient number, it could be optimized.) Direct calculation shows that on $B(\mathcal{O}, R)$

$$(3.11) \qquad \frac{|\nabla \eta|^2}{\eta} \leq \frac{40}{R^2},$$

and also

$$(3.12) \qquad \Delta\eta \geq -\frac{40((n-1)(1+KR)+1)}{R^2}, \quad \text{if} \ \ \operatorname{Ric} \geq -(n-1)K^2,$$

with the help of the Laplacian comparison theorem. In particular $\Delta\eta \geq -40nR^{-2}$ when $\operatorname{Ric} \geq 0$. We give some details about deriving (3.12). Note that

$$\Delta\eta = \frac{\theta'' \, |\nabla r|^2}{R^2} + \frac{\theta' \Delta r}{R}.$$

By the Laplacian comparison theorem when $\operatorname{Ric} \geq -(n-1)K^2$

$$\Delta r \leq (n-1)K \coth(Kr).$$

Since coth is decreasing, and $\theta' = 0$ when $r(x) < \frac{1}{2}R$, this implies

$$\Delta\eta \geq \frac{-40}{R^2} - \frac{\sqrt{40}(n-1)}{R}K\coth(\frac{1}{2}KR)$$

where we have used $|\theta'| \leq \sqrt{40}$. Using the inequality $K\coth(KR) \leq \frac{1}{R}(1+KR)$, one has

$$\Delta\eta \geq \frac{-40}{R^2} - \frac{2\sqrt{40}(n-1)}{R^2}(1+KR).$$

(ii) To obtain the desired estimates, we apply the operator $\mathcal{L}$ to the function $t\eta(-F_\alpha)$, then apply the maximum principle argument. As a first easy case, if $t\eta(-F_\alpha) \leq 0$ on Q, then (3.10) follows. So in the sequel we assume $\max_{(x,t)\in Q} t\eta(-F_\alpha) > 0$. Let (x_0, t_0) be a point where $t\eta(-F_\alpha)$ achieves a positive maximum. Clearly we have $t_0 > 0$, and at (x_0, t_0)

$$\nabla F_\alpha = -\frac{\nabla\eta}{\eta}F_\alpha, \quad \mathcal{L}(t\eta(-F_\alpha)) \geq 0.$$

All further calculations in this proof will be at (x_0, t_0). Let $C_4 := 40((n-1)(1+KR)+1)$, $\tilde{y} := \eta y = \eta\frac{|\nabla v|^2}{v}$ and $\tilde{z} := \eta z = \eta\frac{v_t}{v}$. Combining (3.5) with the above estimates of η, we see that when $\operatorname{Ric} \geq -(n-1)K^2$ on $B(\mathcal{O}, R)$,

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$$0 \leq \eta \mathcal{L}(t\eta(-F_\alpha))$$
$$\leq -t\eta^2 \left(2(m-1)v_{ij}^2 + 2(m-1)R_{ij}v_iv_j\right) + 2mt\eta^2\langle\nabla(-F_\alpha),\nabla v\rangle - (\alpha-1)t\eta^2 z^2$$
$$-t\eta^2 F_1^2 + 2t(m-1)v\eta\frac{|\nabla\eta|^2}{\eta}(-F_\alpha) + \frac{C_4}{R^2}(m-1)tv\eta(-F_\alpha) + \eta^2(-F_\alpha)$$
$$\leq -\frac{n(m-1)+2}{n(m-1)}\cdot t(\tilde{y}-\tilde{z})^2 + 2(n-1)(m-1)K^2 t\tilde{y}\eta v + 2mt(\tilde{y}-\alpha\tilde{z})|\nabla\eta||\nabla v|$$
$$-(\alpha-1)t\tilde{z}^2 + (\tilde{y}-\alpha\tilde{z})\left((m-1)\frac{(80+C_4)}{R^2}\cdot tv+1\right).$$

In the last inequality above we have used $v_{ij}^2 \geq \frac{(\Delta v)^2}{n}$ and $(m-1)\Delta v = z - y = F_1$. Now write $C_5 := 80 + C_4$ and

$$(\tilde{y}-\tilde{z})^2 = \frac{1}{\alpha^2}(\tilde{y}-\alpha\tilde{z})^2 + 2\frac{\alpha-1}{\alpha^2}(\tilde{y}-\alpha\tilde{z})\tilde{y} + \left(\frac{\alpha-1}{\alpha}\right)^2\tilde{y}^2.$$

Also note that

$$2mt(\tilde{y}-\alpha\tilde{z})|\nabla\eta|\cdot|\nabla v| \leq \frac{40}{R}mt(\tilde{y}-\alpha\tilde{z})\tilde{y}^{1/2}v^{1/2}.$$

Putting these together and assuming $\mathrm{Ric} \geq -(n-1)K^2$ on $B(\mathcal{O},R)$, we deduce

$$0 \leq -\frac{t}{a\alpha^2}(\tilde{y}-\alpha\tilde{z})^2 + t(\tilde{y}-\alpha\tilde{z})\left(-\frac{2(\alpha-1)}{a\alpha^2}\tilde{y} + \frac{40m}{R}\cdot\tilde{y}^{1/2}v^{1/2} + (m-1)\frac{C_5}{R^2}v\right)$$
(3.13)
$$+ (\tilde{y}-\alpha\tilde{z}) - \frac{1}{a}\left(\frac{\alpha-1}{\alpha}\right)^2 t\tilde{y}^2 + 2(m-1)(n-1)K^2 t\tilde{y}v\eta - (\alpha-1)t\tilde{z}^2.$$

(1) When $K = 0$, using $-Ax^2 + Bx \leq \frac{B^2}{4A}$, it follows from (3.13)

(3.14) $\quad 0 \leq -\frac{t}{a\alpha^2}(\tilde{y}-\alpha\tilde{z})^2 + (\tilde{y}-\alpha\tilde{z})\left(\frac{tv}{R^2}\left(\frac{200a\alpha^2 m^2}{\alpha-1} + (m-1)C_5\right) + 1\right).$

This gives the first estimate, (3.9).

(2) When $K \neq 0$, in (3.13) we handle the $(\tilde{y}-\alpha\tilde{z})$-term as in (3.14) with $C_6 := \frac{200a\alpha^2 m^2}{\alpha-1} + (m-1)C_5$, and use

$$-\frac{1}{a}\left(\frac{\alpha-1}{\alpha}\right)^2 t\tilde{y}^2 + 2(m-1)(n-1)tK^2\tilde{y}v\eta \leq C_7 tv^2,$$

where

$$C_7 := \frac{(m-1)^2(n-1)^2 a\alpha^2 K^4}{(\alpha-1)^2}.$$

Then the above quadratic inequality (3.13) on $(\tilde{y} - \alpha\tilde{z})$ reduces to

$$0 \leq -\frac{t}{a\alpha^2}(\tilde{y} - \alpha\tilde{z})^2 + \left(C_6\frac{tv}{R^2} + 1\right)(\tilde{y} - \alpha\tilde{z}) + C_7 tv^2.$$

This implies

$$\tilde{y} - \alpha\tilde{z} \leq \frac{a\alpha^2}{2}\left(C_6\frac{v}{R^2} + \frac{1}{t} + \sqrt{\left(C_6\frac{v}{R^2} + \frac{1}{t}\right)^2 + 4\frac{C_7}{a\alpha^2}v^2}\right)$$

$$\leq a\alpha^2\left(C_6\frac{v}{R^2} + \frac{1}{t} + \frac{(m-1)(n-1)K^2}{\alpha - 1}v\right).$$

The claimed result follows easily. $\square$

Using the local estimate one can generalize Proposition 11.12 of [V4] to non-compact complete Riemannian manifolds with nonnegative Ricci curvature.

Corollary 3.4. *Let $u(x,t)$, $t \in [0,T]$, be a smooth positive solution of the PME (1.1) with $m > 1$ on a complete manifold (M^n, g).*

(1) If (M, g) has nonnegative Ricci curvature, then (3.7) holds for $t \in (0, T]$, provided that $v(x,t) = o(r^2(x))$ uniformly in $t \in (0, T]$.

(2) If the Ricci curvature $\mathrm{Ric} \geq -(n-1)K^2$ on M for some $K \geq 0$ and $v_{\max} := \max_{M \times [0,T]} v < \infty$, then for any $\alpha > 1$

$$(3.15) \qquad \alpha\frac{v_t}{v} - \frac{|\nabla v|^2}{v} \geq -(m-1)\kappa\alpha^2\left(\frac{1}{t} + \frac{(m-1)(n-1)}{\alpha - 1}K^2 v_{\max}\right).$$

Proof. (1) Taking $R \to \infty$ and then $\alpha \to 1$ in (3.9) we have the result.

(2) Taking $R \to \infty$ in (3.10) we have the result. $\square$

Integrating along minimizing geodesic paths of the local estimate, one can obtain the following Harnack inequality. Here we just state the most general form. When $K = 0$, if we assume $v_{\min} := \min_{M \times [0,T]} v > 0$, the estimate simplifies by taking $\alpha \to 1$.

Corollary 3.5. *Same notation and assumptions as in Theorem 3.3. Denote $v_{\min}^{R/2,T}$ to be $\min_{B(\mathcal{O}, \frac{R}{2}) \times [0,T]} v$. Assume that $\mathrm{Ric} \geq -(n-1)K^2$ on $B(\mathcal{O}, R)$ for some $K \geq 0$. Then for any $x_1, x_2 \in B(\mathcal{O}, \frac{R}{6})$ and $0 \leq t_1 < t_2 \leq T$, and any*

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$\alpha > 1$

$$\frac{v(x_2, t_2)}{v(x_1, t_1)}$$

$$\geq \left(\frac{t_1}{t_2}\right)^{a\alpha} \exp\left(-\frac{\alpha d^2(x_1, x_2)}{4v_{\min}^{R/2,T}(t_2 - t_1)} - a\alpha(t_2 - t_1)v_{\max}^{R,T}\left(C_3(\alpha)K^2 + \frac{C_2(\alpha) + C_1'(KR)}{R^2}\right)\right).$$

where $d(x_1, x_2)$ is the distance and the constants $C_2(\alpha)$, $C_3(\alpha)$ and $C_1'(KP)$ are as in Theorem 3.3.

Proof. For the minimizing geodesic $\gamma(t)$ joining (x_1, t_1) and (x_2, t_2) we have

$$\log\left(\frac{v(x_2, t_2)}{v(x_1, t_1)}\right) = \int_{t_1}^{t_2}\left(\frac{v_t}{v} + \langle\frac{\nabla v}{v}, \dot\gamma\rangle\right) ds$$

$$\geq \int_{t_1}^{t_2}\left(\frac{v_t}{v} - \frac{|\nabla v|^2}{\alpha v} - \frac{\alpha|\dot\gamma|^2}{4v}\right) ds.$$

The result follows from the observation that $\gamma(s)$ lies completely inside $B(\mathcal{O}, \frac{R}{2})$ and the estimate in Theorem 3.3. $\square$

A different way manipulating the integration on geodesic path can have the following consequence of Corollary 3.4. This estimates are the analogue of the classical one for the positive solutions to the heat equation (in viewing for the heat equation $v = \log u$).

Corollary 3.6. *Same notation and assumptions as in Corollary 3.4. We further assume that $v_{\max} := \max_{M \times [0,T]} v < \infty$. Let $x_1, x_2 \in M$ and $0 < t_1 < t_2 \leq T$.*

(1) *If (M, g) has nonnegative Ricci curvature, then*

$$v(x_2, t_2) - v(x_1, t_1) \geq -(m-1)\kappa v_{\max} \log\frac{t_2}{t_1} - \frac{d^2(x_1, x_2)}{4(t_2 - t_1)}.$$

(2) *If Ricci curvature* $\mathrm{Ric} \geq -(n-1)K^2$ *for some $K \geq 0$, then for any $\alpha > 1$*

$$v(x_2, t_2) - v(x_1, t_1)$$

$$\geq -(m-1)\kappa\alpha v_{\max} \log\frac{t_2}{t_1} - \frac{(m-1)^2(n-1)\kappa\alpha}{\alpha - 1}K^2 v_{\max}^2(t_2 - t_1) - \frac{\alpha d^2(x_1, x_2)}{4(t_2 - t_1)}.$$

Proof. We only prove (2). Let $\gamma(t)$ to be a constant speed geodesic with $\gamma(t_1) =$

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x_1 and $\gamma(t_2) = x_2$. We compute using (3.15)

$$v(x_2, t_2) - v(x_1, t_1) \geq \int_{t_1}^{t_2} v_t + \langle \nabla v, \dot{\gamma} \rangle \, dt$$

$$\geq \int_{t_1}^{t_2} \left(\frac{1}{\alpha} |\nabla v|^2 - (m-1)\kappa\alpha \left(\frac{1}{t} + \frac{(m-1)(n-1)}{\alpha - 1} K^2 v_{\max} \right) v - \frac{1}{\alpha} |\nabla v|^2 - \frac{\alpha}{4} |\dot{\gamma}|^2 \right) dt$$

$$\geq -(m-1)\kappa\alpha v_{\max} \log \frac{t_2}{t_1} - \frac{(m-1)^2(n-1)\kappa\alpha}{\alpha - 1} K^2 v_{\max}^2 (t_2 - t_1) - \frac{\alpha d^2(x_1, x_2)}{4(t_2 - t_1)}.$$

$\square$

Theorem 3.3 can be used to give a local lower estimate for the Laplacian of v^β for $\beta > 1$.

Corollary 3.7. *Same assumptions and same notation as in Theorem 3.3. Let $1 < \beta < m/(m-1)$ fixed. Define α by $\frac{\alpha - 1}{\alpha} = (m-1)(\beta - 1)$.*

(1) *Assume* Ric ≥ 0 *on* $B(\mathcal{O}, R)$, *then we have on* Q'

$$\Delta v^\beta \geq -\kappa\alpha\beta \left(v_{\max}^{R,T} \right)^{\beta - 1} \left(\frac{1}{t} + \frac{v_{\max}^{R,T}}{R^2} (C_1 + C_2(\alpha)) \right)$$

(2) *Assume that* Ric $\geq -(n-1)K^2$ *for some* $K \geq 0$ *on* $B(\mathcal{O}, R)$. *Then we have on* Q'

$$\Delta v^\beta \geq -\kappa\alpha\beta \left(v_{\max}^{R,T} \right)^{\beta - 1} \left(\frac{1}{t} + C_3(\alpha) K^2 v_{\max}^{R,T} \right)$$

$$-\kappa\alpha\beta \frac{\left(v_{\max}^{R,T} \right)^\beta}{R^2} (C_2(\alpha) + C_1'(KR)).$$

where the constants $C_2(\alpha)$, $C_3(\alpha)$ *and* $C_1'(KP)$ *are as in Theorem 3.3.*

Proof. Clearly $\alpha > 1$. We compute

$$\Delta v^\beta = \beta v^{\beta - 1} \left(\Delta v + (\beta - 1) \frac{|\nabla v|^2}{v} \right)$$

$$= \frac{\beta}{\alpha(m-1)} \cdot v^{\beta - 1} \left(\alpha(m-1)\Delta v + (\alpha - 1) \frac{|\nabla v^2|}{v} \right)$$

$$= \frac{\beta}{\alpha(m-1)} \cdot v^{\beta - 1} F_\alpha.$$

The corollary follows easily from Theorem 3.3. $\square$

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4 Local Aronson–Bénilan estimates for FDE

The fast diffusion equation, FDE, is equation (1.1) with $m \in (0, 1)$. However, as we have seen, Aronson–Bénilan estimate for FDE on Euclidean space holds only in the range $1 > m > m_c := 1 - \frac{2}{n}$, where the relevant constant $\kappa = n/(n(m-1)+2)$ is still a positive number. This is also the range where the Barenblatt solutions can be written and play similar role as they play in the theory of PME. Hence the Aronson–Bénilan estimate holds on the range as it would be expected. There is another point needs to be made. Since $m < 1$, the pressure $v = \frac{m}{m-1} u^{m-1}$ is negative and moreover it is an inverse power of u. But $(m-1)v$ is still positive, hence, as shown, the inequalities (2.5) and (2.6) hold for all $m > m_c$.

4.1 Let u be a smooth positive solution to FDE (1.1) with $m \in (m_c, 1)$ on a closed Riemannian manifold (M^n, g) with nonnegative Ricci curvature. From (3.6), and since $m - 1 < 0$ we have

$$\mathcal{L}(F_1) - 2m\langle \nabla F_1, \nabla v \rangle \leq F_1^2 - 2(1-m)v_{ij}^2 \leq -\left(\frac{2}{(1-m)n} - 1 \right) F_1^2.$$

The following follows easily from maximum principle

$$(4.1) \qquad\qquad F_1 \leq -\frac{(m-1)\kappa}{t}, \qquad \text{i.e.,} \quad \Delta v \geq -\frac{\kappa}{t}.$$

Note the direction in the F_1-inequality differs from (3.7) because of $m < 1$; on the other side, the Δv-inequality is the same as in the case $m > 1$.

4.2 Now we prove a new local estimate for FDE with $m \in (m_c, 1)$ on complete manifolds. In this subsection we employ the same notation as in §3.2 and use a similar localization technique as that of Li and Yau. It turns out that this case technically is slightly harder than the previous case. For example we have to make use of the term $(\alpha - 1)t\tilde{z}^2$ (which was simply dropped before). As we have seen from §4.1 for $m \in (m_c, 1)$ we should estimate F_α from the above (instead of from the below). For the local estimate, another difference is that for FDE we will estimate F_α for $\alpha < 1$ (instead of $\alpha > 1$).

Let (x_0, t_0) be a point where function $t\eta F_\alpha$ achieves the positive maximum. Clearly we have $t_0 > 0$ and at (x_0, t_0)

$$\nabla F_\alpha = -\frac{\nabla \eta}{\eta} F_\alpha, \qquad \mathcal{L}(t\eta F_\alpha) \geq 0.$$

All further calculation in this proof will be at (x_0, t_0). Combining (3.5) with the estimates of η, we have that when $\text{Ric} \geq -(n-1)K^2$ on $B(\mathcal{O}, R)$,

$$0 \le \eta \mathcal{L}(t\eta F_\alpha)$$
$$\le t\eta^2 \left(2(m-1)v_{ij}^2 + 2(m-1)R_{ij}v_iv_j\right) + 2mt\eta^2 \langle \nabla F_\alpha, \nabla v\rangle + (\alpha - 1)t\eta^2 z^2$$
$$+ t\eta^2 F_1^2 + 2t(m-1)v\eta \frac{|\nabla \eta|^2}{\eta} F_\alpha + \frac{C_4}{R^2}(m-1)tv\eta F_\alpha + \eta^2 F_\alpha$$
$$\le \frac{n(m-1)+2}{n(m-1)} t(\tilde{y} - \tilde{z})^2 - 2(n-1)(m-1)K^2 t\tilde{y}\eta v + 2mt(\alpha\tilde{z} - \tilde{y})|\nabla \eta||\nabla v|$$
$$+ (\alpha\tilde{z} - \tilde{y})\left((m-1)\frac{(80+C_4)}{R^2} \cdot tv + 1\right) - (1-\alpha)t\tilde{z}^2.$$

Noticing that now we have $(m-1)v > 0$ and $a < 0$ for $m \in (m_c, 1)$. Proceeding as in the proof of (3.13) we then have that when $\mathrm{Ric} \ge -(n-1)K^2$ on $B(\mathcal{O}, R)$,

$$0 \le -\frac{t}{-a\alpha^2}(\alpha\tilde{z} - \tilde{y})^2 + t(\alpha\tilde{z} - \tilde{y})\left(\frac{2(1-\alpha)}{-a\alpha^2}(-\tilde{y}) + \frac{40m}{R} \cdot (-\tilde{y})^{1/2}(-v)^{1/2} + (m-1)\frac{C_5}{R^2}v\right)$$
$$+ (\alpha\tilde{z} - \tilde{y}) - \frac{t}{-a}\left(\frac{1-\alpha}{\alpha}\right)^2 \tilde{y}^2 - 2(m-1)(n-1)K^2 t\tilde{y}\eta v - (1-\alpha)t\tilde{z}^2.$$

Here we can not do what was done in the proof of (3.13) since $\frac{2(1-\alpha)}{-a\alpha^2}$ the coefficient in front of $-\tilde{y}$ (> 0) is positive instead of being negative. For simplicity let $X = \alpha\tilde{z} - \tilde{y} > 0$, $Y = -\tilde{y} > 0$, $\beta = 1 - \alpha > 0$, $\gamma = -a > 0$. By writing $\tilde{z}^2 = \frac{1}{\alpha^2}(\alpha\tilde{z} - \tilde{y} + \tilde{y})^2$, and collecting terms we then have

$$0 \le -\frac{t}{\gamma\alpha^2}\left((1+\gamma\beta)X^2 - 2\beta(1+\gamma)XY + \beta(\beta+\gamma)Y^2\right) + 2(n-1)[(m-1)v]K^2 tY$$
$$+ \frac{40mt}{R}XY^{\frac{1}{2}}(-v)^{\frac{1}{2}} + \frac{tC_5}{R^2}X[(m-1)v] + X.$$

Let $\bar{v}_{\max}^{R,T} := \max_{B(\mathcal{O},R)\times[0,T]}(-v)$. Using that for any $\epsilon_1, \epsilon_2 > 0$

$$\frac{40mt}{R}XY^{\frac{1}{2}}(-v)^{\frac{1}{2}} \le \frac{2\beta\epsilon_1}{\gamma\alpha^2}tXY + \frac{t\gamma\alpha^2}{2\epsilon_1\beta}\frac{1600m^2}{R^2}\bar{v}_{\max}^{R,T}X,$$

and

$$2(n-1)[(m-1)v]K^2 tY \le \gamma\alpha^2 \frac{(n-1)^2(1-m)^2(\bar{v}_{\max}^{R,T})^2 K^4}{\epsilon_2^2\beta^2}t + \frac{\epsilon_2^2\beta^2}{\gamma\alpha^2}Y^2 t,$$

we get

$$0 \le -\frac{t}{\gamma\alpha^2}\left((1+\gamma\beta)X^2 - 2\beta(1+\gamma+\epsilon_1)XY + \beta(\beta+\gamma-\beta\epsilon_2^2)Y^2\right) + X$$
$$+ \frac{t\gamma\alpha^2}{2\epsilon_1\beta}\frac{1600m^2}{R^2}\bar{v}_{\max}^{R,T}X + \frac{tC_5}{R^2}X[(m-1)v] + \gamma\alpha^2 \frac{(n-1)^2(1-m)^2(v_{\max}^{R,T})^2 K^4}{\beta^2\epsilon_2^2}t.$$

Choosing ϵ_2 small such that

(4.2) $$\beta + \gamma - \beta\epsilon_2^2 > 0,$$

then

$$-2\beta(1 + \gamma + \epsilon_1)XY + \beta(\beta + \gamma - \beta\epsilon_2^2)Y^2 \geq -\beta\frac{(1 + \gamma + \epsilon_1)^2}{\beta + \gamma - \beta\epsilon_2^2}X^2$$

hence we have

$$0 \leq -\frac{t}{\gamma\alpha^2}\left(1 + \gamma\beta - \beta\frac{(1 + \gamma + \epsilon_1)^2}{\beta + \gamma - \beta\epsilon_2^2}\right)X^2 + X + \frac{t\gamma\alpha^2}{2\epsilon_1\beta}\frac{1600m^2}{R^2}\bar{v}_{\max}^{R,T}X$$
$$+ \frac{tC_5}{R^2}X\left((1-m)\bar{v}_{\max}^{R,T}\right) + \gamma\alpha^2\frac{(n-1)^2(1-m)^2(\bar{v}_{\max}^{R,T})^2K^4}{\beta^2\epsilon_2^2}t$$

Now for any $\beta \in (0,1)$ we can choose small $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that both (4.2) and

(4.3) $$1 + \gamma\beta - \beta\frac{(1 + \gamma + \epsilon_1)^2}{\beta + \gamma - \beta\epsilon_2^2} > 0$$

hold. For example, any $(\epsilon_1, \epsilon_2) \in \left(0, \frac{\gamma\alpha^2}{6(1+\gamma)}\right] \times \left(0, \sqrt{\frac{\gamma\alpha^2}{3(1+\gamma)}}\right]$ works. Repeating the argument in the proof of Theorem 3.3 after (3.13), we have proved the following result for FDE. (The constant a is defined in (3.8).)

Theorem 4.1. *Let $B(\mathcal{O}, R)$ be a ball in a complete Riemannian manifold (M^n, g), and let u be a positive smooth solution to FDE (1.1) with $m \in (m_c, 1)$ on the cylinder $Q = B(\mathcal{O}, R) \times [0, T]$. Let v be the pressure and let $\bar{v}_{\max}^{R,T} = \max_{B(\mathcal{O},R)\times[0,T]}(-v)$.*

(1) Assume that $\mathrm{Ric} \geq 0$ on $B(\mathcal{O}, R)$. Then for any $0 < \alpha < 1$, $\epsilon_1 > 0$ satisfying

(4.4) $$\bar{C}_1(a, \alpha, \epsilon_1) := 1 + (-a)(1-\alpha) - (1-\alpha)\frac{(1-a+\epsilon_1)^2}{(1-\alpha) - a} > 0,$$

we have

(4.5) $$\frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v} \geq -\frac{(-a)\alpha^2}{\bar{C}_1(a,\alpha,\epsilon_1)}\left(\frac{1}{t} + \frac{\bar{v}_{\max}^{R,T}}{R^2}\left(\bar{C}_2(a,\alpha,\epsilon_1) + \bar{C}_3\right)\right)$$

on $Q' = B(\mathcal{O}, R/2) \times [0, T]$ where

$$\bar{C}_2(a, \alpha, \epsilon_1) := 1600m^2\frac{(-a)\alpha^2}{2\epsilon_1(1-\alpha)} > 0,$$
$$\bar{C}_3 := 40(1-m)(n+2) > 0.$$

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(2) *Assume that* $\mathrm{Ric} \geq -(n-1)K^2$ *on* $B(\mathcal{O}, R)$ *for some* $K \geq 0$. *Then for any* $0 < \alpha < 1$ *and* $\epsilon_1 > 0$, $\epsilon_2 > 0$ *satisfying (4.2) and (4.3) we have*

$$\frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \geq -\frac{(-a)\alpha^2}{\bar{C}_1'}\left(\frac{1}{t} + \bar{C}_4(\alpha, \epsilon_2)\sqrt{\bar{C}_1'}K^2 \bar{v}_{\max}^{R,T}\right)$$

(4.6)
$$-\frac{(-a)\alpha^2}{\bar{C}_1'}\frac{\bar{v}_{\max}^{R,T}}{R^2}\left(\bar{C}_2(a, \alpha, \epsilon_1) + \bar{C}_5(KR)\right)$$

on Q'. *Here*

$$\bar{C}_1' := \bar{C}_1'(a, \alpha, \epsilon_1, \epsilon_2) := 1 + (-a)(1-\alpha) - (1-\alpha)\frac{(1-a+\epsilon_1)^2}{(1-\alpha)-a-(1-\alpha)\epsilon_2^2}$$

$$\bar{C}_4(\alpha, \epsilon_2) := \frac{(n-1)(1-m)}{(1-\alpha)\epsilon_2}$$

$$\bar{C}_5(KR) := 40(1-m)[3 + (n-1)(1+KR)].$$

Notice that $\bar{C}_1(a, \alpha, \epsilon_1) \to 1$ as $\alpha \to 1$. similar to the proof of Corollary 3.4, by taking $R \to \infty$ and then $\alpha \to 1$ Theorem 4.1 has the following consequence for global solution of FDE.

Corollary 4.2. *Let* $u(x,t)$, $t \in [0,T]$, *be a smooth positive solution of the FDE (1.1) with* $m \in (m_c, 1)$ *on a complete manifold* (M^n, g).

(1) *If* (M, g) *has nonnegative Ricci curvature, then*

(4.7)
$$\frac{|\nabla v|^2}{v} - \frac{v_t}{v} \geq -\frac{(1-m)\kappa}{t}$$

holds for $t \in (0, T]$, *provided that* $|v|(x, t) = o(r^2(x))$ *uniformly in* $t \in (0, T]$.

(2) *If Ricci curvature* $\mathrm{Ric} \geq -(n-1)K^2$ *on* M *for some* $K \geq 0$ *and* $\bar{v}_{\max} := \max_{M \times [0,T]}(-v) < \infty$, *then for any* $\alpha \in (0,1)$ *and* $\epsilon_2 > 0$ *satisfying*

$$\bar{C}_1'' := \bar{C}_1''(a, \alpha, \epsilon_2) := 1 + (-a)(1-\alpha) - (1-\alpha)\frac{(1-a)^2}{(1-\alpha)-a-(1-\alpha)\epsilon_2^2} > 0,$$

we have

$$\frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \geq -(1-m)\frac{\kappa\alpha^2}{\bar{C}_1''}\left(\frac{1}{t} + \frac{(1-m)(n-1)}{(1-\alpha)\epsilon_2}\sqrt{\bar{C}_1''}K^2\bar{v}_{\max}\right).$$

Remark 4.3. Demange [D1] found the exact "fundamental solution" for the

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fast diffusion equation $\partial_t u = \Delta u^{1-1/n}$ on the sphere S^n, with initial value δ_{x_0}:

$$u(t,x) = \left(\frac{\sinh((n-1)t)}{\cosh((n-1)t) - \langle x_0, x \rangle} \right)^n,$$

where $\langle x_0, x \rangle$ stands for the scalar product of x_0 and x in $\mathbb{R}^{n+1}$, or equivalently for $\cos(d(x_0, x))$. By direct computation, this solution satisfies

(4.8)
$$\frac{|\nabla v|^2}{v} - \frac{v_t}{v} = -\frac{(n-1)\langle x_0, x \rangle}{\sinh((n-1)t)}$$

$$\geq -\min\left(\frac{(\cos d(x_0, x))_+}{t}, \frac{(n-1)}{\sinh((n-1)t)} \right)$$

$$\geq -\frac{1}{t},$$

and these inequalities are obviously sharp as $t \to 0$ and $x_0 \to x$. The right-hand side of (4.7) also reduces to $-1/t$ since $\kappa(1-m) = 1$ for $m = 1 - 1/n$. In this sense Corollary 4.2 remains sharp even in strictly positive curvature.

Integrating (4.6) along minimal geodesic, we obtain

Corollary 4.4. *Same notation and assumptions as in Theorem 4.1. Denote $\bar{v}_{\min}^{R/2,T}$ to be $\min_{B(\mathcal{O}, \frac{R}{2}) \times [0,T]}(-v)$. Assume that $\mathrm{Ric} \geq -(n-1)K^2$ on $B(\mathcal{O}, R)$ for some $K \geq 0$. Then for any $x_1, x_2 \in B(\mathcal{O}, \frac{R}{6})$ and $0 \leq t_1 < t_2 \leq T$, and any $\alpha > 1$*

$$\frac{-v(x_2, t_2)}{-v(x_1, t_1)} \leq \left(\frac{t_2}{t_1} \right)^{\frac{(-a)\alpha}{\bar{C}'_1}} \cdot \exp\left(\frac{\alpha d^2(x_1, x_2)}{4 \bar{v}_{\min}^{R/2,T}(t_2 - t_1)} \right)$$

$$\cdot \exp\left(\frac{(-a)\alpha}{\bar{C}'_1} \bar{v}_{\max}^{R,T} \left(\bar{C}_4(\alpha, \epsilon_2)\sqrt{\bar{C}'_1}K^2 + \frac{\bar{C}_2(a, \alpha, \epsilon_1) + \bar{C}_5(KR)}{R^2} \right)(t_2 - t_1) \right).$$

where the constants $\bar{C}'_1$, $\bar{C}_2(a, \alpha, \epsilon_1)$, $\bar{C}_4(\alpha, \epsilon_2)$ and $\bar{C}_5(KP)$ are defined as in Theorem 4.1.

Proof. For the minimizing geodesic $\gamma(t)$ joining (x_1, t_1) and (x_2, t_2) we have

$$\log\left(\frac{-v(x_2, t_2)}{-v(x_1, t_1)} \right) = \int_{t_1}^{t_2} \left(\frac{v_t}{v} + \langle \frac{\nabla v}{v}, \dot{\gamma} \rangle \right) ds$$

$$\leq \int_{t_1}^{t_2} \left(\frac{v_t}{v} + \frac{|\nabla v|^2}{\alpha(-v)} + \frac{\alpha|\dot{\gamma}|^2}{4(-v)} \right) ds.$$

The result follows from the observation that $\gamma(s)$ lies completely inside $B(\mathcal{O}, \frac{R}{2})$ and the estimate in Theorem 4.1. $\square$

An integral version of Corollary 4.2 is

Corollary 4.5. *Same notation and assumptions as in Corollary 4.2. We further assume that $\bar{v}_{\max} < \infty$. Let $x_1, x_2 \in M$ and $0 < t_1 < t_2 \le T$.*

(1) *If (M, g) has nonnegative Ricci curvature, then*

$$v(x_2, t_2) - v(x_1, t_1) \ge -(1 - m)\kappa \bar{v}_{\max} \log \frac{t_2}{t_1} - \frac{d^2(x_1, x_2)}{4(t_2 - t_1)}.$$

(2) *If Ricci curvature $\mathrm{Ric} \ge -(n-1)K^2$ for some $K \ge 0$, then for any $\alpha \in (0, 1)$*

$$v(x_2, t_2) - v(x_1, t_1)$$
$$\ge -(1 - m)\frac{\kappa \alpha}{\bar{C}_1''} \bar{v}_{\max} \log \frac{t_2}{t_1} - \frac{(m-1)^2 (n-1)\kappa\alpha}{(1-\alpha)\sqrt{\bar{C}_1''}\epsilon_2} K^2 \bar{v}_{\max}^2 (t_2 - t_1) - \frac{\alpha d^2(x_1, x_2)}{4(t_2 - t_1)}.$$

Corollary 4.6. *Same assumption and same notation as in Theorem 4.1. Let $\bar{\beta} \in (1, \infty)$ fixed. Define $\alpha \in (0, 1)$ by $\frac{\alpha - 1}{\alpha} = (m - 1)(\bar{\beta} - 1)$.*

(1) *Assume $\mathrm{Ric} \ge 0$ on $B(\mathcal{O}, R)$, then we have on Q'*

$$\Delta(-v)^{\bar{\beta}} \le \frac{\kappa \alpha \bar{\beta}}{\bar{C}_1(a, \alpha, \epsilon_1)} \left(\bar{v}_{\max}^{R,T}\right)^{\bar{\beta} - 1} \left(\frac{1}{t} + \frac{\bar{v}_{\max}^{R,T}}{R^2}\left(\bar{C}_2(a, \alpha, \epsilon_1) + \bar{C}_3\right)\right)$$

(2) *Assume that $\mathrm{Ric} \ge -(n-1)K^2$ on $B(\mathcal{O}, R)$ for some $K \ge 0$. Then we have on Q'*

$$\Delta(-v)^{\bar{\beta}} \le \frac{\kappa \alpha \bar{\beta}}{\bar{C}_1'} \left(\bar{v}_{\max}^{R,T}\right)^{\bar{\beta} - 1} \left(\frac{1}{t} + \bar{C}_4(\alpha, \epsilon_2)\sqrt{\bar{C}_1'}K^2 \bar{v}_{\max}^{R,T}\right)$$
$$+ \frac{\kappa \alpha \bar{\beta}}{\bar{C}_1'} \frac{\left(\bar{v}_{\max}^{R,T}\right)^{\bar{\beta}}}{R^2} \left(\bar{C}_2(a, \alpha, \epsilon_1) + \bar{C}_5(KR)\right).$$

5 Entropy formulae

In this section we show that Perelman's entropy formula [P] (see also [N1,N2]) can be adapted to the porous medium equation, with the help of the computation in §3. We also establish similar results for the fast diffusion equation, but the monotonicity of the entropy holds only in the regime $1 - \frac{1}{n} < m < 1$ (it is well-known that the "thermodynamical" properties of the fast diffusion equation change below the critical exponent $1 - 1/n$, see e.g. [O] or [Vi2, Chapter 24]).

Assume that M is a compact manifold. First we derive some auxiliary integral formulae.

Lemma 5.1. *Let u be a positive smooth solution of (1.1) with $m > 0$, and let v, F_1 be as in §3. Then*

$$(5.1)\frac{d}{dt}\int_M vu\,d\mu = \int_M F_1 vu\,d\mu = -m\int_M |\nabla v|^2 u\,d\mu.$$

Proof. By (1.1) and (2.3) we have $\partial_t(vu) = (m-1)vu\Delta v + |\nabla v|^2 u + v\Delta u^m$. Recall now that $F_1 = (m-1)\Delta v$. Then,

$$\frac{d}{dt}\int_M vu\,d\mu = \int_M F_1 vu\,d\mu + \int_M \left(|\nabla v|^2 u + v\Delta u^m\right) d\mu.$$

Using the identity $\nabla u^m = u\nabla v$, a simple integration by parts shows that the second term on the right-hand-side is zero. The first part (5.1) follows. Integration by parts shows that

$$\int_M F_1 vu\,d\mu = \int_M (m-1)(\Delta v)vu\,d\mu = m\int_M (\Delta v)u^m\,d\mu = -m\int_M |\nabla v|^2 u\,d\mu.$$

$\square$

Lemma 5.2.

$$(5.2)\frac{d}{dt}\int_M F_1 vu\,d\mu = 2\int_M \left((m-1)\left(v_{ij}^2 + R_{ij}v_iv_j\right) + F_1^2\right)vu\,d\mu.$$

Proof. Using (1.1), (2.3), and the formula $\partial_t = \mathcal{L} + (m-1)v\Delta$, we have

$$\frac{d}{dt}\int_M F_1 vu\,d\mu = \int_M \partial_t F_1\,vu + F_1\partial_t(vu)\,d\mu$$
$$= \int_M (\mathcal{L}F_1)\,vu\,d\mu + \int_M (m-1)v(\Delta F_1)\,vu\,d\mu$$
$$+ \int_M F_1[(m-1)vu\Delta v + |\nabla v|^2 u + v\Delta u^m]\,d\mu.$$

We will use (3.6) to compute the term with $\mathcal{L}$, and we also use $(m-1)\Delta v = F_1$. Then,

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$$\frac{d}{dt}\int_M F_1 vu\,d\mu = \int_M \left((m-1)v(\Delta F_1)vu + F_1^2 vu + F_1 v\Delta u^m\right)d\mu$$
$$+2(m-1)\int_M \left(v_{ij}^2 + R_{ij}v_i v_j\right)vu\,d\mu + \int_M F_1^2 vu\,d\mu$$
$$+2m\int_M \langle \nabla F_1, \nabla v\rangle vu\,d\mu + \int_M F_1|\nabla v|^2 u\,d\mu.$$

Using the identity $(m-1)\nabla(v^2 u) = (2m-1)vu\nabla v$, and integrating by parts, we get

$$(m-1)\int_M (\Delta F_1)v^2 u\,d\mu = -(2m-1)\int_M \langle \nabla F_1, \nabla v\rangle vu\,d\mu.$$

Finally, since $\nabla u^m = u\nabla v$ we also have

$$\int_M F_1 v\Delta u^m\,d\mu = -\int_M \langle \nabla F_1, \nabla v\rangle vu\,d\mu - \int_M F_1|\nabla v|^2 u\,d\mu.$$

Combining these equalities we prove (5.2). □

Remark 5.3. Formulas 5.1 and 5.2 are particular cases of [Vi2, Theorem 24.2]; they have an interpretation in terms of optimal transport, see [Vi2, Chapters 15 and 24]. This relation between PME and optimal transport goes back to the seminal paper of Otto [O].

Recall the constant $a = \frac{n(m-1)}{n(m-1)+2} = (m-1)\kappa$. We first define

$$(5.3) \qquad \mathcal{N}_u(t) := -t^a \int_M vu\,d\mu.$$

By Lemma 5.1, we have that

$$(5.4) \qquad \frac{d}{dt}\mathcal{N}_u(t) = -t^a \int_M \left(F_1 + \frac{a}{t}\right)vu\,d\mu.$$

Note that the universal estimate (3.7) amounts to $F_1 + \frac{a}{t} \geq 0$. Now we define the **Perelman entropy** associated with (1.1) as

$$(5.5) \qquad \mathcal{W}_u(t) := t\frac{d}{dt}\mathcal{N}_u + \mathcal{N}_u.$$

Remark 5.4. When $m = 1$ this reduces to $tI(t) + S(t)$, where $S(t) = -\int u(t)\log u(t)$ is the Boltzmann entropy from statistical mechanics; and $I(t) = \int u(t)|\nabla \log u(t)|^2$, also known as Fisher information, is its time-derivative. Formula (5.5) generalizes this construction to porous medium equations. Let us note that the functional $-\int uv = -m/(m-1)\int u^m$ has a thermodynamical content; for instance it appears as the macroscopic limit of the micro-

scopic Boltzmann entropy in certain hydrodynamical limits of particle systems [GOVW,K].

Using the last part of (5.1) to compute the term with F_1, we get

$$\mathcal{W}_u(t) = t^{a+1} \int_M \left(m \frac{|\nabla v|^2}{v} - \frac{a+1}{t} \right) vu \, d\mu.$$

Now we show the following Perelman type entropy formula for PME. We put $b = n(m-1)$, so that $a = \frac{b}{b+2}$.

Theorem 5.5. *Let u be a positive smooth solution to (1.1) with $m > 0$. Let v be the pressure and let $\mathcal{W}_u(t)$ be the entropy defined above. Then*

$$\frac{d}{dt} \mathcal{W}_u(t) = -2(m-1)t^{a+1} \int_M \left(\left| v_{ij} + \frac{1}{(b+2)t} g_{ij} \right|^2 + R_{ij} v_i v_j \right) vu \, d\mu$$

$$(5.6) \qquad -2t^{a+1} \int_M \left(F_1 + \frac{a}{t} \right)^2 vu \, d\mu.$$

Proof. Note

$$\frac{d}{dt} \mathcal{W}_u(t) = \frac{d}{dt} \left(t \frac{d}{dt} \mathcal{N}_u \right) - t^a \int_M \left(F_1 + \frac{a}{t} \right) vu \, d\mu.$$

By Lemma 5.1 it is easy to see that

$$\frac{d}{dt} \left(t \frac{d}{dt} \mathcal{N}_u \right) = \frac{d}{dt} \left(-t^{a+1} \int_M F_1 vu \, d\mu + a \mathcal{N}_u \right)$$

$$= -2t^{a+1} \left(\int_M \left((m-1) \left(v_{ij}^2 + R_{ij} v_i v_j \right) + F_1^2 \right) vu \, d\mu \right)$$

$$-(a+1)t^a \int_M F_1 vu \, d\mu - at^a \int_M \left(F_1 + \frac{a}{t} \right) vu \, d\mu.$$

Hence,

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$$\frac{d}{dt}\mathcal{W}_u(t) = -2t^{a+1}\int_M \left((m-1)\left(v_{ij}^2 + R_{ij}v_iv_j\right) + F_1^2\right)vu\,d\mu$$

$$-(a+1)t^a\int_M \left(F_1 + \frac{a}{t}\right)vu\,d\mu - (a+1)t^a\int_M F_1vu\,d\mu$$

$$= -2t^{a+1}\int_M \left((m-1)v_{ij}^2 + (m-1)^2(\Delta v)^2 + (m-1)\frac{a+1}{t}\Delta v\right.$$

$$\left.+ \frac{a^2+a}{2t^2}\right)vu\,d\mu - 2(m-1)t^{a+1}\int_M R_{ij}v_iv_jvu\,d\mu.$$

Observing that $a+1 = \frac{2(b+1)}{b+2}$, $\frac{a^2+a}{2} = \frac{b(b+1)}{(b+2)^2} = (m-1)\frac{n(b+1)}{(b+2)^2}$, hence

$$(m-1)v_{ij}^2 + (m-1)^2(\Delta v)^2 + (m-1)\frac{a+1}{t}\Delta v + \frac{a^2+a}{2t^2}$$

$$= (m-1)v_{ij}^2 + (m-1)^2(\Delta v)^2 + 2(m-1)\frac{b+1}{(b+2)t}\Delta v + (m-1)\frac{n(b+1)}{(b+2)^2t^2}$$

$$= (m-1)\left(v_{ij}^2 + \frac{2}{(b+2)t}\Delta v + \frac{n}{(b+2)^2t^2}\right)$$

$$+ \left((m-1)^2(\Delta v)^2 + \frac{2n(m-1)^2}{(b+2)t}\Delta v + \frac{n^2(m-1)^2}{(b+2)^2t^2}\right).$$

Formula (5.6) follows from completing the squares. $\qquad\square$

Corollary 5.6. *Let (M,g) be a closed Riemannian manifold with nonnegative Ricci curvature. Assume that u is a positive smooth solution to PME (1.1) with $m > 1$. Then*

(1) $\frac{d}{dt}\mathcal{N}_u(t) \leq 0$ and $\frac{d}{dt}\mathcal{W}_u(t) \leq 0$. In particular $\mathcal{N}_u(t)$ is a monotone non-decreasing concave function in $\frac{1}{t}$.

(2) Any ancient positive solution to (1.1) must be a constant.

Proof. We only need to justify (2). For the ancient solution, (3.7) implies $F_1 \geq 0$. On the other hand,

$$\int_M F_1uv\,d\mu = -m\int_M |\nabla v|^2u\,d\mu$$

which implies that $|\nabla v| = 0$. $\qquad\square$

Remark 5.7. The result can be proved for complete noncompact Riemannian manifolds with the help of the gradient estimates from the previous section. The interested reader can find the details for the linear heat equation case in [Ch].

Remark 5.8. In [CT], the following Sobolev type inequality related to PME was proved on $\mathbb{R}^n$. (See also [DD] and [O] for equivalent Sobolev type inequalities related to FDE.)

For any $m > 1$, $f \in L^1(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)$, $|\nabla f^{m-\frac{1}{2}}| \in L^2(\mathbb{R}^n)$, we have

$$\left(n + \frac{1}{m-1}\right) \int_{\mathbb{R}^n} f^m dx \leq \frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^n} \left|\nabla f^{m-\frac{1}{2}}\right|^2 dx + A_m(\|f\|_1),$$

where $A_m(K) := \int_{\mathbb{R}^n} \left(\frac{|x|^2}{2} u_\infty + \frac{1}{m-1} u_\infty^m\right) dx$ with $u_\infty := (C - \frac{m-1}{2m}|x|^2)_+^{\frac{1}{m-1}}$ being the Barenblatt solution of order m and mass $K := \int_{\mathbb{R}^n} u_\infty \, dx$.

The previous defined entropy $\mathcal{W}_u$ is related to the above Sobolev inequality in the following way. First notice that $A(m, n) := A_m(1)$ is a constant depending only on the dimension n and the constant m. Direct calculation shows that Carrillo–Toscani's Sobolev inequality amounts to

$$\mathcal{W}_u\left(\frac{1}{b+2}\right) \geq -2m\left(\frac{1}{b+2}\right)^{a+1} A(m, n).$$

The existence of such a relation is not surprising: the Carrillo–Toscani formula was derived as an inequality between the functional $-m/(m-1) \int u^m$ and its time-derivative.

Remark 5.9. From optimal transport theory we can expect that there are related Lyapunov functionals involving not only $\mathcal{N}_u$ and its time-derivative, but also the Wasserstein distance of order 2; for the heat equation ($m = 1$) examples of such functionals appear e.g. in [Vi2, Theorem 24.2]. The relation between monotonicity properties of the Wasserstein distance on one hand, and Perelman's theorem of monotonicity of the reduced volume on the other hand, has been studied in [Lo,T]. It is likely that also these links could be extended to porous medium equations ($m \neq 1$), but here we shall not explore this possibility.

For the fast diffusion equation, the monotonicity of entropy $\mathcal{W}_u$ is similar to that of PME given in Corollary 5.6.

Corollary 5.10. *Let (M, g) be a closed Riemannian manifold with nonnegative Ricci curvature. Assume that u is a positive smooth solution to FDE (1.1) with $m < 1$. Then*

(1) $\frac{d}{dt}\mathcal{N}_u(t) \leq 0$ for $m \in (m_c, 1)$.

(2) $\frac{d}{dt}\mathcal{W}_u(t) \leq 0$ for $m \in [m_c', 1)$ with $m_c' = 1 - \frac{1}{n}$. In particular $\mathcal{N}_u(t)$ is a monotone non-decreasing concave function in $\frac{1}{t}$ when $m \in [m_c', 1)$.

(3) Any ancient positive solution to (1.1) with $m \in (m_c, 1)$ must be a constant.

Proof. (1) In (5.4), $v < 0$ and by Corollary 4.2(1) $F_1 + \frac{a}{t} \leq 0$.

(2) Notice that

$$-(m-1)\left|v_{ij} + \frac{1}{(b+2)t}g_{ij}\right|^2 \geq -\frac{1}{n(m-1)}\left((m-1)\Delta v + \frac{n(m-1)}{(b+2)t}\right)^2$$

$$= -\frac{1}{n(m-1)}\left(F_1 + \frac{a}{t}\right)^2.$$

Note $v < 0$, the result then follows from (5.6) and $1 + (m-1)n \geq 0$ for $m \in [m'_c, 1)$. $\qquad\qquad\square$

Acknowledgments

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