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Kähler-Ricci flow on complete manifolds

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Abstract. This is a paper based on author's lectures delivered at the 2005 Clay Mathematics Institute summer school at MSRI. It serves as an overview on the Kähler-Ricci flow over complete noncompact manifolds.

1. Introduction

The 2005 Clay Mathematics Institute summer school at MSRI focused on Perelman's work on Ricci flow. The author was asked to give two lectures about Kähler-Ricci flow on complete noncompact manifolds. This paper is the written version of the lectures. Since Perelman did not make any specific claims on this subject, we lectured on materials mostly related to some of the author's recent research results in this subject. The connection with Perelman's work does exist in some of the results, in which the new techniques introduced by Perelman in [**P1**] [**P2**] played important roles. When the manifold is compact, Perelman did talk about the results of bounding the scalar curvature and diameter for Kähler-Ricci flow on compact manifolds with positive first Chern class during his MIT visit in 2003, and claimed some convergence results. Even the author did not attend his MIT lectures, after we learned about the result we did independently work out a complete proof (with limited circulation) on his claimed results on bounding the scalar curvature and the diameter, soon after his Stony-Brook visit of 2003. For readers interested in learning more on this part of Perelman's work, please see Tian's lectures [**T**] (in this volume) for a detailed exposition.

Even on Kähler-Ricci flow over complete noncompact manifolds, the choices of the topics in these lectures are only restricted to those related to author's own research. Efforts have been made, both in re-organizing the results so that they appears in the most natural order, and in the choice of the cleanest proof. Since the subject is still in the middle of rapid development the presentation/proofs here may not be in their ultimate best forms. The lectures mainly serve as an introduction, especially to graduate students who are interested in the subject. Due to our limited knowledge and restricted length of the lectures, we could not cover many important works on the subject in the lectures. For example we did not discuss in

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details about the recent important progress made by Chau and Tam [**CT**] on the uniformization problem. We apologize for many such omissions. The interested readers are encouraged to study the other related literatures for a more complete overview on this subject.

2. A maximum principle

The maximum principle probably is the most useful tool in the study of the geometric flow. When the manifold is not compact, it is well-known that the maximum principle does not always hold. In this section we show a maximum principle for the time-dependent heat equation on a family of complete Riemannian manifolds $(M, g_{ij}(t))$ satisfying certain conditions. By comparing with the special case that the metric is a fixed one, it is easy to see that the result in the section is optimal.

Assume that the family of metric $g_{ij}(x,t)$ on $M \times [0,T]$ satisfying the equation

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2\Upsilon_{ij}(x,t)$$

with $R(x,t) := g^{ij}(x,t)\Upsilon_{ij}(x,t)$. We assume that $R_m := \inf_{x\in M} R(x,t) > -\infty$ for all t and $R_*(t) := \frac{1}{2}\min\{0, R_m\}$ is integrable on $[0, T_1]$. We also assume that $g_{ij}(x,t) \geq g^*_{ij}(x)$, where $g^*_{ij}(x)$ is a complete Riemannian metric. This setting certainly covers the case when the metrics are deformed by the corresponding Ricci tensor (namely Ricci flow) and the Ricci tensors are bounded uniformly.

Let $r_*(x,y)$ be the distance between x and y, with respect to g^*_{ij}. Consider the time-dependent heat equation $(\frac{\partial}{\partial t} - \Delta)$ (where the Δ is with respect to $g(t)$). We formulate a general maximum principle in the following theorem.

THEOREM 2.1. *Let $f(x,t)$ be a Lipschitz function satisfying*

$$\left(\Delta - \frac{\partial}{\partial t}\right) f(x,t) \geq 0 \qquad \text{whenever } f(x,t) \geq 0.$$

Assume that there exists a constant $b > 0$ such that for some fixed $o \in M$

$$\int_0^{T_1}\int_M \exp(-b\, r_*^2(o,x)) f_+^2(x,t)\, d\mu_t(x)\, dt < \infty,$$

where $f_+ = \max\{f, 0\}$. If $f(x,0) \leq 0$, then $f(x,t) \leq 0$ on $M \times [0, T_1]$.

PROOF. Let $h(x,t) = \exp(\int_0^t R_*(s)\, ds) f(x,t)$. It is easy to see that

$$\left(\Delta - \frac{\partial}{\partial t} + R_*\right) h(x,t) \geq 0 \tag{2.1}$$

whenever $h(x,t) \geq 0$. For any T with $0 < T \leq T_1$, let

$$\eta(x,t) = -\frac{r_*^2(o,x)}{4(2T - t)}.$$

Without the loss of the generality we may assume that $T \leq \frac{1}{8b}$, since we can always split $[0,T]$ into smaller intervals (such that each has the length less than $\frac{1}{8b}$) and apply the induction. Therefore we have that

$$\int_0^T \int_M h_+^2 e^{\eta}\, d\mu_t\, dt < \infty.$$

Using the fact that $g_{ij}(x,t) \geq g^*_{ij}(x)$ we have that

$$|\nabla \eta|^2 + \frac{\partial}{\partial t}\eta \leq 0. \tag{2.2}$$

Let $\psi(t) : [0,\infty) \to [0,1]$ be a cut-off function so that $\psi(t) = 0$ for $t \geq 1$ and $\psi(t) = 1$ for $t \leq \frac{1}{2}$. Let $\varphi(x) = \psi(\frac{r_*(o,x)}{a})$. It is easy to see that there exists a constant $C > 0$ independent of a such that $|\nabla\varphi|^2 \leq \frac{C}{a^2}$, using again the assumption that $g_{ij}(x,t) \geq g^*_{ij}(x)$. By (2.1) and (2.2) we have that

$$\begin{aligned}
0 &\leq \int_0^T \int_M \varphi^2 e^\eta h_+ \left(\Delta - \frac{\partial}{\partial t} + R_*\right) h \, d\mu_t \, dt \\
&= \int_0^T \int_M \left(-2\langle \nabla\varphi, \nabla h_+\rangle h_+ e^\eta \varphi - \langle \nabla h, \nabla \eta\rangle e^\eta \varphi^2 h_+ - |\nabla h_+|^2 e^\eta \varphi^2\right) d\mu_t \, dt \\
&+ \int_0^T \int_M \left(-\frac{1}{2}\left(\frac{\partial}{\partial t} h_+^2\right)\varphi^2 e^\eta + R_* \varphi^2 e^\eta h_+^2\right) d\mu_t \, dt \\
&\leq \int_0^T \int_M \left(2|\nabla\varphi|^2 h_+^2 e^\eta + \frac{1}{2}|\nabla\eta|^2 e^\eta \varphi^2 h_+^2 + \frac{1}{2} h_+^2 \varphi^2 e^\eta \eta_t\right) d\mu_t \, dt \\
&- \int_M h_+^2 \varphi^2 e^\eta \, d\mu_t|_0^T + \int_0^T \int_M \left(h_+^2 e^\eta \varphi^2 (-\frac{1}{2}R + R_*)\right) d\mu_t \, dt \\
&\leq 2\int_0^T \int_M |\nabla\varphi|^2 h_+^2 e^\eta d\mu_t \, dt - \left(\frac{1}{2}\int_M h_+^2 e^\eta \varphi^2 \, d\mu_t\right)(T).
\end{aligned} \tag{2.3}$$

Letting $a \to \infty$, we have that

$$\left(\frac{1}{2}\int_M h_+^2 e^\eta \, d\mu_t\right)(T) \leq 0.$$

This implies that $f(x,T) \leq 0$. Since T is arbitrary we have the claimed result. □

Despite of the simplicity of its proof, the theorem is in the most general form comparing with all the previous known maximum principles (cf. [**Sh1**] [**Sh2**] [**NT2**]).

3. A long time existence result

From this section on, we shall focus on the case that M has bounded non-negative bisectional curvature. There are also works under different assumptions. Most of them reduce the problem to a single Monge-Ampère equation and apply the parabolic version of the earlier established elliptic techniques for solving the Monge-Ampère equation, which is originated from the fundamental work of Yau [**Y1**]. Let us start with the following result, which was first proved by Shi [**Sh2**].

THEOREM 3.1. *Let $(M, g(x))$ be a complete Kähler manifold with bounded non-negative bisectional curvature. Assume that there exist $2 \geq \theta > 0$ and $C > 0$ such that for any $x \in M$,*

$$\frac{1}{V(x,r)}\int_{B(x,r)} R(y)\, d\mu_y \leq \frac{C}{(1+r)^\theta}, \tag{3.1}$$

where $B(x,r)$ is the ball of radius r centered at x and $V(x,r)$ is the volume of such a ball. Then the Kähler-Ricci flow with initial data $g(x,0) = g(x)$ has long time solution on $M \times [0,\infty)$.

In the following we are going to present a simplified (different from [**Sh2**]) proof of the above theorem for the case $\theta > 1$. The original work of Shi mainly adapted the techniques from the study of the Monge-Ampère equation. While the proof here makes only a simple use of the maximum principle for the linear parabolic equation. For the general case, there also exists a simplification of Shi's work in [**NT2**]

The proof is based on the existence of a 'good' solution to the so-called Poincaré-Lelong equation whose existence was proved in [**NST**]. The following one is the further refined result from [**NT1**].

THEOREM 3.2. *Let $(M, g(x))$ be a complete Kähler manifold with nonnegative bisectional curvature. Let ρ be a real closed $(1,1)$ form with trace f. Let $o \in M$ be a fixed point. Assume that $f \geq 0$ and ρ satisfies*

$$\int_0^\infty \left(\frac{1}{V(o,s)} \int_{B(o,s)} \|\rho\| \, d\mu \right) ds < \infty, \tag{3.2}$$

and

$$\liminf_{r\to\infty} \left(\exp(-ar^2) \int_{B(o,r)} \|\rho\|^2 d\mu \right) < \infty \tag{3.3}$$

for some $a > 0$, where $B(o,r)$ is the ball of radius r centered at o and $V(o,r)$ is its volume. Then there exists a solution u to the Poincaré-Lelong equation $\sqrt{-1}\partial\bar{\partial}u = \rho$. Moreover, there exists estimate

$$|\nabla u|(x) \leq C(n) \int_0^\infty k_f(x,s) \, ds \tag{3.4}$$

with $k_f(x,s) = \frac{1}{V(x,s)} \int_{B(x,s)} f \, d\mu$.

The proof of the above result uses a simple idea originally due to Mok, Siu and Yau, by which one solves the Poisson equation first and then to show that the solution in fact also solves the over-determinate Poincaré-Lelong equation. The original result was improved in the author's thesis later. The more refined result was later proved in [**NST**]. The reader may want to consult [**NST**] and [**NT1**] for the detailed account on this approach.

By Theorem 3.2 we know that there exists a solution to the Poincaré-Lelong equation

$$\frac{\partial^2 u_0}{\partial z^\alpha \partial z^{\bar{\beta}}}(x) = R_{\alpha\bar{\beta}}(x) \tag{3.5}$$

where $R_{\alpha\bar{\beta}}(x)$ is the Ricci tensor of $g(x)$. Moreover under the assumption that $\theta > 1$, the gradient estimate (3.4) implies that

$$|\nabla_0 u_0|(x) \leq C_1 \tag{3.6}$$

for some $C_1 > 0$, where $C_1 = C_1(m, \theta)$. Here we use ∇_0 to denote the gradient with respect to the initial metric $g(x)$ and reserve ∇ for the time-dependent metric $g(x,t)$. The same convention applies to the distance function $r_t(x,y)$ $(r_0(x,y))$ between two points $x, y \in M$ as well as the volume element $d\mu_t$ $(d\mu_0)$. Let

$$F(x,t) = \log\left(\frac{\det(g_{\alpha\bar{\beta}}(x,t))}{\det(g_{\alpha\bar{\beta}}(x,0))} \right).$$

Then for the solution of Kähler-Ricci flow

$$d\mu_t = e^F d\mu, \tag{3.7}$$

$$F(x,\bar{t}) = -\int_0^{\bar{t}} R(x,t)dt \tag{3.8}$$

where $R(x,t)$ is the scalar curvature of the metric $g(x,t)$.

First by [**Sh1**], there exists a short time solution with the existence time span depending on the C^0-bound of the curvature tensor such that for any fixed time slice the curvature tensor is bounded. By the results of Bando [**B**] and Mok [**M**] we know that the short time solution $(M, g(t))$ to the Kähler-Ricci flow still has nonnegative bisectional curvature. Therefore, in order to prove the long time existence, we only need to obtain a uniform estimate on the scalar curvature, therefore a uniform upper bound on the curvature tensor, which rules out the possibility of finite singularities. This can be proved by applying the general maximum principle, Theorem 2.1 after checking the following lemma.

Lemma 3.3. *Suppose there is a function $u_0(x)$ such that*

$$\sqrt{-1}\partial\bar{\partial}u_0 = Ric(g(\cdot,0)) \tag{3.9}$$

where $Ric(g(0))$ is the Ricci form of the initial metric $g(0)$. Let F be the ratio of the volume element as above and let $u(x,t) = u_0(x) - F(x,t)$. Then

$$\sqrt{-1}\partial\bar{\partial}u = Ric(g(t)), \tag{3.10}$$

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(x,t) = 0, \tag{3.11}$$

$$\left(\Delta - \frac{\partial}{\partial t}\right) |\nabla u|^2 = \|u_{\alpha\beta}\|^2 + \|u_{\alpha\bar{\beta}}\|^2, \tag{3.12}$$

$$\left(\Delta - \frac{\partial}{\partial t}\right) \left(|\nabla u|^2 + 1\right)^{\frac{1}{2}} \geq 0, \tag{3.13}$$

and

$$\left(\Delta - \frac{\partial}{\partial t}\right) R = \left(\Delta - \frac{\partial}{\partial t}\right) u_t = -\|u_{\alpha\bar{\beta}}\|^2. \tag{3.14}$$

Here $\|u_{\alpha\bar{\beta}}\|^2(x,t) = g^{\alpha\bar{\beta}}(x,t)g^{\gamma\bar{\delta}}(x,t)u_{\alpha\bar{\delta}}(x,t)u_{\gamma\bar{\beta}}(x,t)$, $\|u_{\alpha\beta}\|^2(x,t) = g^{\alpha\bar{\beta}}(x,t)$ $g^{\gamma\bar{\delta}}(x,t)\ u_{\alpha\gamma}(x,t)\ u_{\bar{\beta}\bar{\delta}}(x,t)$.

The proof of lemma is tedious but routine computations. We leave them as exercises. The reader can consult [**NT2**] for the details. Now combining (3.12) and (3.14) we have that

$$\left(\Delta - \frac{\partial}{\partial t}\right) \left(|\nabla u|^2 + R\right) = \|u_{\alpha\beta}\|^2 \geq 0.$$

One can also check that $|\nabla u|^2(x,t) + R(x,t)$ satisfies the growth control condition needed for the maximum principle in Theorem 2.1. For this sake one only needs to estimate the growth of $\int_{B(x,r)} |\nabla u|^2\, d\mu_0$ (in terms of r, which can be bounded by the growth of R and u itself). The complete detailed estimate was written in [**NT2**]. We then have the following slightly stronger result.

Theorem 3.4. *Let $(M^m, g_{\alpha\bar\beta}(x,t))$ be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature such that its scalar curvature R_0 is bounded and satisfies*

$$\frac{1}{V(x,r)}\int_{B(x,r)} R(y)\,d\mu_y \le \frac{C_1}{(1+r)^\theta}$$

for some constants $\theta > 1$ and $C_1 > 0$ for all x and r. Then the Kähler-Ricci flow $\frac{\partial}{\partial t} g_{\alpha\bar\beta} = -R_{\alpha\bar\beta}$ has long time existence. Moreover, there is a function $u(x,t)$ such that

$$\sqrt{-1}\partial\bar\partial u(\cdot,t) = Ric(g(t)),$$
$$|\nabla u| \le C(m)C_1,$$

and

$$R(x,t)+|\nabla u|^2(x,t) \le \sup_{x\in M}\left(R_0(x) + |\nabla_0 u_0|^2(x)\right) \le \sup_{x\in M} R_0(x)+(C(m)C_1)^2 \tag{3.15}$$

for some constant positive $C(m)$ depending only on m and for all (x,t). Moreover, the equality holds in (3.15) for some (x_0,t_0), with $t_0 > 0$ if and only if $g_{\alpha\bar\beta}(x,t)$ is a Kähler-Ricci soliton.

4. Uniqueness

The solution obtained in Theorem 3.4 can also be shown unique. This is a result of Fan [**F**]. In fact, the result in [**F**] is more general than the uniqueness on the solution constructed by Theorem 3.4. We present the argument of [**F**] in this section.

Assume that $g(x,t)$ and $\tilde g(x,t)$ are two solutions to the Kähler-Ricci flow. Since the curvature of both solution are uniformly bounded, at least on $M\times[0,T]$ for some small $T>0$, it is easy to see that

$$C^{-1} g_{\alpha\bar\beta}(x) \le g_{\alpha\bar\beta}(x,t), \tilde g_{\alpha\bar\beta}(x,t) \le C g_{\alpha,\bar\beta}(x) \tag{4.1}$$

for some $C>0$. We also denote by $\tilde R_{\alpha\bar\beta}(x,t)$ the Ricci tensor of metric $\tilde g(x,t)$. By the proof (especially, Lemma 3.3) above we know that there exist potential function $u(x,t)$ and $\tilde u(x,t)$ for the Ricci form of the metrics $g(x,t)$ and $\tilde g(x,t)$. Form the equation of Kähler-Ricci flow we have that

$$g_{\alpha\bar\beta}(x,\bar t) = g_{\alpha\bar\beta}(x,0) - \int_0^{\bar t} u_{\alpha\bar\beta}(x,t)\,dt.$$

If we denote $\varphi(x,\bar t) = -\int_0^{\bar t} u(x,t)\,dt$, we have that $g_{\alpha\bar\beta}(x,t) = g_{\alpha\bar\beta}(x,0)+\varphi_{\alpha\bar\beta}(x,t)$. Now we can write the Kähler-Ricci flow as a single parabolic equation exactly as in the compact case:

$$\frac{\partial}{\partial t}\varphi = \log\left(\frac{\det(g_{\alpha\bar\beta}(x,0)+\varphi_{\alpha\bar\beta}(x,t))}{\det(g_{\alpha\bar\beta}(x,0))}\right) - u_0(x). \tag{4.2}$$

Exactly the same we also have $\tilde\varphi(x,t)$ for $\tilde g(x,t)$ and the equation (4.2) for $\tilde\varphi$.

Now let $\psi(x,t) = \varphi(x,t) - \tilde\varphi(x,t)$. Then the uniqueness would follow if we can show that $\psi(x,t)\equiv 0$. It is easy to see that

$$\frac{\partial}{\partial t}\psi(x,t) = \log\left(\frac{\det(g_{\alpha\bar\beta}(x,0)+\varphi_{\alpha\bar\beta}(x,t))}{\det(g_{\alpha\bar\beta}(x,0)+\tilde\varphi_{\alpha\bar\beta}(x,t))}\right).$$

While the righthand side can be written as

$$\log\left(\det(g_{\alpha\bar\beta}(x,0)+(s\varphi+(1-s)\tilde\varphi)_{\alpha\bar\beta}(x,t))\right)|_{s=0}^{s=1}$$

which can be further written as

$$\int_0^1 g_{(s)}^{\alpha\bar\beta}\left(\varphi_{\alpha\bar\beta}(x,t)-\tilde\varphi_{\alpha\bar\beta}(x,t)\right)\,ds=\left(\int_0^1 g_{(s)}^{\alpha\bar\beta}\,ds\right)\psi_{\alpha\bar\beta}(x,t)$$

where $g_{(s)}^{\alpha\bar\beta}(x,t)$ is the inverse of the definite matrix (in local coordinate)

$$\left(sg_{\alpha\bar\beta}(x,t)+(1-s)\tilde g_{\alpha\bar\beta}(x,t)\right).$$

If we denote

$$\Delta_{g,\tilde g}=\left(\int_0^1 g_{(s)}^{\alpha\bar\beta}\,ds\right)(x,t)\frac{\partial^2}{\partial z^\alpha\partial z^{\bar\beta}}$$

the above computation gives

$$\frac{\partial}{\partial t}\psi=\Delta_{g,\tilde g}\psi.$$

Theorem 2.1 can be applied to the family of metrics

$$\bar g_{\alpha\bar\beta}(x,t)=\left(\int_0^1 (sg+(1-s)\tilde g)_{\alpha\bar\beta}\,ds\right)(x,t)$$

since we do have that uniform lower bound for $\bar g(x,t)$ and $-\bar g^{\alpha\bar\beta}\frac{\partial}{\partial t}\bar g_{\alpha\bar\beta}\geq 0$. Since one can also verify the growth condition on ψ, the maximum principle implies that $\psi\equiv 0$, therefore the uniqueness of the solution.

The above argument in fact proves a slightly more general result than [**F**] since we only need to assume the lower bound on the Ricci curvature for g and $\tilde g$, instead of the curvature tensor being bounded.

5. Long time existence without curvature decay assumption

The assumption (3.1) requires the information both on the volume growth and the curvature growth. It is desirable to replace the uniform average curvature decay assumption in Theorem 3.1 by a single geometric condition. In this direction we have the following result [**N3**], which seems to be the first long time existence result without assuming any curvature decay.

THEOREM 5.1. *Let (M^m,g_0) ($m=\dim_{\mathbb{C}}(M)$) be a complete Kähler manifold with bounded nonnegative bisectional curvature. Assume that M is of maximum volume growth. Then the Kähler-Ricci flow $\frac{\partial}{\partial t}g_{\alpha\bar\beta}=-R_{\alpha\bar\beta}$, with $g(x,0)=g_0(x)$ has a long time solution. Moreover the solution has no slowly forming (Type II) singularity as t approaches ∞ (namely the solution is nonsingular). Furthermore, M is diffeomorphic (homeomorphic) to $\mathbb{C}^m$, for $m>2$ ($m=2$), and is biholomorphic to a pseudoconvex domain in $\mathbb{C}^m$.*

The following general result on complete Kähler manifolds with nonnegative bisectional curvature was proved in Theorem 0.1 of [**NT1**]. We do not need the full strength of the above result for the sake of Theorem 5.1. However since it provides an effective tool to do the reduction for the study of complete Kähler manifolds with nonnegative bisectional curvature we present its most general form with the hope that it may be useful for other purposes.

Theorem 5.2. *Let $(M, g(x))$ be a complete Kähler manifold with nonnegative bisectional curvature and let u be a continuous plurisubharmonic function on M satisfying*

$$|u|(x) \le C \exp(ar^2(x))$$

for some positive constants a and C, where $r(x)$ is the distance function to some fixed point. There exists a positive constants T_0 and T_1 so that the heat equation $\left(\frac{\partial}{\partial t} - \Delta\right) v(x,t) = 0$ with $v(x,0) = u(x)$ has a plurisubharmonic solution $v(x,t)$ on $M \times [0, T_0]$. Moreover, the null space

$$\mathcal{K}(x,t) = \left\{ w \in T_x^{1,0}(M) | v_{\alpha\bar{\beta}}(x,t) w^{\alpha} = 0 \right\}$$

is a distribution on M for all $0 < t < T_1$. $\mathcal{K}(x,t)$ is also invariant under the parallel translation.

Applying this general result to the Busemann function, which has been known to be a Lipschitz continuous plurisubhamonic function, we can have the following corollary [**NT1**].

Corollary 5.3. *Let M^m be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Suppose that M is simply-connected. Then $M = N \times M'$, isometric-biholomorphically, where N is compact and M' supports a smooth strictly plurisubharmonic function with bounded gradient. Moreover, for any $x \in M'$, there exists $C > 0$ (may depends on x) such that*

$$\frac{1}{V(x,r)} \int_{B(x,r)} R(y)\, d\mu_y \le \frac{C}{r} \tag{5.1}$$

where $R(y)$ is the scalar curvature function, $B_x(r)$ is the ball of radius r centered at x and $V_x(r)$ is the volume of such ball. If M has positive bisectional curvature somewhere then (5.1) still holds on M without assuming the simply-connectedness.

We omit the proof of the above two results since they are not results on Ricci flow even though the proof is certainly motivated by Hamilton's work on tensor maximum principle. The interested readers can consult the original paper [**NT1**] for the details. The curvature decay information is obtained through understanding the solution (sub-solution) to the Poisson equation $\Delta u = -R(y)$, which goes back to the papers [**NST**] and [**N1**]. There exists a splitting of M according to the minimal order of nonconstant holomorphic functions. The interested reader may consult [**N2**] for the statement of such a result.

A direct consequence from this result is that on a Kähler manifold of maximum volume growth, whose Busemann function can be easily shown to be exhaustion, there can not be a compact factor N in the splitting provided through Corollary 5.3. Therefore the estimate (5.1) always holds on M. (Strictly speaking one has to work on the universal coever of M. Since the fundamental group of a complete Riemannian manifold with nonnegative Ricci curvature and maximum volume growth is finite, the estimate (5.1) descends to M.) In general, when we study the Kähler-Ricci flow, we may as well assume that (5.1) holds since the Kähler-Ricci flow preserves the product structure and the Kähler-Ricci flow on the compact manifolds has been better understood.

The following lemma, making use of the decay (5.1), is useful. For example it implies that the asymptotical volume ratio is preserved under Ricci flow. The result stated in the lemma is considerably more general than that the asymptotical volume ratio is preserved under the Kähler-Ricci flow. (Under some point-wise decay assumption on the curvature tensor, the asymptotical volume ratio being preserved by Ricci flow was proved by Hamilton first in Theorem 18.3 of [**H**]).

LEMMA 5.4. *Assume that $(M, g(t))$ be a solution to Kähler Ricci flow on $M \times [0, T]$ with bounded curvature so that $g(x, 0)$ satisfies (5.1). Then*

$$\lim_{r\to\infty} \frac{V_t(o,r)}{V_0(o,r)} = 1 \tag{5.2}$$

for any $t > 0$. Here $V_t(o, r)$ is the volume (with respect to $g(t)$) of $B_t(o, r)$ (the ball with respect to $g(t)$ again).

PROOF. We first show that $\limsup_{r\to\infty} \frac{V_t(o,r)}{V_0(o,r)} \le 1$. By Theorem 17.1 of [**H**], (or Lemma 8.2 of [**P1**]), we know that $r_t(o, x) \ge r_0(o, x) - Ct$ for some C depending on the Ricci curvature bound. This implies that

$$B_t(o, r) \subset B_0(o, r + Ct)$$

Noticing that the metric is shrinking we have that

$$V_t(0, r) \le V_t(B_0(o, r + Ct)) \le V_0(B_0(o, r + Ct))$$

which implies that

$$\limsup_{r\to\infty} \frac{V_t(o,r)}{V_0(o,r)} \le \lim_{r\to\infty} \frac{V_0(B_0(o, r + Ct))}{V_0(o,r)} \le 1.$$

To prove the result we only need to show that $\liminf_{r\to\infty} \frac{V_t(o,r)}{V_0(o,r)} \ge 1$. We first observe that $B_0(o, r) \subset B_t(o, r)$. Hence

$$V_t(o, r) \ge \int_{B_0(o,r)} d\mu_t.$$

Therefore, for any $\bar{t} > 0$,

$$V_{\bar{t}}(o, r) - V_0(o, r) \ge \int_{B_0(o,r)} \left(e^{F(x,\bar{t})} - 1\right) d\mu_0,$$

The right hand side above can be written as

$$\int_{B_0(o,r)} e^{F(x,t)}|_0^{t=\bar{t}}\, d\mu_0 = \int_0^{\bar{t}} \int_{B_0(o,r)} \frac{\partial}{\partial t}\left(e^{F(x,t)}\right) d\mu_0 dt.$$

Direct computation shows that

$$\frac{\partial}{\partial t}\left(e^{F(x,t)}\right) = e^F F_t = -e^F R(x, t) \ge -g^{\alpha\bar{\beta}}(x, 0)R_{\alpha\bar{\beta}}(x, t) = \Delta_0 F(x, t) - R(x, 0). \tag{5.3}$$

Combining the above we have that

$$V_{\bar{t}}(o, r) - V_0(o, r) \ge \int_0^{\bar{t}} \int_{B_0(o,r)} (\Delta_0 F(x, t) - R(x, 0))\, d\mu_0\, dt.$$

Therefore,

$$\frac{V_{\bar{t}}(o,r)}{V_0(o,r)} - 1 \ge -\frac{\bar{t}}{V_0(o,r)} \sup_{0\le t\le\bar{t}} \int_{\partial B_0(o,r)} |\nabla_0 F|(x, t)\, dA - \frac{\bar{t}}{V_0(o,r)} \int_{B_0(o,r)} R(x, 0)\, d\mu_0.$$

It is easy to see that $\frac{\partial}{\partial z^{\gamma}} F = \Gamma^{\beta}_{\beta\gamma}(t) - \Gamma^{\beta}_{\beta\gamma}(0)$. Differentiate the equation $\Gamma^{\alpha}_{\beta\gamma} = g^{\alpha\bar{\delta}} \frac{\partial g_{\beta\bar{\delta}}}{\partial z^{\gamma}}$ we have that

$$\frac{\partial}{\partial t}\Gamma^{\gamma}_{\alpha\beta} = -g^{\gamma\bar{\delta}} \nabla_{\alpha} R_{\beta\bar{\delta}}.$$

By Shi's derivative estimate [**Sh1**] we have that $|\nabla_0 F| \leq C$ for some constant depending on the initial bound of the curvature tensor. It is now easy to see that

$$\frac{\bar{t}}{V_0(o,r)} \sup_{0\leq t\leq \bar{t}} \int_{\partial B_0(o,r)} |\nabla_0 F|(x,t)\, dA \leq C\bar{t}\frac{A_0(o,r)}{V_0(o,r)} \leq \frac{C\bar{t}}{r}$$

and

$$\frac{\bar{t}}{V_0(o,r)} \int_{B_0(o,r)} R(x,0)\, d\mu_0 \leq \frac{C\bar{t}}{r}.$$

The claimed result follows by letting $r \to \infty$. □

Notice that the proof presented here is technically easier than the proof of [**NT2**].

Under the assumption of maximum volume growth, the above lemma together with the local injectivity radius estimate of Cheeger-Gromov-Taylor, in terms of the volume (cf. Theorem 4.3 in [**CGT**]), ensures the injectivity radius lower bound estimate in the case that the curvature is locally bounded. This is enough for applying the singularity analysis of Hamilton in Section 16 of [**H**]. Since both Type I and II singularity models are ancient solutions, the first half part of Theorem 5.1 follows from the following result on ancient solutions, which was proved originally in [**N3**].

THEOREM 5.5. *Let* $(M^m, g(t))$ *(*$m = \dim_{\mathbb{C}}(M), n = \dim_{\mathbb{R}}(M) = 2m$*) be a non-flat ancient solution to Kähler-Ricci flow. Assume that* $(M, g(t))$ *has bounded nonnegative bisectional curvature. Then the asymptotic volume ratio* $\mathcal{V}(g(t)) = 0$.

Once we have the longtime existence result one can obtain the further geometric information by adapting the techniques of [**Sh2**]. This gives the second part of Theorem 5.1. Interested readers please refer to [**N3**] for more details.

6. Ancient solutions and shrinking solitons

The proof to Theorem 5.5 needs the following classification result on the gradient shrinking solitons. The similar result for the Riemannian case under the stronger assumption on the positivity of the sectional curvature was proved by Perelman for dimension 3 [**P2**] (See also [**CLN**]).

THEOREM 6.1. *Let* (M^m, g) *be a non-flat gradient shrinking soliton to Kähler-Ricci flow.*
(i) If the bisectional curvature of M *is positive then* M *must be compact and isometric-biholomorphic to* $(\mathbb{P}^m, \omega_{FS})$.
(ii) If M *has nonnegative bisectional curvature then the universal cover* $\tilde{M}$ *splits as* $\tilde{M} = N_1 \times N_2 \times \cdots \times N_l \times \mathbb{C}^k$ *isometric-biholomorphically, where* N_i *are compact irreducible Hermitian Symmetric Spaces equipped with the canonical Kähler-Einstein metrics.*

The following proposition on the gradient shrinking solitons, which is used in the proof of the above theorem, may be of other uses in the study of gradient shrinking solitons.

Recall that a complete Riemannian manifold (M, g) is called a gradient shrinking soliton if there exists a smooth function f such that, for some positive constant a,

$$\nabla_i \nabla_j f + R_{ij} - a g_{ij} = 0.$$

The following result generalizes the result of Perelman by removing the uniform curvature bound.

PROPOSITION 6.2. *Let (M, g) be a Ricci non-flat gradient shrinking soliton. Assume that the Ricci curvature of M is nonnegative. Then there exists a $\delta = \delta(M)$ $(1 \geq \delta 0)$ such that*

$$R(x) \geq \delta > 0. \tag{6.1}$$

PROOF. It is easy to see, from the strong maximum principle, that the scalar curvature $R(x) > 0$. Differentiating the defining equation of the solitons and applying the second Bianchi identity, we have that

$$\nabla_i R = 2 R_{ij} f_j. \tag{6.2}$$

This implies that

$$\nabla_i R + 2 f_{ij} f_j - 2a f_i = 2\left(R_{ij} + f_{ij} - a g_{ij}\right) f_j = 0$$

which further implies that there exists a constant $C_1 = C_1(M)$ such that

$$R + |\nabla f|^2 - 2af = C_1. \tag{6.3}$$

These equations are well known for the gradient shrinking solitons.

Let $o \in M$ be a fixed point. For any $x \in M$ we denote the distance of x from o by $r(x)$. Let $\gamma(s)$ be minimal geodesic joining x from o, parametrized by the arc-length. For simplicity we often also denote $r(x)$ by s_0. Let $\{E_i(s)\}$ $(0 \leq i \leq n-1)$ be a parallel frame along $\gamma(s)$ such that $E_0(s) = \gamma'(s)$. If $s_0 \geq 2$, for $s_0 \geq r_0 \geq 1$, define $n-1$-variational vector fields $Y_i(s)$ $(1 \leq i \leq n-1)$ along $\gamma(s)$ by

$$Y_i(s) = \begin{cases} sE_i(s), & 0 \leq s \leq 1 \\ E_i(s), & 1 \leq s \leq s_0 - r_0. \\ \frac{s_0 - s}{r_0} E_i(s), & s_0 - r_0 \leq s \leq s_0 \end{cases}$$

From the second variation consideration in Lemma 8.3 (b) of [**P1**] (see also Theorem 17.4 of [**H**]), we have that

$$\sum_{i=1}^{n-1} \int_0^{s_0} |Y_i'(s)|^2 - R(\gamma'(s), Y_i(s), \gamma'(s), Y_i(s))\, ds \geq 0.$$

In particular we can find $C(M)$, which depends only the upper bound of the Ricci curvature of M on $B_o(1)$, such that

$$\begin{aligned} \int_0^{s_0 - r_0} Ric(\gamma'(s), \gamma'(s))\, ds &\leq C(M) + \frac{n-1}{r_0} - \int_{s_0 - r_0}^{s_0} \left(\frac{s_0 - s}{r_0}\right)^2 Ric(\gamma'(s), \gamma'(s))\, ds \\ &\leq C(M) + \frac{n-1}{r_0}. \end{aligned}$$

Here we have used the fact the Ricci curvature is nonnegative. We claim that there exists a positive constant $A = A(M)$, if $s_0 \geq A$ and $R(x) \leq 1$, there exists another constant, still denoted by $C(M)$, such that

$$\int_0^{s_0} Ric(\gamma'(s), \gamma'(s))\, ds \leq \frac{a}{2} s_0 + C(M). \tag{6.4}$$

Assume that we have proved the claim (6.4). Then there exists $C_2 = C_2(M) > 0$ such that

$$\begin{aligned}\langle \nabla f, \gamma'(s)\rangle|_o^{\gamma(r(x))} &= \int_0^{s_0} \frac{d}{ds}(\langle \nabla f, \gamma'(s)\rangle)\, ds \\ &= \int_0^{s_0} (\nabla_i \nabla_j f)\frac{d\gamma^i(s)}{ds}\frac{d\gamma^j(s)}{ds}\, ds \\ &= \int_0^{s_0} (a - Ric(\gamma'(s), \gamma'(s)))\, ds \\ &= ar(x) - \int_0^{s_0} Ric(\gamma'(s), \gamma'(s))\, ds \\ &\geq \frac{a}{2} r(x) - C_2,\end{aligned}$$

which implies that for every $x \in M \setminus B_o(A)$ with $R(x) \leq 1$,

$$\langle \nabla f, \nabla r\rangle(x) \geq \frac{a}{2} r(x) - C_2 - |\nabla f|(o).$$

It in particular implies that for every such x, with $r(x) \geq \frac{4}{a}(C_2 + |\nabla f|(o))$, $\nabla f(x) \neq 0$.

Now we first prove the proposition assuming the claim (6.4). For any $x \in M \setminus (B_o(A) \cup B_o(\frac{4}{a}(C_2 + |\nabla f|(o))))$, without the loss of generality we can assume that $R(x) \leq 1$, let $\sigma(\eta)$ be the integral curves of ∇f, passing x with $\sigma(0) = x$. By (6.3) we have that

$$-\frac{d}{d\eta}(R(\sigma(\eta)) = -2R_{ij}\frac{d\sigma^i}{d\eta}\frac{d\sigma^j}{d\eta} \leq 0. \tag{6.5}$$

This implies that $R(x) \geq R(\sigma(\eta))$, for $\eta < 0$. On the other hand,

$$-\frac{d}{d\eta} r(\sigma(\eta)) = -\langle \nabla r, \nabla f\rangle \leq -(C_2 + |\nabla f|(o)) \leq 0 \tag{6.6}$$

as far as $r(\sigma(\eta)) \geq \max(A, \frac{4}{a}(C_2 + |\nabla f|(o)))$, noticing that we always have $R(\sigma(\eta)) \leq 1$. This implies that the integral curve σ exists for all $\eta < 0$ since $|\nabla f|$ is bounded inside the closed ball $\overline{B_o(2r(x))}$. The estimate (6.6) also implies that there exists $\eta_1 < 0$ such that $r(\sigma(\eta_1)) = \max(A, \frac{4}{a}(C_2 + |\nabla f|(o)))$. Applying (6.5) we have that

$$R(x) \geq \inf_{y \in B_o(r(\sigma(\eta_1)))} R(y).$$

This proves the proposition assuming the claim (6.4).

Now we prove the claim (6.4). First by soliton equation and the fact $R_{ij} \geq 0$ we have that $f_{ij} \leq a g_{ij}$. This implies that along any minimizing geodesic $\gamma(s)$ from o, $f''(s) \leq a$. Hence there exists $B = B(M)$ such that

$$f(x) \leq (a+1) r^2(x)$$

for $r(x) \geq B$. Using (6.3) and the fact that $R > 0$ we have that

$$|\nabla f|(x) \leq 2(a+1) r(x)$$

for $r(x) \geq B$. On the other hand, (6.2) also implies that

$$|\nabla R|^2 \leq 4R^2 |\nabla f|^2.$$

The above two inequality implies the the estimate

$$|\nabla \log R|(x) \leq 2(a+1) r(x)$$

for $r(x) \geq B$. Now we adapt the notations and situations right before (6.4) and choose r_0 in the second variational computation right before (6.4) such that $\frac{n-1}{r_0} = \epsilon s_0$ with some fixed positive constant $\epsilon \leq \min(1, \frac{a}{2})$. Then the second variational computation right before (6.4) implies that

$$\int_0^{s_0 - r_0} Ric(\gamma'(s), \gamma'(s))\, ds \leq C(M) + \epsilon s_0. \tag{6.7}$$

Notice that $r_0 = \frac{n-1}{\epsilon s_0} \leq \frac{n-1}{\epsilon} \leq \frac{s_0}{2}$ if $s_0 \geq A$ for some $A = A(M) \geq \max(1, 2B, 2\frac{n-1}{\epsilon})$. Now using the gradient estimate on $\log R$ above we have that

$$\begin{aligned} \log \frac{R(\gamma(s_1))}{R(\gamma(s_0))} &= -\int_{s_1}^{s_0} \frac{d}{ds} \log R(\gamma(s))\, ds \\ &\leq \int_{s_1}^{s_0} |\nabla \log R|\, ds \\ &\leq 2(a+1)s_0(s_0 - s_1) \end{aligned}$$

for $s_1 \leq s_0$. Hence

$$R(\gamma(s)) \leq R(\gamma(s_0)) \exp(\frac{2(a+1)(n-1)}{\epsilon}) \leq \exp(\frac{2(a+1)(n-1)}{\epsilon})$$

for any $s \geq s_0 - r_0$. Here we have used the assumption $R(x) = R(\gamma(s_0)) \leq 1$. This further implies that

$$\begin{aligned} \int_{s_0 - r_0}^{s_0} Ric(\gamma'(s), \gamma'(s))\, ds &\leq \int_{s_0 - r_0}^{s_0} R(\gamma(s))\, ds \\ &\leq r_0 \exp(\frac{2(a+1)(n-1)}{\epsilon}) \\ &= \frac{n-1}{\epsilon s_0} \exp(\frac{2(a+1)(n-1)}{\epsilon}) \\ &\leq C(\epsilon, M). \end{aligned}$$

Together with (6.7), we prove our claim (6.4). Hence we complete the proof of the proposition. □

The first statement of Theorem 6.1 follows from Proposition 6.2 and the last part of Corollary 5.3 immediately since the noncompactness of M would result in two contradicting conclusions on the behavior of the scalar curvature. For the second statement, first observe that the gradient shrinking soliton equation preserves after lifting to the universal cover $\tilde{M}$. By Corollary 5.3, the manifold $\tilde{M}$ splits into two factors. One of them is compact and the other is noncompact. By the uniformization theorem of Mori, Siu-Yau and Mok (cf. [**Zh**]) we can conclude that the compact factor can be written as product of the Hermitian symmetric spaces. They also are equipped with the canonical metrics except the ones of rank one. It is also easy to see that the restriction (of the metric and the potential function f) to the compact factor still satisfying a shrinking soliton equation. We deduce from this that the potential functions are constant on those factors of rank greater than one. So we only have to take care of the compact factors of rank ones. However, since those factors of rank one are biholomorphic to the complex projective spaces, the existence of a Kähler-Einstein metric in its Kähler class resulting the vanishing of the Futaki invariants [**Fu**]. This further implies that f is a constant function, since that ∇f is a holomorphic vector field and the Futaki invariant $F(\omega, f) = \int_M \nabla f(f)\,\omega^m = \int_M |\nabla f|^2 d\mu = 0$. On the noncompact factor, since the restriction is still a gradient shrinking soliton, we can conclude that it must be Ricci flat by Proposition 6.2 and the curvature decay estimate (5.1). This completes the proof of Theorem 6.1.

The proof of Theorem 5.1 follows from Theorem 6.1 since by Proposition 11.2 of [**P1**] by which one can blow down a non-collapsed (which is the case if we assume that the asymptotical volume ratio is positive) ancient solution to obtain (as the limit) a shrinking soliton. While Theorem 6.1 implies that the limiting shrinking soliton can not have positive asymptotical volume ratio. This proves that the ancient solution can not have positive asymptotical volume ratio. There are other consequences of Theorem 5.1 including the following corollary.

Corollary 6.3. *For a fixed $\kappa > 0$, the set of κ-solution to Kähler-Ricci flow is compact module scaling.*

For a more detailed account on the above result please consult [**N3**]. Theorem 6.1 make the argument work for all dimensions while the earlier similar results of [**P1**] [**P2**] (in the Riemannian case) are only valid for dimension three. It is an interesting question whether such result still hold in high dimension for the Riemannian case, where the lack of a result similar to Corollary 5.3 is the main difficulty.

7. Large time behavior

The following result on the asymptotical behavior of Kähler-Ricci flow is useful in the recent work of Chau and Tam [**CT**].

Proposition 7.1. *Let $(M, g(t))$ be the solution to Kähler-Ricci flow provided by Theorem 5.1. Then for any (x_j, t_j) with $t_j \to \infty$ and $\frac{r_0^2(x_j, x_0)}{t_j} \le C$ for some fixed point $x_0 \in M$ and $C > 0$ (where $r_0(x, y)$ is the distance function with respect to the initial metric $g(0)$), define $g_j(t) = \frac{1}{t_j} g(t_j t)$. Then the pointed sequence $(M, x_j, g_j(x, t))$ sub-sequentially converges to a gradient expanding Kähler-Ricci soliton $(M_\infty, x_\infty, g_\infty(t))$.*

Notice that by a result of H.-D. Cao [**C**], if the blow-down is taken places at the points where the re-scaled curvature $tR(x, t)$ assumes the maximum over some space-time region. In a recent work [**CT**], Chau and Tam have further proved that the manifold M in Theorem 7.1 is biholomorphic to $\mathbb{C}^m$. The above Proposition 7.1 is useful (the above mentioned result of H.-D. Cao will not be enough) in [**CT**] for obtaining the compactness on the constructed biholomorphic maps. In fact, in their paper, Chau and Tam proved a special case (for space-time points (p, t_j)) of Proposition 7.1 independently.

Motivated by the work of Kodaira and Hirzebruch on the compact case, we propose the following question on the complex structure of complete Kähler manifolds.

Problem 7.2. Assume that M is a complete Kähler manifold which is diffeomorphic (homeomorphic) to the Euclidean space $\mathbb{C}^m$. Also assume that M does not support any nonconstant bounded holomorphic functions. Is M biholomorphic to $\mathbb{C}^m$?

It is still unknown that if a complete Kähler manifold with positive bisectional curvature is homeomorphic to $\mathbb{C}^m$. It is even still unknown if the manifold with positive bisectional curvature is Stein (this is a conjecture in [**GW**]). The above problem is differently formulated from the the well-known one [**Y2**] (see also [**GW**]) assuming the positivity on the curvature. It is simply a natural question about the uniqueness of the complex structure on a differentiable manifold with a fixed differential structure. Notice that without assuming that the manifold is Kähler, the question on the uniqueness of complex structure on the (topological) projective spaces is still open and the answer to the corresponding question for noncompact manifolds, namely Problem 7.2 without Kählerity, is negative.

The proof of Proposition 7.1 is based on a new 'reduced volume' monotonicity observed in [**N4**], which is dual to Perelman's reduced volume. The result is in certain sense a dual statement of Proposition 11.2 of [**P1**].

More precisely, for a fixed x_0, let γ be a path $(x(\eta), \eta)$ joining $(x_0, 0)$ to (y, t). Following [**P1**] (see also [**FIN**]) we define

$$\widetilde{\mathcal{L}}(\gamma) = \int_0^t \sqrt{\eta}\left(R + 4|\gamma'(\eta)|^2\right)\, d\eta.$$

Let $X = \gamma'(t) = \frac{dz^\alpha(t)}{dt}\frac{\partial}{\partial z^\alpha}$ and let Y be a variational vector field along γ. Here $|\gamma'(t)|^2 = g_{\alpha\bar{\beta}}\frac{dz^\alpha(t)}{dt}\frac{dz^{\bar{\beta}}(t)}{dt}$. Using $\widetilde{\mathcal{L}}$ as energy we can define the $\widetilde{\mathcal{L}}$-geodesics and we denote $L_+(y,t)$ to be the length of a shortest geodesics jointing $(x_0, 0)$ to (y, t). We also define

$$\ell_+(y, t; x_0, 0) := \frac{1}{2\sqrt{t}} L_+(y, t).$$

Following the first and second variation calculation of [**P1**] (see also [**FIN**] [**N4**]) we have that

$$|\nabla \ell_+|^2 = -R + \tfrac{\ell_+}{t} + \tfrac{K}{t^{3/2}}, \tag{7.1}$$

$$\frac{\partial \ell_+}{\partial t} = R - \tfrac{K}{2t^{3/2}} - \tfrac{\ell_+}{t}, \tag{7.2}$$

$$\Delta \ell_+ \le R + \tfrac{n}{2t} - \tfrac{K}{2t^{3/2}}. \tag{7.3}$$

Here

$$K := \int_0^t \eta^{3/2} H(X)\, d\eta,$$

where $H(X) := \partial R/\partial t + 2\langle \nabla R, X\rangle + 2\langle X, \nabla R\rangle + 4Rc(X, X) + R/t$, is exactly the traced Li-Yau-Hamilton differential Harnack expression of H.-D. Cao [**C**] applying to the $(1, 0)$ vector field $2X$.

Proposition 7.3. *Let $(M^m, g(t))$ be a complete solution to Kähler-Ricci flow with bounded nonnegative bisectional curvature (or a complete solution to Ricci flow with bounded and nonnegative curvature operator). Let $H(x, t; x_0, 0)$ be the fundamental solution to the forward conjugate heat equation $\left(\frac{\partial}{\partial t} - \Delta - R\right) u = 0$, centered at $(x_0, 0)$. Then*

$$\tilde{u}(x, t; x_0, 0) := \frac{1}{(\pi t)^m} \exp\left(-\ell_+(x, t; x_0, 0)\right)$$

satisfies

$$\left(\frac{\partial}{\partial t} - \Delta - R\right)\tilde{u}(x, t) \le 0. \tag{7.4}$$

In particular,

$$\tilde{u}(x,t;x_0,0) \le H(x,t;x_0,0) \tag{7.5}$$

and

$$\tilde{\theta}_+^{(x_0,0)}(t) := \int_M \tilde{u}(x,t)\, d\mu_t$$

is monotone non-increasing in t. Moreover, the equality in (7.4), or (7.5) implies that M is a gradient expanding soliton.

Proof. First (7.1)–(7.3) imply that

$$\left(\frac{\partial}{\partial t} - \Delta - R\right)\left(\frac{1}{(\pi t)^m}\exp\left(-\ell_+(y,t)\right)\right) = -\frac{K}{t^{\frac{3}{2}}}\left(\frac{1}{(\pi t)^m}\exp\left(-\ell_+(y,t)\right)\right) \le 0. \tag{7.6}$$

Here we have used fact that $K \ge 0$ under the assumption that M has bounded non-negative bisectional curvature. Also if the equality holds it implies that $K \equiv 0$. This further implies that M is an expanding soliton from the computation in [**FIN**]. In order to prove (7.5) one just needs to apply the maximum principle as before and notice that $\lim_{t\to 0} \frac{1}{(\pi t)^m}\exp(-\ell_+(y,t)) = \delta_{x_0}(y)$. The equality case follows from the consideration of the equality in [**FIN**]. □

The proof of Proposition 7.1 follows from (7.6) and a similar argument as in the mean curvature flow case [**Hu**] was first done by Huisken. Interested reader should consult [**Hu**] [**N4**] for more details.

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