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Isoperimetric Comparisons via Viscosity

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Abstract Viscosity solutions are suitable notions in the study of nonlinear PDEs justified by estimates established via the maximum principle or the comparison principle. Here we prove that the isoperimetric profile functions of Riemannian manifolds with Ricci lower bound are viscosity supersolutions of some nonlinear differential equations. From these one can derive the isoperimetric inequalities of Lévy-Gromov and Bérard–Besson–Gallot, as well as an upper bound of Morgan–Johnson.

Keywords Viscosity solutions · Isoperimetric profile function · Isoperimetric inequality

Mathematics Subject Classification 53C21

1 Introduction

Viscosity solutions are solutions with usually less regularity. However, this flexibility is important in the development and the study of nonlinear PDEs. For motivation, examples and techniques, see, e.g., [7,11]. One particular advantage of the concept is that it allows effective uses of the comparison principle so that crucial estimates can be established for existence and uniqueness even though the viscosity solutions are a much broader class of solutions.

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Let (M^n, g) be a compact Riemannian manifold. The isoperimetric profile function is defined as follows. For any $\beta \in (0, 1)$, consider smooth regions $\Omega \subset M$ such that the volume $|\Omega|$ satisfies $|\Omega| = \beta|M|$, and let $h_1(\beta, g) = \inf_\Omega \frac{|\partial\Omega|}{|M|}$. Here $|\partial\Omega|$ denotes the $n-1$-dimensional area of $\partial\Omega$, and the infimum is taken over all Ω satisfying the volume constraint. The profile function generally is not smooth, but is continuous.

In this short note we shall prove that isoperimetric profile functions are viscosity supersolutions of some nonlinear differential equations. From these one can derive the isoperimetric inequalities of Lévy-Gromov [8] and Bérard–Besson–Gallot [5], as well as some new comparison results.

The consideration here is motivated by a paper of Andrews–Bryan [2], where the authors established some comparisons for the isoperimetric profile functions for metrics on a two-sphere deformed by the Ricci flow motivated by the earlier work of Hamilton [9]. The comparison here is for static metrics satisfying some conditions on the Ricci curvature. There are two differential equations involved (see Theorem 2.2 in Sect. 2 and Theorem 4.1 in Sect. 4). One is of second order, which can be viewed as a stability result for the isoperimetric profile function. This equation has more or less been shown previously in the works of [3,14] (see [13] and [4] as well). The other is of first order, which can be viewed as a Hamilton–Jacobi type equation. The consideration also leads to an alternate proof of a comparison theorem in Sect. 3, which was originally proved in [14]. We hope to investigate further in the future the application of this approach to the study of the isoperimetric profile functions. The interested reader should consult the survey article [1] and the references therein for related and more recent developments.

2 Isoperimetric Profile Function as a Viscosity Supersolution

Let (M^n, g) be a compact Riemannian manifold. The isoperimetric profile function is defined as follows. For any $\beta \in (0, 1)$, consider smooth region $\Omega \subset M$ such that its volume $|\Omega|$ satisfies $|\Omega| = \beta|M|$, and let $h_1(\beta, g) = \inf_\Omega \frac{|\partial\Omega|}{|M|}$. Here $|\partial\Omega|$ denotes the $n-1$-dimensional area of $\partial\Omega$, and the infimum is taken for all Ω satisfying the volume constraint. It is known (cf. [16, Chapter VI]) that $h_1(\beta, g)$ is continuous (in fact Hölder continuous), satisfying the symmetry $h_1(\beta, g) = h_1(1-\beta, g)$. Moreover, it has the asymptotics (cf. [16, Proposition 1.3 of Chapter VI]):

$$\lim_{\beta\to 0} \frac{h_1(\beta, g)}{\beta^{\frac{n-1}{n}}} = n\frac{\sigma_n^{1/n}}{|M|^{1/n}}, \tag{2.1}$$

where σ_n denotes the volume of the unit ball in the Euclidean space $\mathbb{R}^n$.

The isoperimetric inequality of Lévy-Gromov [8] asserts the following:

Theorem 2.1 (Lévy-Gromov) *Assume that the Ricci curvature of (M, g), $Ric_g \geq (n-1)\kappa g$ for some $\kappa > 0$. Then*

$$h_1(\beta, g) \geq h_1(\beta, g_\kappa),$$

where (M_k, g_k) is the space form of constant sectional curvature k.

We prove the following result which implies the above inequality via a maximum principle for viscosity solutions. This argument avoids the estimate of Heintze–Karcher.

Theorem 2.2 *Assume that the Ricci curvature of the compact manifold* (M^n, g), $Ric_g \geq \kappa(n-1)g$. *The isoperimetric profile function* $h_1(\beta, g)$, *as a function of* β, *is a positive viscosity supersolution (over* $(0,1)$*) of the differential equation:*

$$-\psi''\psi = (n-1)\left(k + \left(\frac{\psi'}{n-1}\right)^2\right). \tag{2.2}$$

Before we prove the result we first derive Theorem 2.1 from the above. First observe that $h_1(\beta, g_1)$ is a smooth solution to (2.2) on $(0,1)$. By scaling, it suffices to prove it for $k = 1$. Assume that the claimed estimate in Theorem 2.1 fails. Then by the asymptotics and the symmetry, there exists $\beta_0 \in (0,1)$ such that $h_1(\beta, g) - h_1(\beta, g_1)$ attains its negative minimum. Now in a small neighborhood of β_0, there exists a smooth $\varphi(\beta)$ such that $\varphi(\beta) \leq h_1(\beta, g)$ and $\varphi(\beta_0) = h_1(\beta_0, g)$. This support function can be constructed easily from $h_1(\beta, g_1)$, which is smooth, as follows. Let $-\alpha \doteqdot \min_{\beta\in[0,]} h_1(\beta, g) - h_1(\beta, g_1)$. Then one may choose $\varphi(\beta) = h_1(\beta, g_1) - \alpha$. Moreover, we have that

$$-\varphi''\varphi \geq (n-1)\left(1 + \left(\frac{\varphi'}{n-1}\right)^2\right). \tag{2.3}$$

On the other hand, by the above $\varphi(\beta) - h_1(\beta, g_1)$ attains a local negative minimum at β_0. Hence we have that $h_1(\beta_0, g_1) > \varphi(\beta_0) > 0$, $\varphi'(\beta_0) = h_1'(\beta_0, g_1)$ and $\varphi''(\beta_0) \geq h_1''(\beta_0, g_1)$. By writing $h_1(\beta, g_1)$ as $h_{1,g_1}(\beta)$, this implies that at β_0,

$$\begin{aligned}
-\varphi(\beta)\varphi''(\beta) &\leq -\varphi(\beta)h''_{1,g_1}(\beta) \\
&= \frac{\varphi(\beta)}{h_{1,g_1}(\beta)}(-h_{1,g_1}(\beta)h''_{1,g_1}(\beta)) \\
&= \frac{\varphi(\beta)}{h_{1,g_1}(\beta)} \cdot (n-1)\left(1 + \left(\frac{\varphi'(\beta)}{n-1}\right)^2\right),
\end{aligned}$$

which contradicts (2.3), by noting that $\frac{\varphi(\beta_0)}{h_{1,g_1}(\beta_0)} < 1$.

Now we prove Theorem 2.2. By definition, we need to verify that for any β_0, and a small neighborhood U of it, a smooth function $0 < \psi(\beta) \leq h_1(\beta, g)$ in U with $\psi(\beta_0) = h_1(\beta_0, g)$, the equation (2.3) holds at $\beta = \beta_0$. Let Ω be the domain minimizing $|\partial\Omega|$ with $|\Omega| = \beta_0|M|$. Let $\partial\Omega$ denote the boundary of Ω. By the regularity theorem [17], $\partial\Omega$ is a smooth hypersurface except for a singular set of Hausdorff codimension 7. The mean curvature of N, the smooth part, is defined and is a constant. For a small region D of N, we may consider the variation given by $\exp_x(t\eta(x)\nu(x))$ with ν being the unit outward normal, η being a function supported in D. Let N_t be this *variation* of N and let Ω_t be the domain bounded by N_t (together with the irregular part of $\partial\Omega$, which is not altered). Recall that $\exp_N((x,t)) = \exp_x(t\nu(x))$. Simple calculation shows that if $J(\exp_N)|_{(x,s)} = a(x,s)$ with $a(x,0) = 1$,

$$|\Omega_t| = |\Omega| + \int_D \int_0^{t\eta} a(x,s)\, ds d\mu_{g_N},$$
$$\frac{d}{dt}|\Omega_t|\Big|_{t=0} = \int_N \eta\, d\mu_{g_N}.$$

Recall that the first variation formula for the submanifolds also gives

$$\frac{d}{dt}|N_t|\Big|_{t=0} = (n-1)\int_N \eta H\, d\mu_{g_N}.$$

Let $\beta(t) = \frac{|\Omega_t|}{|M|}$. It is easy to see that $\psi(\beta(t)) \le \frac{|N_t|}{|M|}$ and $\psi(\beta(0)) = \psi(\beta_0) = \frac{|N_0|}{|M|}$. Now let $F(t) = \frac{|N_t|}{|M|} - \psi(\beta(t))$ which attains a local minimum at $t = 0$. The first variation formula yields that

$$(n-1)H = \psi'(\beta_0). \tag{2.4}$$

Note that $\frac{d}{dt}|_{t=0}\psi(\beta(t)) = \psi'(\beta_0)\frac{1}{|M|}\int_N \eta$. The fact that $F''(0) \ge 0$ and the second variational formula (cf. [12, p. 8]) yields at $t = 0$ $(\beta = \beta_0)$

$$\frac{1}{|M|}\int_N |\nabla\eta|^2 + (\eta(n-1)H)^2 - \eta^2 h_{ij}^2 - \eta^2 \operatorname{Ric}(\nu,\nu)$$
$$\ge \psi''\left(\frac{1}{M}\int_N \eta\right)^2 + \psi'\frac{(n-1)H}{|M|}\int_N \eta. \tag{2.5}$$

The smallness of the singular set allows $\eta = 1$, via approximations. Hence we have that for $\beta = \beta_0$

$$-\psi''\psi^2 \ge (n-1)\psi\left(\frac{\psi'}{n-1}\right)^2 + \frac{1}{|M|}\int_N \operatorname{Ric}(\nu,\nu). \tag{2.6}$$

This proves the claimed differential inequality by cancelation and using $\operatorname{Ric}(\nu,\nu) \ge k(n-1)$.

Consequences include the following result for the case of $\kappa = 0$ and $\kappa = -1$.

Corollary 2.3 (i) *Assume that the Ricci curvature of (M, g), $Ric_g \ge 0$. The isoperimetric profile function $h_1(\beta, g)$, as a function of β, is a positive supersolution of the differential equation:*

$$-\psi''\psi = (n-1)\left(\frac{\psi'}{n-1}\right)^2. \tag{2.7}$$

(ii) *Assume that the Ricci curvature of (M, g), $Ric_g \ge -(n-1)g$. The isoperimetric profile function $h_1(\beta, g)$, as a function of β, is a positive supersolution of the differential equation:*

$$-\psi''\psi = (n-1)\left(-1 + \left(\frac{\psi'}{n-1}\right)^2\right). \tag{2.8}$$

3 Comparisons from Above on Manifolds with Ricci Lower Bound

Motivated with the consideration of the last section we consider (M, g) with $\mathrm{Ric}_g \geq (n-1)\kappa$, where κ is a constant, but not necessarily satisfying $\kappa > 0$. This of courses allows manifolds with infinite volume. Now we define $h_2(\beta, g)$, another profile function which is natural for this setting, as $\inf |\partial\Omega|$ among all Ω such that $|\Omega| = \beta$. Clearly $h_2(\beta, g)$ is now defined for $(0, |M|)$. When $|M| < \infty$, $h_2(\beta, g) = |M| h_1(\frac{\beta}{|M|}, g)$.

The following comparison result holds for the profile function $h_2(\beta, g)$.

Theorem 3.1 *Let (M, g) be a complete Riemannian manifold with* $\mathrm{Ric}_g \geq (n-1)\kappa$. *Then*

$$h_2(\beta, g) \leq h_2(\beta, g_\kappa)$$

for $\beta \in (0, |M|)$. If the equality ever holds somewhere, (M, g) must be isometric to the space form (M, g_κ).

Note that the famous Cartan–Hadamard conjecture asserts the opposite estimate if (M, g) is a Cartan–Hadamard manifold with the sectional curvature $K_M \leq \kappa \leq 0$. The result is an analogue of the eigenvalue comparison result of Cheng [6]. This result was first proved in [14]. Below is an alternate argument.

This profile function satisfies the scaling law $h_2(\beta, cg) = c^{\frac{n-1}{2}} h_2(c^{-\frac{n}{2}}\beta, g)$. Hence it suffices to prove for cases $\kappa = -1, 0, 1$. For the proof, we need the following simple lemma (one can find its proof, for example, in [15]).

Lemma 3.1 *Let $\rho(t)$ be a continuous function on $[0, b]$. Assume that $\rho(0) \leq 0$ and there exist some positive constants ϵ, C such that $D^-\rho \leq C\rho$, whenever $0 < \rho(t) \leq \epsilon$. Then $\rho(b) \leq 0$. The same result holds if D^- is replaced by D^+, D_- or D_+.*

To prove Theorem 3.1, let $p \in M$ be a fixed point and introduce $I_p(\beta, g) = |\partial B_p(r)|$ with $|B_p(r)| = \beta$. Clearly $h_2(\beta, g) \leq I_p(\beta, g)$ while $h_2(\beta, g_\kappa) = I_{\bar{p}}(\beta, g_\kappa)$ where $\bar{p} \in M_\kappa$ is a fixed point in the space form M_κ. The claimed result follows if we can establish that $I_p(\beta, g) \leq h_2(\beta, g_\kappa)$. Let $f(\beta) = I_p(\beta, g) - I_{\bar{p}}(\beta, g_\kappa)$. Since $f(0) = 0$ it suffices to show that $f' \leq 0$ by Lemma 3.1. Let $B_{\bar{p}}(\bar{r})$ be the ball in M_k such that $|B_{\bar{p}}(\bar{r})| = \beta$. Now direct calculation shows that

$$I_p' = \frac{n-1}{|\partial B_p(r)|}\int_{\partial B_p(r)} H(r, \theta); \quad I_{\bar{p}}' = \frac{n-1}{|\partial B_{\bar{p}}(\bar{r})|}\int_{\partial B_{\bar{p}}(\bar{r})} \bar{H}(\bar{r}) = (n-1)\bar{H}(\bar{r}).$$

Here $H(r, \theta)$ denotes the mean curvature of $\partial B_p(r)$ in terms of polar coordinates and $\bar{H}(\bar{r})$ is the mean curvature of $\partial B_{\bar{p}}(\bar{r})$ in the space form M_κ. By the volume comparison theorem $|B_p(r)| \leq |B_{\bar{p}}(r)|$, which implies that $\bar{r} \leq r$ since $\beta = |B_p(r)| = |B_{\bar{p}}(\bar{r})|$.

Also by the Laplacian comparison theorem $H(r,\theta) \le \bar{H}(r)$. Noting that $\bar{H}(s)$ (mean curvature of the spheres in M_κ) is a monotone non-increasing function. We then have

$$\begin{aligned}\frac{n-1}{|\partial B_p(r)|}\int_{\partial B_p(r)} H(r,\theta) &\le (n-1)\bar{H}(r)\\ &\le (n-1)\bar{H}(\bar{r}).\end{aligned}$$

This proves $f' \le 0$, hence the claimed inequality in Theorem 3.1. The equality case follows from the equality in the volume comparison (applying to balls with varying centers).

Combining Theorem 2.1 and Theorem 3.1 we have the two-sided bounds below.

Corollary 3.2 *Assume that the Ricci curvature of (M, g), $Ric_g \ge (n-1)g$. Then*

$$h_1(\beta, g_1) \le h_1(\beta, g) \le \frac{|\mathbb{S}^n|}{|M|}\cdot h_1\left(\frac{|M|}{|\mathbb{S}^n|}\beta, g_1\right).$$

The scaling relation between $h_2(\beta, g)$ and $h_1(\beta, g)$ yields that $h_2(\beta, g)$ also satisfies Theorem 2.2 when $\kappa = 1$. Namely, on $(0, |M|)$, $h_2(\beta, g)$ is a viscosity supersolution of (2.2). This implies the following corollary.

Corollary 3.3 *Let (M, g) be a compact Riemannian manifold with $\mathrm{Ric}_g \ge (n-1)\kappa$. Then if $\kappa > 0$, $\frac{h_2(\beta,g)}{h_2(\beta,g_\kappa)}$ is a monotone non-increasing function on $(0, |M|)$.*

Proof For $\kappa > 0$, without loss of generality we assume $\kappa = 1$. Now notice that $h_2(\beta, g_\kappa)$ is a smooth solution of (2.2) and $h_2(\beta, g)$ is a viscosity supersolution of (2.2). By Theorem 3.1 we have that $\frac{h_2(\beta,g)}{h_2(\beta,g_\kappa)} \le 1$. If the claimed result does not hold, then one can find $0 < \beta_1 < \beta_2 < |M|$ such that

$$\frac{h_2(\beta_1, g)}{h_2(\beta_1, g_\kappa)} < \frac{h_2(\beta_2, g)}{h_2(\beta_2, g_\kappa)}.$$

Then $\frac{h_2(\beta,g)}{h_2(\beta_1,g_\kappa)}$ achieves minimum in $(0, \beta_2)$ which is strictly smaller than 1, by Theorem 3.1. Now one can repeat the argument in the proof of the Lévy-Gromov isoperimetric estimate to arrive at a contradiction! Precisely, assume that the minimum is attained at β_0, for a neighborhood U and a support function $\psi > 0$ of $h_2(\beta, g)$, we have that $\frac{\psi(\beta)}{h_2(\beta_1,g_1)}$ attains a local minimum at β_0. Then at β_0

$$\frac{\psi'(\beta)}{h_2'(\beta, g_1)} = \frac{\psi(\beta)}{h_2(\beta, g_1)} \doteqdot \lambda < 1 \tag{3.1}$$

$$\frac{\psi''(\beta)}{h_2(\beta, g_1)} - \frac{\psi(\beta)h_2''(\beta, g_1)}{h_2^2(\beta, g_1)} \ge 0. \tag{3.2}$$

Combining (3.1), (3.2) and that ψ satisfies (2.3) we have

$$\begin{aligned}(n-1)\left(1+\left(\frac{\psi'}{n-1}\right)^2\right) &\le -\psi''\psi\\ &\le \lambda^2\left(-h_2h_2''\right)\\ &= (n-1)\left(\lambda^2+\left(\frac{\lambda h_2'}{n-1}\right)^2\right)\\ &= (n-1)\left(\lambda^2+\left(\frac{\psi'}{n-1}\right)^2\right).\end{aligned}$$

This is a contradiction since $\lambda < 1$. □

4 Bérard–Besson–Gallot Comparison via the Viscosity

Here using the ideas from Sect. 2, we derive some first order equation satisfied by the profile function. This together with the maximum principle argument implies the improved lower estimate of Bérard–Besson–Gallot [5]. As in [5] we need to use the Heintze–Karcher estimates (cf. [16, Theorem 3.8 of Chapter IV]), unlike in the case for Lévy-Gromov's estimate.

For manifold (M, g) with $\mathrm{Ric}(g) \ge (n-1)\kappa$, let

$$s_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}t, & \kappa > 0,\\ t, & \kappa = 0,\\ \frac{1}{\sqrt{|\kappa|}}\sinh\sqrt{|\kappa|}t, & \kappa < 0; \end{cases} \qquad c_\kappa(t) = \begin{cases} \cos\sqrt{\kappa}t, & \kappa > 0,\\ 1, & \kappa = 0,\\ \cosh\sqrt{|\kappa|}t, & \kappa < 0. \end{cases}$$

Let d denote the diameter of the manifold. Since for the consideration in this section, one only gets the sharp result for $\kappa > 0$, we shall focus on this case first. Define

$$\lambda^\kappa_{n,d} = \int_{-\frac{d}{2}}^{\frac{d}{2}} c_\kappa^{n-1}(t)\,dt = \frac{1}{\sqrt{\kappa}}\int_{-\frac{\sqrt{\kappa}d}{2}}^{\frac{\sqrt{\kappa}d}{2}} \cos^{n-1}(t)\,dt.$$

Theorem 4.1 *Assume that the Ricci curvature of (M, g), $Ric_g \ge (n-1)\kappa g$, with $\kappa > 0$. Let d be the diameter of (M, g). The isoperimetric profile function $h_1(\beta, g)$, as a function of β, is a positive viscosity supersolution of the differential equation:*

$$\psi\left(1+\frac{1}{\kappa}\left(\frac{\psi'}{n-1}\right)^2\right)^{\frac{n-1}{2}} = \frac{1}{\lambda^\kappa_{n,d}}. \tag{4.1}$$

Proof The derivation follows essentially the argument in [5]. By scaling invariance of the result, we may assume that $\kappa = 1$. By definition, we need to verify that for any

β_0, and a small neighborhood U of it, a smooth function $0 < \psi(\beta) \le h_1(\beta, g)$ in U with $\psi(\beta_0) = h_1(\beta_0, g)$, the inequality

$$\psi\left(1 + \left(\frac{\psi'}{n-1}\right)^2\right)^{\frac{n-1}{2}} \ge \frac{1}{\lambda^1_{n,d}} \tag{4.2}$$

holds at $\beta = \beta_0$. Let Ω be the domain minimizing $|\partial\Omega|$ with $|\Omega| = \beta_0|M|$. Let $\partial\Omega$ denote the boundary of Ω. Let η, D, N_t, Ω_t be as those quantities in the proof of Theorem 2.2. Similarly as before we have that

$$(n-1)H = \psi'(\beta_0), \tag{4.3}$$

where H is the mean curvature of the regular part of $\partial\Omega$. As in [5], let

$$r_0 = \max\{\mathrm{dist}(x, \partial\Omega) \,|\, x \in \Omega\}.$$

It is easy to see that

$$r_1 \doteqdot \max\{\mathrm{dist}(x, \partial\Omega) \,|\, x \in M \setminus \Omega\} \le d - r_0.$$

In fact, for any $x_1 \in \Omega$ and $x_2 \in M \setminus \Omega$, let $\gamma(s)$ be the minimum geodesic joining from $x_1 = \gamma(0)$ to $x_2 = \gamma(l)$. Hence $l = L(\gamma) \le d$. On the other hand, assume that $s_1 > 0$ is the first time $\gamma(s) \in \partial\Omega$ and s_2 is the last time $\gamma(s) \in \partial\Omega$. Then

$$d \ge l = L(\gamma) \ge s_1 + l - s_2 \ge r_0 + r_1.$$

Now we use Heintze–Karcher's estimate (cf. [16, Theorem 3.8 of Chapter IV]) to conclude that

$$|\Omega| \le |N| \int_0^{r_0} (\cos t - H \sin t)_+^{n-1}\, dt;$$
$$|M \setminus \Omega| \le |N| \int_0^{d-r_0} (\cos t + H \sin t)_+^{n-1}\, dt.$$

Putting them together we have that

$$1 \le \psi(\beta_0) \int_{r_0-d}^{r_0} (\cos t - H \sin t)_+^{n-1}\, dt. \tag{4.4}$$

Writing $\cos\theta_0 = \frac{1}{\sqrt{1+H^2}}$, $\sin\theta_0 = \frac{H}{\sqrt{1+H^2}}$, we have that

$$1 \le \psi(\beta_0)(1 + H^2)^{\frac{n-1}{2}} \int_{r_0-d}^{r_0} [\cos(t + \theta_0)]_+^{n-1}\, dt$$

$$\leq \psi(\beta_0)\left(1+\left(\frac{\psi'}{n-1}\right)^2\right)^{\frac{n-1}{2}} \int_{-\frac{d}{2}}^{\frac{d}{2}} \cos^{n-1} t\, dt.$$

This implies the claimed result. □

A direct consequence is the Bérard–Besson–Gallot's estimate. By scaling, without loss of generality we may assume $k = 1$. It is well known that the diameter of the manifold d is bounded from above by π. Define

$$\gamma_n \doteqdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-1} t\, dt, \quad \alpha(n, d) \doteqdot \left(\frac{\gamma_n}{\lambda^1_{n,d}}\right)^{\frac{1}{n}}.$$

Theorem 4.2 (Bérard–Besson–Gallot) *Let* (M^n, g) *be a compact Riemannian manifold with* Ric $\geq (n-1)g$*, and* $\gamma_n, \lambda^1_{n,d}, \alpha$ *be as above. Then*

$$h_1(\beta, g) \geq \alpha \cdot h_1(\beta, g_1). \tag{4.5}$$

This improves Lévy-Gromov's Theorem 2.1 since $\alpha \geq 1$ with the equality if and only if M is isometric to the round sphere.

To prove (4.5) we first observe that $h_1(\beta, g_1)$ is a solution of (4.1) with $d = \frac{\pi}{2}$. For simplicity we denote $h_1(\beta, g_1)$ by $\varphi(\beta)$. Hence φ satisfies

$$\varphi\left(1+\left(\frac{\varphi'}{n-1}\right)^2\right)^{\frac{n-1}{2}} = \frac{1}{\gamma_n}. \tag{4.6}$$

Assume that the claimed result fails. By the asymptotics we conclude that $\frac{h_1(\beta,g)}{\alpha\varphi(\beta)}$ attains the minimum $\lambda < 1$ at some interior point β_0. At this point apply Theorem 4.1 to the support function $\psi(\beta) > 0$ with $\psi(\beta_0) = \lambda\alpha\varphi(\beta_0)$ we conclude that at β_0

$$\psi' = \lambda\alpha\varphi',$$

and

$$\begin{aligned}
\lambda\alpha\varphi\left(1+\left(\frac{\lambda\alpha\varphi'}{n-1}\right)^2\right)^{\frac{n-1}{2}} &= \psi\left(1+\left(\frac{\psi'}{n-1}\right)^2\right)^{\frac{n-1}{2}} \\
&\geq \frac{\gamma_n}{\lambda_n}\varphi\left(1+\left(\frac{\varphi'}{n-1}\right)^2\right)^{\frac{n-1}{2}} \\
&= (\alpha\varphi)\left(\alpha^2+\left(\frac{\alpha\varphi'}{n-1}\right)^2\right)^{\frac{n-1}{2}}.
\end{aligned}$$

The above estimate yields a contradiction since $\lambda < 1$, and $\alpha \geq 1$, $\varphi(\beta_0) > 0$.

When $\kappa = 0$, a similar argument proves the following result.

Corollary 4.3 *Let* (M^n, g) *be a compact Riemannian manifold with* Ric ≥ 0. *Let*

$$\lambda^0_{n,d} \doteqdot \int_0^d (1+t^2)^{\frac{n-1}{2}}\, dt, \quad \alpha'(n,d) \doteqdot \left(\frac{\gamma_n}{\lambda^0_{n,d}}\right)^{\frac{1}{n}}.$$

Then $h_1(\beta, g)$ *is a viscosity positive supersolution of the equation*

$$\psi\left(1+\left(\frac{\psi'}{n-1}\right)^2\right)^{\frac{n-1}{2}} = \frac{1}{\lambda^0_{n,d}}. \tag{4.7}$$

In particular, it implies that $h_1(\beta, g) \geq \alpha'(n,d) \cdot h_1(\beta, g_1)$.

It seems natural to formula the third profile function $h_3(\beta, g) \doteqdot \inf_\Omega \frac{|\partial\Omega|}{|M|^{\frac{n-1}{n}}}$ for all Ω with $\beta = \frac{|\Omega|}{|M|}$ and hope that $h_3(\beta, g) \geq h_3(\beta, g_\kappa)$ as before. Indeed such an estimate would improve the result in Theorem 4.2. However, such a result turns out to be false. The example can be found even for dimension two. Recall the so-called Rosenau solution of the Ricci flow equation on the Riemann sphere: $g(t) = u(x,t)(dx^2 + d\theta^2)$ with $u(x,t) = \frac{\sinh(-t)}{\cosh x + \cosh t}$. The solution tends to the geometry obtained by gluing two cigar solutions arbitrarily far out (cf. [10]) as $t \to -\infty$.

Note that $h_3(\beta, \lambda g) = h_3(\beta, g)$. Hence for $t < 0$ one can rescale the metric $g(t)$ such that its Ricci curvature would satisfy $\mathrm{Ric}_{\lambda g(t)} \geq (n-1)$. But on the other hand $h_3(\frac{1}{2}, g(t)) \to 0$ as $t \to \infty$, while $h_3(\frac{1}{2}, g_1)$ is a fixed definite positive constant.

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