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Kähler manifolds and cross quadratic bisectional curvature

Abstract. In this article we continue the study of the two curvature notions for Kähler manifolds introduced by the first named author earlier: the so-called cross quadratic bisectional curvature (CQB) and its dual (^{d}CQB) (which is a Hermitian form on maps between $T'M$ and $T''M$) . We first show that compact Kähler manifolds with $\mathrm{CQB}_1 > 0$ (CQB_1 is the restriction on rank one maps) or ${}^d\mathrm{CQB}_1 > 0$ are Fano, while nonnegative CQB_1 or ${}^d\mathrm{CQB}_1$ leads to a Fano manifold as well, provided that the universal cover does not contain a flat de Rham factor. For the latter statement we employ the Kähler-Ricci flow to deform the metric. We conjecture that all Kähler C-spaces will have nonnegative CQB and positive ^{d}CQB. By giving irreducible such examples with arbitrarily large second Betti numbers we show that the positivity of these two curvature put no restriction on the Betti number. A strengthened conjecture is that any Kähler C-space will actually have positive CQB unless it is a $\mathbb{P}^1$ bundle. Finally we give an example of non-symmetric, irreducible Kähler C-space with $b_2 > 1$ and positive CQB, as well as compact non-locally symmetric Kähler manifolds with CQB< 0 and ^{d}CQB< 0.

Keywords. Kähler homogenous spaces, Cross quadratic bisectional curvature, Generalized Hartshorne conjecture, Kähler-Ricci flow

1. Introduction

In a recent work [22] by the first named author, the concept of *cross quadratic bisectional curvature* (denoted as CQB from now on) and its dual notion (denoted by ^{d}CQB) for Kähler manifolds were introduced (they shall be defined shortly below). Both concepts are closely related to the notion of quadratic bisectional curvature (abbreviated as QB, see [31], [6], [32], [14], [7], [23], and [22] for the definition and results related to it). One of the reasons for the consideration of these different notions of curvature is to find suitable differential geometric characterizations for the Kähler C-spaces motivated by the generalized Hartshorne conjecture as illustrated below.

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First recall that a simply connected compact complex manifold is called a C-space (following H. C. Wang [29]) if its group of biholomorphisms acts transitively. A Kähler C-space is such a manifold which admits a Kähler metric such that the group of the holomorphic isometries also acts transitively. Namely it is Kähler homogenous [2]. It was proved that [17] any Kähler homogenous manifold is a product of a torus with a Kähler C-space. The Hartshorne conjecture (Mori's theorem [19]) asserts that a compact complex manifold with ample tangent bundle must be biholomorphic to $\mathbb{P}^n$. A weaker result is the so-called Frankel's conjecture (a theorem of Siu-Yau [26]) asserts that any Kähler manifold with positive bisectional curvature must be $\mathbb{P}^n$. These works together with [18] provide a curvature characterization of Hermitian symmetric spaces. As an attempt of providing a curvature characterization of homogenous Kähler manifolds, the generalized Hartshorne conjecture asserts that *A Fano manifold has nef tangent bundle if and only if it is a Kähler C-space*. This was proposed in [5] (Conjectures 11.1 and 11.2), and in his Harvard thesis [33] by the second author. The conjecture is only known for dimensions 2 and 3 by appealing to the classification theory of low dimensions.

In [22], inspired partially by the connection between the positive orthogonal Ricci (denoted by $\mathrm{Ric}^\perp > 0$, and studied in [24, 25]) and QB > 0, and partially by the work of Calabi-Vesentini [3], among other things the first named author proved that CQB > 0 implies $\mathrm{Ric}^\perp > 0$, which leads to the vanishing of holomorphic $(p, 0)$- forms and simply-connectedness of the compact Kähler manifolds. The positivity of ^{d}CQB, on the other hand, leads to the vanishing of the first cohomology group of the holomorphic tangent bundle, thus the manifold must be infinitesimally rigid, i.e., without nontrivial small deformations. It is also proved in [22] that, any classical Kähler C-space M^n with $b_2 = 1$ and $n \geq 2$ its canonical Einstein metric admits positive CQB and positive ^{d}CQB. This makes the two conditions (namely CQB>0 and ^{d}CQB>0) better candidates than QB in terms of describing Kähler C-spaces, as only about eighty percent of the above spaces have positive or nonnegative QB by the calculation (cf.[13]) in the excellent work of Chau and Tam [7].

Inspired by the perspective of a curvature characterization of the Kähler C-spaces, in this paper we continue the effort of understanding (with the aim of classifying) compact Kähler manifolds with positive or nonnegative CQB (or ^{d}CQB). Recall that by [22], on a Kähler manifold (M^n, g), if we denote by $T'M$ and $T''M$ the holomorphic and anti-holomorphic tangent bundle of M, then CQB is a Hermitian quadratic form on linear maps $A : T''M \to T'M$:

$$\mathrm{CQB}_R(A) = \sum_{\alpha,\beta=1}^{n} R(A(\overline{E}_\alpha), \overline{A(\overline{E}_\alpha)}, E_\beta, \overline{E}_\beta) - R(E_\alpha, \overline{E}_\beta, A(\overline{E}_\alpha), \overline{A(\overline{E}_\beta)}) \tag{1.1}$$

where R is the curvature tensor of M and $\{E_\alpha\}$ is a unitary frame of $T'M$. The expression is independent of the choice of the unitary frame. When the meaning is clear we simply write CQB or CQB(A). The manifold (M^n, g) is said to have positive (nonnegative) CQB, if at any point $x \in M$, and for any non-trivial linear map $A : T''_x M \to T'_x M$, the value

$\mathrm{CQB}(A)$ is positive (nonnegative). We say that $CQB_k > 0$ if (1.1) holds for all A with rank no greater than k.

Similarly, the dual notion (${}^d\mathrm{CQB}$) introduced in [22] is a Hermitian quadratic form on linear maps $A : T'M \to T''M$:

$$
{}^d\mathrm{CQB}_R(A) = \sum_{\alpha,\beta=1}^{n} R(\overline{A(E_\alpha)}, A(E_\alpha), E_\beta, \overline{E}_\beta) + R(E_\alpha, \overline{E}_\beta, \overline{A(E_\alpha)}, A(E_\beta)) \tag{1.2}
$$

where R again is the curvature tensor of M and $\{E_\alpha\}$ is a unitary frame of $T'M$. The manifold (M^n, g) is said to have positive (nonnegative) ${}^d\mathrm{CQB}$, if ${}^d\mathrm{CQB}(A) > 0$ (≥ 0) at any point in $x \in M$, and for any non-trivial linear map $A : T'_xM \to T''_xM$. Related to this there is a tensor analogous to the Ricci: $\mathrm{Ric}^+(X, \overline{X}) = \mathrm{Ric}(X, \overline{X}) + H(X)/|X|^2$, where H is the holomorphic sectional curvature. We say that ${}^d\mathrm{CQB}_k > 0$ if (1.2) holds for all A with rank no greater than k.

It is proved in [22] that compact Kähler manifold M^n with $\mathrm{Ric}^+ > 0$ is projective and simply connected. Also, if ${}^d\mathrm{CQB}> 0$, then $H^1(M, T'M) = \{0\}$, so M is locally deformation rigid. Moreover ${}^d\mathrm{CQB}_1 > 0$ implies $\mathrm{Ric}^+ > 0$.

Serving as a further step of the study our first result of this article is that the positivity of either CQB_1 (or ${}^d\mathrm{CQB}_1$) implies the positivity of the Ricci curvature. Thus a compact manifold with either $\mathrm{CQB}_1 > 0$ or ${}^d\mathrm{CQB}_1 > 0$ is Fano, answering positively a question asked in [22].

Theorem 1.1. *Let (M, g) be a Kähler manifold with either $CQB_1 > 0$ or ${}^dCQB_1 > 0$. Then its Ricci curvature is positive. So compact Kähler manifolds with positive CQB_1 or ${}^dCQB_1$ are Fano.*

As a corollary, the above theorem implies that a product Kähler manifold has positive (or nonnegative) CQB or ^{d}CQB if and only if each of its factors is so:

Corollary 1.2. *Let $M = M_1 \times M_2$ be a product Kähler manifold. Then M has $CQB > 0$ (or ≥ 0) if and only if both M_1 and M_2 are so. Also, for any positive integer k, M has $CQB_k > 0$ (or ≥ 0) if and only if both M_1 and M_2 are so. The same statements hold for ${}^dCQB$ or ${}^dCQB_k$ as well.*

By deforming the metric via the Kähler-Ricci flow we further show that if M has $\mathrm{CQB}_1 \geq 0$ (or ${}^d\mathrm{CQB}_1 \geq 0$) and its universal cover does not contain a flat de Rham factor then M is Fano as well. Note that the finiteness of the fundamental group of M implies the nonexistence of the flat de Rham factor. Namely in particular, if M has $\mathrm{CQB}_1 \geq 0$ (or ${}^d\mathrm{CQB}_1 \geq 0$) and $\pi_1(M)$ is finite, then M is a Fano manifold:

Theorem 1.3. *Let (M, g) be a compact Kähler manifold with $CQB_1 \geq 0$ (or ${}^dCQB_1 \geq 0$) and its universal cover does not contain a flat de Rham factor. Then M is Fano. In fact, the Kähler-Ricci flow will evolve the metric g to ones with positive Ricci curvature.*

To prove this we adopt a nice technique of Böhm-Wilking [1] of deforming the metric via the Kähler-Ricci flow into $g(t)$ with positive Ricci curvature to our curvature conditions. In [1], the authors deformed a Riemannian metric with nonnegative sectional curvature (also assuming finiteness of the fundamental group) into one with positive Ricci via the Ricci flow. Since $\mathrm{CQB}_1 \geq 0$ (or ${}^d\mathrm{CQB}_1 \geq 0$) is different from the sectional curvature being nonnegative, a different collection of invariant time-dependent convex sets is constructed to serve the purpose. We also need somewhat different estimates to show that $\mathrm{Ric}(g(t)) > 0$ for $t > 0$, where $g(t)$ is a short time solution of the Kähler-Ricci flow. In fact our curvature conditions here are much weaker than the bisectional curvature being nonnegative (which is weaker than the sectional curvature), since the result of Mok [18] asserts that the nonnegativity of bisectional curvature of an irreducible compact Kähler manifold must be locally Hermitian symmetric, and that the first author proved in [22] that all classical Kähler C-spaces with $b_2 = 1$ admits Einstein metrics with CQB> 0 and ^{d}CQB> 0 (see also further examples with $b_2 > 1$ in this paper).

As suggested by R. Hamilton, the condition CQB≥ 0 and ^{d}CQB≥ 0 have their analogous versions for Riemannian manifolds, and the above theorem also holds in that case. See §3 for more details.

By the structure theorem of [4] for compact Kähler manifolds with nonnegative Ricci, we have the following:

Corollary 1.4. *Let (M, g) be a compact Kähler manifold with $CQB_1 \geq 0$ (or ${}^dCQB_1 \geq 0$). Then there exists a finite cover of M' of M, such that M' is a holomorphic and metric fiber bundle over its Albanese variety, which is a flat complex torus, with the fiber being a Fano manifold.*

Note that for a compact Kähler manifold with nonnegative QB, any harmonic $(1,1)$ form is parallel, and the positivity of QB implies that $b_2 = 1$. The positivity/nonnegativity of CQB or ^{d}CQB however does not put any restrictions on b_2 (see Theorem 1.6 below). On the other hand, since CQB> 0 implies positive $\mathrm{Ric}^{\perp}$ by [22], while $\mathbb{P}^1$ bundles do not admit any Kähler metric with positive $\mathrm{Ric}^{\perp}$ by [24], so for Kähler C-spaces with $b_2 > 1$, we could only hope for nonnegative CQB instead of positive CQB in general. We propose the following:

Conjecture 1.5. *Any Kähler C-space (with the canonical Kähler-Einstein metric) has nonnegative CQB and positive ^{d}CQB.*

As a supporting evidence to Conjecture 1.5, we prove the following:

Theorem 1.6. *There are irreducible Kähler C-spaces with arbitrarily large b_2 which have nonnegative CQB and positive ^{d}CQB.*

To prove this result as an initial study towards the conjecture, we look into the simplest kind of irreducible Kähler C-spaces with $b_2 > 1$, namely, Type A flag manifolds: $M^n = SU(r+1)/\mathbb{T}$, where $\mathbb{T}$ is a maximal torus in $SU(r+1)$. The complex dimension is $n = \frac{1}{2}r(r+1)$ and $b_2 = r$. Equip M^n with the canonical Kähler-Einstein metric g, we show that it has nonnegative CQB and positive ^{d}CQB. This answers negatively another question asked in [22] regarding b_2.

As shown in [24], any $\mathbb{P}^1$ bundle cannot admit a Kähler metric with positive orthogonal Ricci curvature, thus cannot have positive CQB. We speculate that *any Kähler C-space which is not a $\mathbb{P}^1$ bundle has a metric with positive CQB.*

For compact Hermitian symmetric spaces, this speculation holds true (see Corollary 2.3 in the next section). For non-symmetric Kähler C-spaces, result below gives at least an example of irreducible Kähler C-space of $b_2 > 1$ with positive CQB. Such a space is necessarily not a $\mathbb{P}^1$ bundle.

Consider irreducible Kähler C-spaces of Type A in general, namely, $SU(r+1)/K$, where K is the centralizer of some sub-torus of $\mathbb{T}$. The smallest dimensional such space which is not a $\mathbb{P}^1$ bundle nor symmetric is $M^{12} = SU(6)/S(U(2)\times U(2)\times U(2))$. It has $b_2 = 2$. Equip it with the Kähler-Einstein metric, we show that it indeed has positive CQB:

Theorem 1.7. *Let $M^{12} = SU(6)/S(U(2)\times U(2)\times U(2))$ be the irreducible Kähler C-space which is non-symmetric, with $b_2 = 2$, and equip it with the Kähler-Einstein metric. Then it has positive CQB and positive ^{d}CQB.*

We should point out that understanding the curvature behavior of Kähler C-spaces is a nontrivial matter, despite the fact that such spaces are classical objects of study since 1950s and are fully classified from the Lie algebraic point of view. As an illustrating example, recall the following folklore conjecture:

Conjecture 1.8. *Any Kähler C-space (with the canonical Kähler-Einstein metric) has positive holomorphic sectional curvature H.*

This question is still widely open. For Kähler C-spaces with $b_2 = 1$, all the classical types plus a few exceptional ones are known to have $H > 0$ by the work of Itoh [13]. In a recent thesis [16], Simon Lohove underwent a highly sophisticated approach and was able to show that all irreducible Kähler C-spaces of classical type with rank less than or equal to 4 have $H > 0$. Note that the rank here means that of the group, so all such spaces have $b_2 \leq 4$ in particular. Through isometric embedding, he also reduced the question largely to the case of flag manifolds with Kähler-Einstein metrics.

In the more challenging opposite direction, we propose the following:

Conjecture 1.9. *Let (M, g) be a Kähler (Kähler-Einstein) manifold with CQB ≥ 0 and ^{d}CQB > 0. Then M is biholomorphic (isometric) to a Kähler C-space (with the canonical Kähler-Einstein metric).*

This conjecture, if affirmed, would be the first curvature characterization of compact homogeneous Kähler manifolds, which has been long missing but hoped for, in relation to the generalized Hartshorne conjecture (cf. [5]). A more general conjecture is to drop the Kähler-Einstein assumption above. The simply-connectedness, projectivity, and deformation rigidity result proved recently in [22], and Theorem 1.3 above are positive evidences towards this conjecture. Theorem 1.3 and Corollaries 1.2 and 1.4 also serve an initial step towards the classification conjecture as the main result of [12] towards the classification of Kähler manifolds with nonnegative bisectional curvature. The examples in Theorems 1.6

and 1.7 indicate that the situation here is more delicate. There are also attempts of investigating the generalized Hartshorne conjecture by a Hermitian curvature flow (cf. [28]), which aims to classify Hermitian Fano manifolds with Griffiths nonnegative curvature.

Note that most results mentioned above, except the construction of examples, hold for the non-positive cases by flipping the sign of the curvature. These results are summarized in the last section. In the last section we also show that the two dimensional Mostow-Siu example [20] had CQB<0 and ^{d}CQB<0. This is a non-Hermitian symmetric example to which Theorem 4.1 of [22] can be applied, hence is locally deformational rigid (it is in fact strongly rigid by the work of Siu [27]). The existence of non-symmetric examples with CQB<0 and ^{d}CQB<0 also shows that the local rigidity result of [22] is in deed more general than that of [3]. The examples naturally lead to the question of the role played by CQB and ^{d}CQB in the strong rigidity and holomorphicity of harmonic maps. We leave this to a future study.

2. Cross quadratic bisectional curvature and its dual

It is proved in [22] that positive CQB$_1$ implies that the orthogonal Ricci curvature $\mathrm{Ric}^\perp$ is positive, and CQB$_2>0$ implies that the Ricci curvature Ric is 2-positive, namely, the sum of any two of its eigenvalues is positive. We first show that the Ricci curvature is also positive under the CQB$_1>0$ assumption:

Theorem 2.1. *Let (M^n,g) $(n\geq 2)$ be a Kähler manifold with positive (or nonnegative) CQB$_1$, then its Ricci curvature is also positive (or nonnegative). Moreover* $\mathrm{Ric}(X,\overline{X})\geq \frac{1}{n-1}\,\mathrm{Ric}^\perp(X,\overline{X})$.

Proof. First we claim that, under the assumption that CQB$_1$ is positive, then for any unit vectors X, Y in $T'M$ such that $X\perp Y$, we must have $\mathrm{Ric}(X,\overline{X})>R(X,\overline{X},Y,\overline{Y})$. To see this, let E be a unitary frame for $T'M$ with $X=E_1$ and $Y=E_2$, and let A be the map such that $A(\overline{E}_2)=E_1$ and $A(\overline{E}_i)=0$ for any $i\neq 2$. Applying (2, 1) we get $\mathrm{Ric}_{1\bar{1}}>R_{2\bar{2}1\bar{1}}$. By the same token, $\mathrm{Ric}_{1\bar{1}}>R_{i\bar{i}1\bar{1}}$ for any $i>1$. Add up these inequalities for i from 2 to n, we get $(n-1)\,\mathrm{Ric}_{1\bar{1}}>\mathrm{Ric}^\perp_{1\bar{1}}$, so the Ricci curvature is positive since the orthogonal Ricci is known to be positive by [22]. The nonnegative case goes similarly. □

The proof also implies that

Corollary 2.2. *Let (M^n,g) $(n\geq 2)$ be a Kähler manifold with positive (or nonnegative) CQB$_1$, then Ric_{n-1} is also positive (or nonnegative).*

Corollary 2.3. *Let $M^n=M_1\times M_2$ be a product Kähler manifold. Then M has positive (or nonnegative) CQB$_k$ if and only if both M_1 and M_2 have positive (or nonnegative) CQB$_k$ for any $1\leq k\leq n$.*

Proof. Since CQB is independent of the choice of the unitary frames E we take the unitary frame E to be compatible with the product structure:

$$E=\{E_1,\dots,E_r;E_{r+1},\dots,E_n\},$$

where r is the dimension of M_1 and the first r elements give a frame for M_1. We will use the index convention that $i, j, \ldots$ run from 1 and r, while $\alpha, \beta, \ldots$ run from $r+1$ and n. Denote by R', R'' the curvature tensor of M_1, M_2, respectively, and write

$$A(\overline{E}_i) = A'(\overline{E}_i) + B(\overline{E}_i), \quad A(\overline{E}_\alpha) = C(\overline{E}_\alpha) + A''(\overline{E}_\alpha)$$

for the decomposition into $T'M = T'M_1 \times T'M_2$, then by definition, we have

$$\begin{aligned}
\mathrm{CQB}^M(A) &= \sum_{a,b,c=1}^{n} \mathrm{Ric}_{a\bar{b}}\, A_{ca}\overline{A_{cb}} - \sum_{a,b,c,d=1}^{n} R_{a\bar{b}c\bar{d}} A_{ac}\overline{A_{bd}} \\
&= \sum_{i,j,c} \mathrm{Ric}_{i\bar{j}}\, A_{ci}\overline{A_{cj}} + \sum_{\alpha,\beta,c} \mathrm{Ric}_{\alpha\bar{\beta}}\, A_{c\alpha}\overline{A_{c\beta}} - \sum_{a,b,c,d=1}^{n} R_{a\bar{b}c\bar{d}} A_{ac}\overline{A_{bd}} \\
&= \mathrm{CQB}^{M_1}(A') + \mathrm{CQB}^{M_2}(A'') + \sum_{i,j,\alpha} \mathrm{Ric}_{i\bar{j}}\, A_{\alpha i}\overline{A_{\alpha j}} + \sum_{\alpha,\beta,i} \mathrm{Ric}_{\alpha\bar{\beta}}\, A_{i\alpha}\overline{A_{i\beta}},
\end{aligned}$$

so the conclusion follows. Note that the positivity of CQB_k implies that the dimension of the manifold must be at least 2. □

Since every irreducible compact Hermitian symmetric space with dimension bigger than one has positive CQB and ^{d}CQB by [22], the above corollary allows us to conclude that

Corollary 2.4. *Every compact Hermitian symmetric has positive ^{d}CQB and nonnegative CQB, and it has positive CQB if and only if it does not have any $\mathbb{P}^1$ factor.*

If (M^n, g) is a compact Kähler manifold with nonnegative Ricci curvature, then by the work of Campana, Demailly and Peternell [4], the universal cover $\widetilde{M}$ of M is holomorphically and isometrically the product $\mathbb{C}^k \times M_1 \times M_2$, where the first factor (if $k > 0$) is the flat de Rham factor, and M_1 is Calabi-Yau (simply connected with trivial canonical line bundle), while M_2 is rationally connected. Also, there exists a finite cover M' of M, such that the Albanese map $\pi : M' \to \mathrm{Alb}(M')$ is surjective and is a holomorphic and metric fiber bundle with fiber $M_1 \times M_2$. Here the bundle being metric means that any point in the base is contained in a neighborhood over which the bundle is isometric to the product of the fiber with the base neighborhood.

Now if (M^n, g) is a compact Kähler manifold with $\mathrm{CQB}_1 \geq 0$, then since it has nonnegative Ricci, the above structure theorem applies. We claim that the Calabi-Yau factor cannot occur in this case:

Theorem 2.5. *Let (M^n, g) be a compact Kähler manifold with $CQB_1 \geq 0$. Then a finite cover M' of M is a holomorphic and metric fiber bundle over its Albanese torus, with fiber being a rationally connected manifold. In particular, if M has no flat de Rham factor, then it is rationally connected.*

Proof. The goal is to rule out the Calabi-Yau factor, namely, to show that if M_1 is a simply-connected compact complex manifold with $c_1 = 0$, then it cannot admit any Kähler metric with $\mathrm{CQB}_1 \geq 0$. To see this, notice that we have shown that $(n-1)\,\mathrm{Ric} \geq \mathrm{Ric}^{\perp} \geq 0$. So if $\mathrm{Ric}(X, \overline{X}) = 0$ for $X \in T'M_1$, then $\mathrm{Ric}^{\perp}(X, \overline{X}) = 0$ and $R(X, \overline{X}, X, \overline{X}) = 0$. Let η be the Ricci $(1,1)$-form of M_1, then by

$$c_1 \cdot [\omega]^{n-1} = \int_{M_1} \eta \wedge \omega^{n-1} = \frac{1}{n}\int_{M_1} S\omega^n,$$

where ω is the Kähler form and S the scalar curvature, we see that the vanishing of the first Chern class c_1 plus the nonnegativity of Ricci imply that M_1 has to be scalar flat hence Ricci flat. So the holomorphic sectional curvature is identically zero, contradicting the fact that M_1 is simply connected. □

In fact for any pair of X and Y by choosing $\{E_i\}$ such that $E_1 = \frac{X}{|X|}$, and letting A be the map with $A(\overline{E}_1) = Y$, $A(\overline{E}_i) = 0$ for $i \geq 2$ the argument above implies the following corollary.

Corollary 2.6. *The assumption $CQB_1 \geq 0$ is equivalent to that for any X and Y,*

$$|X|^2\,\mathrm{Ric}(Y, \overline{Y}) - R(X, \overline{X}, Y, \overline{Y}) \geq 0. \tag{2.1}$$

If $CQB_1 > 0$, then the above holds as a strict inequality if X, Y are nonzero.

Remark: It is not hard to see that under the $\mathrm{CQB} \geq 0$ assumption, any tangent vector $X \in T'M$ with $\mathrm{Ric}(X, \overline{X}) = 0$ must be in the kernel of the curvature tensor R, namely, $R(X, \overline{Y}, Z, \overline{W}) = 0$ for any $Y, Z, W \in T'M$.

Next, let us recall the notion of *dual cross quadratic bisectional curvature* (${}^d\mathrm{CQB}$) introduced in [22]. It is a Hermitian quadratic form on linear maps $A : T'M \to T''M$:

$$^d\mathrm{CQB}(A) = \sum_{\alpha,\beta=1}^{n} R(\overline{A(E_\alpha)}, A(E_\alpha), E_\beta, \overline{E}_\beta) + R(E_\alpha, \overline{E}_\beta, \overline{A(E_\alpha)}, A(E_\beta)) \tag{2.2}$$

where R again is the curvature tensor of M and $\{E_\alpha\}$ is a unitary frame of $T'M$. The manifold (M^n, g) is said to have positive (or nonnegative) ${}^d\mathrm{CQB}$, if at any point in M, for any unitary frame E of $T'M$ at p, and for any non-trivial linear map $A : T'M \to T''M$, the value ${}^d\mathrm{CQB}_E(A)$ is positive (or nonnegative). Related to this there is a $\mathrm{Ric}^+(X, \overline{X}) = \mathrm{Ric}(X, \overline{X}) + H(X)/|X|^2$.

It is proved in [22] that compact Kähler manifold M^n with positive $\mathrm{Ric}^+ > 0$ is projective and simply connected. If ${}^d\mathrm{CQB} > 0$ it also satisfies $H^1(M, T'M) = \{0\}$, so it is locally deformation rigid. Moreover ${}^d\mathrm{CQB}_1 > 0$ implies $\mathrm{Ric}^+ > 0$. Strictly analogous to the nonnegative CQB case, we have the following

Theorem 2.7. *A Kähler manifold with positive (or nonnegative) ${}^dCQB_1 > 0$ will have positive (or nonnegative) Ricci. A compact Kähler manifold with nonnegative ${}^dCQB_1 \geq 0$ and without flat de Rham factor is rationally connected. Moreover*

$$\mathrm{Ric}(X, \overline{X}) \geq \frac{1}{n+1}\,\mathrm{Ric}^+(X, \overline{X}).$$

In fact $^dCQB_1 \geq 0$ *is equivalent to the estimate:*

$$|X|^2 \operatorname{Ric}(Y, \overline{Y}) + R(X, \overline{X}, Y, \overline{Y}) \geq 0 \tag{2.3}$$

for any pair of $(1,0)$*-type vectors* X, Y*. If* $^dCQB_1 > 0$*, then the above holds as a strict inequality if* X, Y *are nonzero.*

Corollary 2.8. *Let* $M^n = M_1 \times M_2$ *be a product Kähler manifold. Then for any* $1 \leq k \leq n$*,* M *has positive (or nonnegative)* $^dCQB_k$ *if and only if both* M_1 *and* M_2 *have positive (or nonnegative)* $^dCQB_k$*.*

As noted in [22], when (M^n, g) is Kähler-Einstein, the CQB or ^{d}CQB conditions are given by the eigenvalue information for the curvature operator Q introduced by Calabi-Vessentini [3] and Itoh [13], which is the adjoint operator from $S^2(T'M)$ into itself, defined by

$$\langle Q(X \cdot Y), \overline{Z \cdot W} \rangle = R(X, \overline{Z}, Y, \overline{W})$$

for any type $(1,0)$ tangent vectors X, Y, Z, W in $T'M$, where $X \cdot Y = \frac{1}{2}(X \otimes Y + Y \otimes X)$ and the induced metric on $S^2(T'M)$ is given by

$$\langle X \cdot Y,\ \overline{Z \cdot W} \rangle = \frac{1}{2}\left(g(X, \overline{Z}) g(Y, \overline{W}) + g(X, \overline{Z}) g(Y, \overline{W}) \right).$$

If we denote by μ the constant Ricci curvature of M, and by λ_1, λ_N the smallest and largest eigenvalue of Q, respectively, then

$$\text{CQB} > 0 \iff \mu > \lambda_N, \quad \text{and} \quad {}^d\text{CQB} > 0 \iff \lambda_1 > -\mu.$$

In section 4, we shall examine the eigenvalue bounds for the simplest kind of Kähler C-spaces, namely, the Type A spaces, and check the sign for CQB and ^{d}CQB.

3. Fanoness of the nonflat factor

In this section we study further the factor in the splitting provided by Theorem 2.5. If we assume that the manifold (M, g) in Theorem 2.5 is simply-connected we show that M is a Fano manifold. Precisely we have the following slightly stronger result.

Theorem 3.1. *Assume that* (M, g) *be a compact Kähler manifold with* $CQB_1 \geq 0$ *(or* $^dCQB_1 \geq 0$*). Assume that the universal cover* $\tilde{M}$ *does not have a flat de Rham factor. Then* M *must be Fano. In fact the Kähler-Ricci flow evolves the metric* g *into a Kähler metric* $g(t)_{t \in (0, \epsilon)}$ *with positive Ricci curvature for some* ϵ*.*

Proof. Here we adapt an idea of Böhm-Wilking in [1] where the authors proved that the Ricci flow deformation of a metric with nonnegative sectional curvature of a compact manifold with finite fundamental group evolves the initial metric into one with positive Ricci curvature for some short time. The assumption on the fundamental group is to effectively rule out the flat de Rham factor in its universal cover. A dynamic version of Hamilton's maximum principle (cf. §1 of [1], Chapter 10 of [8], as well as [21]) was employed. Since $\mathrm{CQB}_1 \geq 0$ (or ${}^d\mathrm{CQB}_1 \geq 0$) is different from the sectional curvature being nonnegative, we need to construct a different collection of invariant time-dependent convex sets and prove the corresponding estimates to show that $\mathrm{Ric}(g(t)) > 0$. We shall focus on the case $\mathrm{CQB}_1 \geq 0$ since the other case is similar.

Let $g(t)$ be the solution to Kähler-Ricci flow with initial metric g satisfying $\mathrm{CQB}_1 \geq 0$:

$$\frac{\partial}{\partial t} g_{\alpha\bar{\beta}}(t) = -R_{\alpha\bar{\beta}}, \quad g(0) = g$$

where $R_{\alpha\bar{\beta}}$ denoted the Ricci curvature of $g(t)$. By Hamilton's maximum principle we can focus on the study of a collection of sets $\{C(t)\}$, each being a convex subset of the space of algebraic curvature operators satisfying the following conditions:

$$0 \leq \mathrm{Ric}(X, \overline{X}), \ \forall\, X \in T'_x M; \tag{3.1}$$

$$\left|\mathrm{Ric}(X, \overline{Y}) - R^{g(t)}_{X\overline{Y}Z\overline{Z}}\right|^2 \leq (D_1 + tE_1)\,\mathrm{Ric}(X, \overline{X})\,\mathrm{Ric}(Y, \overline{Y}), \ \forall\, X, Y, Z, |Z| = 1; \tag{3.2}$$

$$\|R\| \leq D_2 + tE_2. \tag{3.3}$$

Here in (3.3) R is viewed as the curvature operator and $\|\cdot\|$ is the natural norm extended to the corresponding tensors from the Kähler metric on $T'_x M$.

First we need to check that the sets $C(t)$ are convex. Clearly (3.1) and (3.3) are convex conditions. For (3.2) let R and S be two Kähler curvature operators. We shall check that if (3.2) holds for R and S then it holds for $\eta R + (1-\eta)S$ for $\eta \in [0,1]$. Given Z with $|Z| = 1$, $\mathrm{Ric}(X, \overline{Y}) - R_{X\overline{Y}Z\overline{Z}}$ is a Hermitian symmetric form on $T'_x M$, which we denote it as A, and denote the corresponding one for the curvature operator S as B. We also denote $R(X, \overline{X})$ and $R(Y, \overline{Y})$ as a_1 and a_2. Similarly we have b_1 and b_2 for the corresponding Ricci of the curvature operator S. Then

$$\begin{aligned}
|\eta A + (1-\eta)B|^2 &= \eta^2|A|^2 + \eta(1-\eta)(A\overline{B} + B\overline{A}) + (1-\eta)^2|B|^2 \\
&\leq \eta^2|A|^2 + 2\eta(1-\eta)|A|\,|B| + (1-\eta)^2|B|^2 \\
&\leq (D_1 + tE_1)\left(\eta^2 a_1 a_2 + 2\eta(1-\eta)\sqrt{a_1 a_2 b_1 b_2} + (1-\eta)^2 b_1 b_2\right) \\
&\leq (D_1 + tE_1)\,(\eta a_1 + (1-\eta)b_1)\,(\eta a_2 + (1-\eta)b_2)\,.
\end{aligned}$$

This completes the proof of the convexity of $C(t)$. Recall that after applying the Uhlenbeck's trick [10] the Kähler-Ricci flow evolves the curvature tensor R by the following PDE:

$$\left(\frac{\partial}{\partial t} - \Delta\right) R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\alpha\bar{\beta}p\bar{q}} R_{\gamma\bar{\delta}q\bar{p}} + R_{\alpha\bar{\delta}p\bar{q}} R_{\gamma\bar{\beta}q\bar{p}} - R_{\alpha\bar{p}\gamma\bar{q}} R_{p\bar{\beta}q\bar{\delta}}. \tag{3.4}$$

Here computation is with respect to a unitary frame. The first term on the right hand side can be written as $(2\,\mathrm{Rm}^2)_{\alpha\bar{\beta}\gamma\bar{\delta}}$, the second and third terms combined can be expressed as $(2\,\mathrm{Rm}^{\#})_{\alpha\bar{\beta}\gamma\bar{\delta}}$ with $\mathrm{Rm}^{\#} := \mathrm{ad}\cdot(\mathrm{Rm}\wedge\mathrm{Rm})\cdot\mathrm{ad}^*$. Here we identify $\wedge^2 T_p M$ with $\mathfrak{so}(T_p M)$, and view Rm as a symmetric map of $\mathfrak{so}(T_p M)$; $\mathrm{ad} : \wedge^2(\mathfrak{so}(T_p M)) \to \mathfrak{so}(T_p M)$ is the adjoint representation. Tracing it gives the evolution equation of the Ricci curvature:

$$\left(\frac{\partial}{\partial t} - \Delta\right) R_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}p\bar{q}} R_{q\bar{p}}. \tag{3.5}$$

We shall show that the set $C(t)$ defined by (3.1), (3.2) and (3.3) are invariant under the equation (3.4) and (3.5). Hamilton's maximum principle (see §1 of [1]) allows us to drop the diffusion term in verifying the invariance.

We first show that (3.2) holds at $t = 0$ since by Theorem 2.1 we have that (3.1) holds at $t = 0$, and it is easy to choose D_2 and E_2 to make (3.3) hold if ϵ is sufficiently small. By Theorem 2.1, in particular (2.1), we have that for any Z with $|Z| = 1$, $A(X, \overline{Y}) \doteqdot \mathrm{Ric}(X, \bar{Y}) - R_{X\bar{Y}Z\bar{Z}}$ is a Hermitian symmetric tensor which is nonnegative. Diagonalize A with a unitary frame $\{E_i\}$ and eigenvalues $\{\lambda_i\}$. Then we compute that for $X = x^i E_i$ and $Y = y^j E_j$

$$\begin{aligned}
\left|A_{i\bar{j}}x^i\bar{y}^i\right|^2 &= \left|\sum \lambda_i x^i \bar{y}^j\right|^2 \leq \sum \lambda_i |x^i|^2 \sum \lambda_j |y^j|^2 \\
&= (\mathrm{Ric}(X, \bar{X}) - R_{X\bar{X}Z\bar{Z}}) \cdot (\mathrm{Ric}(Y, \bar{Y}) - R_{Y\bar{Y}Z\bar{Z}}) \\
&\leq \sum_{i=1}^{n} \left(\mathrm{Ric}(X, \bar{X}) - R_{X\bar{X}E'_i\bar{E}'_i}\right) \sum_{j=1}^{n} \left(\mathrm{Ric}(Y, \bar{Y}) - R_{Y\bar{Y}E'_j\bar{E}'_j}\right) \\
&= (n-1)^2 \,\mathrm{Ric}(X, \overline{X})\,\mathrm{Ric}(Y, \overline{Y}).
\end{aligned}$$

Here $\{E'_j\}$ is another unitary frame so chosen that $E'_1 = Z$. Hence if we choose $D_1 = (n-1)^2$ the estimate (3.2) holds at $t = 0$.

Now we need to verify that the PDE/ODE preserves the set $C(t)$. For that we only need to prove that the time derivative of the convex condition lies inside the tangent cone of the convex set. The trick of [1] is to chose E_1 sufficiently large (compared with D_1, D_2, E_2) to make sure that (3.2) stay invariant under the PDE (3.4) (or the corresponding ODE $\frac{d}{dt}\mathrm{Rm} = \mathrm{Rm}^2 + \mathrm{Rm}^{\#}$) for $t \in [0, \epsilon]$ if ϵ is very small. With a suitably chosen D_2, it is easy to have (3.3). In fact we may choose $E_2 = 1$ if ϵ is small. For (3.1), if $\mathrm{Ric}(X, \bar{X})$ ever becomes zero for some X, then within $C(t)$ by (3.2), we have

$$\mathrm{Ric}(X, \overline{Y}) - R_{X\overline{Y}Z\overline{Z}} = 0, \quad \forall\, Y, Z.$$

This then via the polarization implies that $R_{X\overline{Y}Z\overline{W}} = 0$, $\forall\, Y, Z, W$. Thus (3.5) implies $\frac{\partial}{\partial t}\mathrm{Ric}(X, \overline{X}) \geq R_{X\overline{X}p\bar{q}}R_{q\bar{p}} = 0$. This shows that (3.1) is preserved by (3.5).

As in [1], the main issue is to show that (3.2) is preserved under the flow, namely (3.4) and (3.5). For this it suffices to show that as long as R is in $C(t)$,

$$\frac{\partial}{\partial t}\left((D_1 + tE_1)\,\mathrm{Ric}(X, \overline{X}) \cdot \mathrm{Ric}(Y, \overline{Y}) - \left|\mathrm{Ric}(X, \overline{Y}) - R_{X\overline{Y}Z\overline{Z}}\right|^2\right) \geq 0. \tag{3.6}$$

Direct calculation shows that the left hand side of the above inequality is

$$\begin{aligned}
&E_1\,\mathrm{Ric}(X, \overline{X})\,\mathrm{Ric}(Y, \overline{Y}) + (D_1 + tE_1)\left(R_{X\bar{X}p\bar{q}}\,\mathrm{Ric}(Y, \overline{Y}) + R_{Y\bar{Y}p\bar{q}}\,\mathrm{Ric}(X, \overline{X})\right) R_{q\bar{p}} \\
&-2\Re\left(\left(\frac{\partial}{\partial t}\mathrm{Ric}(X, \overline{Y}) - \frac{\partial}{\partial t}R_{X\overline{Y}Z\overline{Z}}\right)\overline{(\mathrm{Ric}(X, \overline{Y}) - R_{X\overline{Y}Z\overline{Z}})}\right).
\end{aligned}$$

We shall show that for ϵ small and $t \in [0, \epsilon]$ the above is nonnegative. Namely the first term dominates the rest. By (3.2), by letting $\epsilon \leq \frac{1}{E_1}$ (with E_1 to be decided later), $(D_1 + tE_1) \leq$

$2D_1$. In the mean time E_1 is chosen to be large comparing with $D_1^2 D_2$. First (3.2), together with $|\operatorname{Ric}(X,\overline{Y})| \le \sqrt{\operatorname{Ric}(X,\overline{X})\operatorname{Ric}(Y,\overline{Y})}$, imply that

$$|R_{X\overline{Y}Z\overline{Z}}| \le 4D_1\sqrt{\operatorname{Ric}(X,\overline{X})\operatorname{Ric}(Y,\overline{Y})} \tag{3.7}$$

which then implies that

$$\left|R_{X\bar{X}p\bar{q}}R_{q\bar{p}}\right| \le 4nD_1D_2\operatorname{Ric}(X,\overline{X}).$$

This, together with $tE_1 \le 1$, implies that

$$(D_1+tE_1)\left(R_{X\bar{X}p\bar{q}}\operatorname{Ric}(Y,\overline{Y}) + R_{Y\bar{Y}p\bar{q}}\operatorname{Ric}(X,\overline{X})\right)R_{q\bar{p}} \ge \tag{3.8}$$
$$-16nD_1^2D_2\operatorname{Ric}(X,\overline{X})\cdot\operatorname{Ric}(Y,\overline{Y}).$$

To handle the term involving $\frac{\partial}{\partial t}R_{X\overline{Y}Z\overline{Z}}$ we observe the following estimates:

$$|R_{X\overline{U}Z\overline{W}}| \le 32nD_1\sqrt{nD_2}\sqrt{\operatorname{Ric}(X,\overline{X})}, \tag{3.9}$$
$$|R_{Y\overline{U}Z\overline{W}}| \le 32nD_1\sqrt{nD_2}\sqrt{\operatorname{Ric}(Y,\overline{Y})}, \quad \forall\, U,Z,W,\ |U|=|Z|=|W|=1. \tag{3.10}$$

These can be derived easily out of (3.7) and (3.3). Now note that

$$\left|\overline{(\operatorname{Ric}(X,\overline{Y}) - R_{X\overline{Y}Z\overline{Z}})}\right| \le \sqrt{2D_1}\sqrt{\operatorname{Ric}(X,\overline{X})\operatorname{Ric}(Y,\overline{Y})}.$$

Hence we only need to establish that

$$\left|\left(\frac{\partial}{\partial t}\operatorname{Ric}(X,\overline{Y}) - \frac{\partial}{\partial t}R_{X\overline{Y}Z\overline{Z}}\right)\right| \le C(D_1,D_2,n)\sqrt{\operatorname{Ric}(X,\overline{X})\operatorname{Ric}(Y,\overline{Y})}$$

for some positive C depends on D_1, D_2 and n. By (3.4) and (3.5) we have that

$$\begin{aligned}\left(\frac{\partial}{\partial t}\operatorname{Ric}(X,\overline{Y}) - \frac{\partial}{\partial t}R_{X\overline{Y}Z\overline{Z}}\right) &= R_{X\overline{Y}Z\overline{W}}\operatorname{Ric}_{Z\overline{W}} - R_{Z\overline{Z}p\bar{q}}R_{q\bar{p}X\overline{Y}} \\ &\quad - R_{Z\overline{Y}q\bar{p}}R_{p\bar{q}X\overline{Z}} + R_{Z\bar{p}X\bar{q}}R_{p\overline{Z}q\overline{Y}}.\end{aligned}$$

Putting Estimates (3.7), (3.9) and (3.10) together we have the estimate we want. Taking $E_1 \ge 100C(D_1,D_2,n)$ we have proved (3.6). Hence $\{C(t)\}$ is an invariant collection of convex subsets under the Kähler-Ricci flow.

If for some $t \in (0,\epsilon)$, $\operatorname{Ric}(g(t))$ has a nontrivial kernel, the strong maximum principle (see for example, pages 675-676 of [1]) takes effect to imply that the universal cover splits a factor according to the distribution provided by the vectors in the kernel of the Ricci curvature. The factor is flat since by (3.2) the kernel of Ric would be the kernel of the curvature tensor. If there exists a sequence of such $t_i \to 0$ this implies that the universal cover contains a flat de Rham factor. This is a contradiction. Thus we have proved that $\operatorname{Ric}(g(t)) > 0$ for any $t \in (0,\epsilon')$ for some ϵ' small. $\square$

The result indicates a connection with Kähler C-spaces in view of the splitting theorem of Matsushima [17]. An argument similar as [1] was also employed by Liu in [15] to the non-positive setting to conclude that the deformed metric has negative Ricci curvature if the initial metric has non-positive bisectional curvature.

The conditions CQB≥ 0 and ^{d}CQB≥ 0 can have their corresponding Riemannian versions: We say that a Riemannian manifold (M^n, g) has CQB$^{\mathcal{R}}\geq 0$, if for any $x \in M$ and an orthonormal frame $\{e_i\}$, it holds that

$$\sum_{j=1}^{n} \mathrm{Ric}(A(e_j), A(e_j)) - \sum_{i,j=1}^{n} R(A(e_i), e_j, e_i, A(e_j)) \geq 0, \forall \text{ linear maps } A: T_xM \to T_xM. \tag{3.11}$$

For ^{d}CQB$^{\mathcal{R}}\geq 0$ we require that

$$\sum_{j=1}^{n} \mathrm{Ric}(A(e_j), A(e_j)) + \sum_{i,j=1}^{n} R(A(e_i), e_j, e_i, A(e_j)) \geq 0, \forall \text{ linear maps } A: T_xM \to T_xM. \tag{3.12}$$

Here to be consistent with the Kähler notations[1], the curvature tensor is defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

(which implies that $R(X,Y,Y,X) = \langle R(X,Y)Y, X\rangle$ is positive for spheres). If we restrict to A of rank one we have similar conditions as (2.1) and (2.3). Namely, CQB$_1^{\mathcal{R}} \geq 0$ is equivalent to

$$|X|^2 \,\mathrm{Ric}(Y,Y) - R(X,Y,Y,X) \geq 0. \tag{3.13}$$

Similarly, ^{d}CQB$_1^{\mathcal{R}} \geq 0$ is equivalent to

$$|X|^2 \,\mathrm{Ric}(Y,Y) + R(X,Y,Y,X) \geq 0. \tag{3.14}$$

It is easy to see that (3.13) and (3.14) will each imply the nonnegativity of the Ricci curvature. By adapting the proof of Theorem 3.1 we have the following result.

Theorem 3.2. *Assume that (M, g) be a compact Riemannian manifold with CQB$^{\mathcal{R}}{}_1 \geq 0$ (or ^{d}CQB$^{\mathcal{R}}{}_1 \geq 0$). Assume that the universal cover $\tilde{M}$ does not have a flat de Rham factor. Then M admits a metric with positive Ricci. In particular its fundamental group is finite. In fact the flow evolves the metric g into a metric $g(t)_{t\in(0,\epsilon)}$ with positive Ricci curvature for some ϵ.*

It is easy to check that the nonnegativity of the sectional curvature implies CQB$^{\mathcal{R}}{}_1 \geq 0$ and ^{d}CQB$^{\mathcal{R}}{}_1 \geq 0$. In fact CQB$^{\mathcal{R}}{}_1 \geq 0$ is the same as the $(n-2)$-nonnegativity of curvature in the sense of H. Wu (namely for any $n-1$ orthonormal frame $\{e_0, \cdots, e_{n-2}\}$, $\sum_{j=1}^{n-2} R(e_0, e_j, e_j, e_0) \geq 0$).[2] Hence the result above provides a generalization of the result

[1]Thanks to N. Wallach for suggesting this.

[2]See the Appendix. We are grateful to B. Wilking for pointing this out to us.

of Böhm-Wilking in [1]. The notions of $\mathrm{CQB}^{\mathcal{R}}$ and ${}^{d}\mathrm{CQB}^{\mathcal{R}}$ are not geometrically motivated as CQB and ^{d}CQB. We are grateful to Professor R. Hamilton for suggesting (3.13) and that Theorem 3.2 may still hold to the first named author. The nonnegative/positive conditions of these curvature respect the product structure (hence there is no difficult problem of a corresponding Hopf's conjecture for these curvatures).

Proposition 3.1. *Let $M^n = M_1 \times M_2$ be a product manifold. Then for any $1 \le k \le n$, M has positive (or nonnegative) ${}^{d}CQB_k^{\mathcal{R}}$ if and only if both M_1 and M_2 have positive (or nonnegative) ${}^{d}CQB_k^{\mathcal{R}}$.*

It is also not hard to check $\mathrm{CQB}^{\mathcal{R}} \ge 0$ and ${}^{d}\mathrm{CQB}^{\mathcal{R}} \ge 0$ for the locally symmetric spaces. A study of these conditions perhaps should begin with the homogenous Riemannian manifolds. The homogenous manifolds with positive sectional curvature is quite scarce [30]. We expect that $\mathrm{CQB}^{\mathcal{R}} \ge 0$ and ${}^{d}\mathrm{CQB}^{\mathcal{R}} \ge 0$ are more inclusive. We leave the more detailed study in this direction to a future project.

4. Kähler C-spaces

Recall that Kähler C-spaces are the orbit spaces of the adjoint representation of compact semisimple Lie groups [2]. Any such space is the product of simple Kähler C-spaces, and all simple Kähler C-spaces can be obtained in the following way.

Let $\mathfrak{g}$ be a simple complex Lie algebra. They are classified as four classical sequences $A_r = \mathfrak{sl}_{r+1}$ $(r \ge 1)$, $B_r = \mathfrak{so}_{2r+1}$ $(r \ge 2)$, $C_r = \mathfrak{sp}_{2r}$ $(r \ge 3)$, $D_r = \mathfrak{so}_{2r}$ $((r \ge 4)$ and the exceptional ones E_6, E_7, E_8, F_4 and G_2.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra with corresponding root system $\Delta \subset \mathfrak{h}^*$, so we have $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathbb{C}E_\alpha$ where E_α is a root vector of α. Let $r = \dim_{\mathbb{C}} \mathfrak{h}$ and fix a fundamental root system $\{\alpha_1, \ldots, \alpha_r\}$. This gives an ordering in Δ, and let Δ^+, Δ^- be the set of positive or negative roots. Each $\beta \in \Delta^+$ can be expressed as $\beta = \sum_{i=1}^r n_i(\beta)\alpha_i$. For a fixed nonempty subset $\Phi \subseteq \{\alpha_1, \ldots, \alpha_r\}$, denote by

$$\Delta_\Phi^+ = \{\beta \in \Delta^+ \mid n_i(\beta) > 0 \text{ for some } \alpha_i \in \Phi\}.$$

Let G be the simple complex Lie group with Lie algebra $\mathfrak{g}$ and L the closed subgroup with Lie subalgebra $\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\beta\in\Delta\setminus\Delta_\Phi^+} \mathbb{C}E_\beta$. Then $M^n = G/L$ is a simple Kähler C-space, and all simple Kähler C-spaces can be obtained that way. The complex dimension n of M is equal to the cardinality $|\Delta_\Phi^+|$, while $b_2(M) = |\Phi|$. The tangent space $T'M$ at the point eL can be identified with the subspace $\mathfrak{m}^+ = \bigoplus_{\beta\in\Delta_\Phi^+} \mathbb{C}E_\beta$ of $\mathfrak{g}$. Following Itoh [13], we will denote this simple Kähler C-space as $M^n = (\mathfrak{g}, \Phi)$.

Next let us recall the Chevalley basis (see [11] or Prop. 11 of [16]). Let B be the Killing form of $\mathfrak{g}$. For each $\alpha \in \Delta$, let H_α be the unique element in $\mathfrak{h}$ such that $B(H_\alpha, H) = \alpha(H)$ for any $H \in \mathfrak{h}$. One can always choose root vectors E_α of $\mathfrak{g}_\alpha$ so that $[E_\alpha, E_{-\alpha}] = H_\alpha$, $\overline{E}_\alpha = -E_{-\alpha}$, and $N_{-\alpha,-\beta} = -N_{\alpha,\beta}$, where $N_{\alpha,\beta}$ is defined by $[E_\alpha, E_\beta] = N_{\alpha,\beta}E_{\alpha+\beta}$ for any $\alpha, \beta \in \Delta$ with $\alpha \ne -\beta$. When $\alpha + \beta$ is not a root, then $N_{\alpha,\beta} = 0$.

Denote by $z_\alpha = B(E_\alpha, E_{-\alpha})$. Then $[E_\alpha, E_{-\alpha}] = z_\alpha H_\alpha$, and z_α are all real and $z_{-\alpha} = z_\alpha$ for each α. Now we describe the invariant Kähler metrics on M. Such a metric g makes the tangent frame $F := \{E_\alpha, \alpha \in \mathfrak{m}^+\}$ an orthogonal frame, with $g(E_\alpha, \overline{E}_\alpha) = g_\alpha z_\alpha$ where g_α satisfy the following additive condition with respect to Φ:

Write $\Phi = \{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$, where $1 \leq i_1 < \cdots < i_m \leq r$. Assign $g_{\alpha_{i_j}} = c_j > 0$ arbitrarily, and require $g_\beta = n_{i_1}(\beta)c_1 + \cdots + n_{i_m}(\beta)c_m$ for any $\beta = n_1(\beta)\alpha_1 + \cdots + n_r(\beta)\alpha_r$ in Δ_Φ^+. Denote this metric as $g = g_{(c_1,\ldots,c_m)}$. So the invariant Kähler metrics on M are determined by $m = b_2$ positive constants $c_1, \ldots, c_m$. It turns out (see §3.2 of [16]) that the metric is Einstein if and only if up to scaling, $g_\alpha = \sum_{\beta \in \Delta_\Phi^+} B(\alpha, \beta)$ for any $\alpha \in \Delta_\Phi^+$.

Following the computation initiated in [13], Lohove ([16], Prop 16) completed the curvature formula for (M^n, g) under the Chevalley frame F, which we will describe below. For $\alpha, \beta, \gamma, \delta \in \Delta_\Phi^+$, write $R(E_\alpha, \overline{E}_\beta, E_\gamma, \overline{E}_\delta)$ as $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$. A highly distinctive property of the curvature of M is that

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = 0 \quad \text{unless} \quad \alpha + \gamma = \beta + \delta. \tag{4.1}$$

To take advantage of the symmetry of curvature for Kähler metrics, let us consider the order relation $<$ in Δ: for $\alpha \neq \beta \in \Delta$, write $\alpha < \beta$ if $n_s(\alpha) < n_s(\beta)$ but $n_i(\alpha) = n_i(\beta)$ for all $1 \leq i < s$ (if $s > 1$).

For $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$ with $\alpha + \gamma = \beta + \delta$, by Kähler symmetries, we may assume that α is the smallest, and $\beta \leq \delta$. If $\alpha = \beta$, then $\gamma = \delta$, so we are left with $R_{\alpha\overline{\alpha}\gamma\overline{\gamma}}$ where $\alpha \leq \gamma$. If $\alpha \neq \beta$, then we are left with the case $\alpha < \beta \leq \delta < \gamma$. In the first case, Lohove obtained that, for any $\alpha, \gamma \in \Delta_\Phi^+$ with $\alpha \leq \gamma$:

$$R_{\alpha\overline{\alpha}\gamma\overline{\gamma}} = \begin{cases} g_\alpha z_\alpha z_\gamma B(H_\alpha, H_\gamma) + \frac{g_\alpha g_\gamma}{g_{\alpha+\gamma}} z_{\alpha+\gamma} N_{\alpha,\gamma}^2, & \text{if } \gamma - \alpha \in \Delta_\Phi^+; \\ g_\gamma z_\alpha z_\gamma B(H_\alpha, H_\gamma) + \frac{g_\gamma^2}{g_{\alpha+\gamma}} z_{\alpha+\gamma} N_{\alpha,\gamma}^2, & \text{if } \gamma - \alpha \notin \Delta_\Phi^+. \end{cases} \tag{4.2}$$

For the second case, he obtained that, for any $\alpha, \beta, \gamma, \delta \in \Delta_\Phi^+$ with $\alpha < \beta \leq \delta < \gamma$ and $\alpha + \gamma = \beta + \delta$,

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = \begin{cases} g_\alpha z_{\alpha-\beta} N_{\alpha,-\beta} N_{\gamma,-\delta} + \frac{g_\alpha g_\beta}{g_{\alpha+\gamma}} z_{\alpha+\gamma} N_{\alpha,\gamma} N_{\beta,\delta}, & \text{if } \gamma - \beta \in \Delta_\Phi^+; \\ g_\delta z_{\alpha-\beta} N_{\alpha,-\beta} N_{\gamma,-\delta} + \frac{g_\gamma g_\delta}{g_{\alpha+\gamma}} z_{\alpha+\gamma} N_{\alpha,\gamma} N_{\beta,\delta}, & \text{if } \gamma - \beta \notin \Delta_\Phi^+. \end{cases} \tag{4.3}$$

Note that in [16] the curvature R differs from here by a minus sign, as he is using a different sign convention. Next let us specialize to the simplest case, namely, when

$$\mathfrak{g} = A_r = \mathfrak{sl}(r+1)$$

is the space of all traceless complex $(r+1)$ square matrices. A Cartan subalgebra $\mathfrak{h}$ is given by all (traceless) diagonal matrices. The Killing form B is $B(X, Y) = \mathrm{tr}(XY)$. The root system is given by $\Delta = \{\alpha_{ij} : 1 \leq i, j \leq r+1\}$, where $\alpha_{ij}(H) = H_{ii} - H_{jj}$ for any $H \in \mathfrak{h}$, with a fundamental basis $\{\alpha_1, \ldots, \alpha_r\}$ where $\alpha_i = \alpha_{i(i+1)}$. The positive roots are $\Delta^+ = \{\alpha_{ij} : 1 \leq i < j \leq r+1\}$, with $-\alpha_{ij} = \alpha_{ji}$.

Denote by E_{ij} the $(r+1)\times(r+1)$ matrix whose only nonzero entry is 1 at the (i, j)-th position, and write $H_{ij} = E_{ii} - E_{jj}$. Then $\{H_{ij}, E_{ij}\}$ forms a Chevalley basis. Since $[E_{ij}, E_{ji}] = H_{ij}$, we know that $z_\alpha = 1$ for all $\alpha \in \Delta$. Thus the square norm $g(E_\alpha, \overline{E}_\alpha) = g_\alpha$.

To simplify our further discussions, let us introduce the following notations. For any $\alpha < \gamma$ in Δ^+, we will denote by

$$\begin{aligned}
&\gamma \sqcup \alpha \iff \gamma = \alpha_{ij},\ \alpha = \alpha_{jk} \ \text{ for some } 1 \leq i < j < k \leq r+1;\\
&\gamma \sqsupset' \alpha \iff \gamma = \alpha_{ik},\ \alpha = \alpha_{ij} \ \text{ for some } 1 \leq i < j < k \leq r+1;\\
&\gamma \sqsupset'' \alpha \iff \gamma = \alpha_{ik},\ \alpha = \alpha_{jk} \ \text{ for some } 1 \leq i < j < k \leq r+1;\\
&\gamma \sqsupset \alpha \iff \gamma \sqsupset' \alpha \ \text{ or } \ \gamma \sqsupset'' \alpha.
\end{aligned}$$

Since $B(H_{ij}, H_{kl}) = \mathrm{tr}\{(E_{ii} - E_{jj})(E_{kk} - E_{ll})\} = \delta_{ik} + \delta_{jl} - \delta_{il} - \delta_{jk}$, we get

$$B(H_\alpha, H_\alpha) = 2, \ \text{for each } \alpha,$$

and for any $\alpha < \gamma$ in Δ^+, we have

$$B(H_\alpha, H_\gamma) = \begin{cases} -1, & \text{if } \gamma \sqcup \alpha; \\ 1, & \text{if } \gamma \sqsupset \alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Also, since $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$, we get that for any $\alpha < \gamma$ in Δ^+,

$$N_{\alpha,\gamma} = \begin{cases} -1, & \text{if } \gamma \sqcup \alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Also, for any $\alpha < \beta \in \Delta^+$,

$$N_{\alpha,-\beta} = \begin{cases} -1, & \text{if } \gamma \sqsupset'' \alpha; \\ 1, & \text{if } \gamma \sqsupset' \alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Note that for $\delta < \gamma \in \Delta^+$, we have $N_{\gamma,-\delta} = -N_{-\gamma,\delta} = N_{\delta,-\gamma}$. Putting all these info into the Itoh-Lahove curvature formula, we get $R_{\alpha\overline{\alpha}\alpha\overline{\alpha}} = 2g_\alpha$, and for any $\alpha < \gamma$ in Δ^+_Φ,

$$R_{\alpha\overline{\alpha}\gamma\overline{\gamma}} = \begin{cases} -\frac{g_\alpha g_\gamma}{g_{\alpha+\gamma}}, & \text{if } \gamma \sqcup \alpha; \\ g_\alpha, & \text{if } \gamma \sqsupset \alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Also, for $\alpha < \beta < \delta < \gamma$ in Δ^+_Φ with $\alpha + \gamma = \beta + \delta$, only two cases will result in nonzero values for $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$, namely, either when $\gamma \sqcup \alpha = \delta \sqcup \beta$ and $\gamma \sqsupset' \delta$, or when $\beta \sqsupset' \alpha$, $\delta \sqsupset'' \alpha$, and $\gamma = \beta + \delta - \alpha$. In the first case the curvature equals to $-\frac{g_\alpha g_\delta}{g_{\alpha+\gamma}}$, and in the second case the curvature equals to g_α. Note that these two cases can be described equivalently as: there exist $1 \leq i < p < q < k \leq r+1$ such that $\delta = \alpha_{ip}, \beta = \alpha_{pk}, \gamma = \alpha_{iq}, \alpha = \alpha_{qk}$ for the first case, while $\delta = \alpha_{iq}, \beta = \alpha_{pk}, \gamma = \alpha_{ik}, \alpha = \alpha_{pq}$ for the second case.

Now let us switch to the unitary frame $\tilde{E}_\alpha = \frac{E_\alpha}{\sqrt{g_\alpha}}$ of $\mathfrak{m}^+$. For the sake of convenience, we will still use $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$ to denote the curvature component $R(\tilde{E}_\alpha, \overline{\tilde{E}_\beta}, \tilde{E}_\gamma, \overline{\tilde{E}_\delta})$. Also, to avoid clumsy notations, we will write $g_{\alpha_{ik}}$ simply as g_{ik}. Up to the Kähler symmetries, the only non-zero components of the curvature are

$$\begin{aligned} R_{\alpha\overline{\alpha}\alpha\overline{\alpha}} &= \frac{2}{g_\alpha}, \quad \alpha \in \Delta_\Phi^+; && (4.4) \\ R_{\alpha\overline{\alpha}\gamma\overline{\gamma}} &= \begin{cases} -\frac{1}{g_{ik}}, & \text{if } \exists\, i<j<k: \ \gamma = \alpha_{ij},\ \alpha = \alpha_{jk}; \\ \frac{1}{g_{ik}}, & \text{if } \exists\, i<j<k: \ \gamma = \alpha_{ik},\ \alpha = \alpha_{ij} \text{ or } \alpha_{jk}, \end{cases} && (4.5) \end{aligned}$$

where we assumed $\alpha < \gamma$. For $\alpha < \beta < \delta < \gamma$ in Δ_Φ^+, the curvature component $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$ will be equal to the following non-zero values only when there are $1 \leq i < p < q < k \leq r+1$ such that

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = \begin{cases} -\frac{\sqrt{g_{ip}g_{qk}}}{g_{ik}\sqrt{g_{iq}g_{pk}}}, & \text{if } \delta = \alpha_{ip}, \beta = \alpha_{pk}, \gamma = \alpha_{iq},\ \alpha = \alpha_{qk}; \\ \frac{\sqrt{g_{pq}}}{\sqrt{g_{ik}g_{iq}g_{pk}}}, & \text{if } \delta = \alpha_{iq}, \beta = \alpha_{pk}, \gamma = \alpha_{ik},\ \alpha = \alpha_{pq}. \end{cases} \tag{4.6}$$

Now check the sign for CQB or ^{d}CQB. First let us consider the case when $\Phi = \{\alpha_1, \ldots, \alpha_r\}$, namely, when $M^n = SU(r+1)/\mathbb{T}$ is the flag manifold, where $\mathbb{T}$ is a maximal torus. We have $n = \frac{1}{2}r(r+1)$, $b_2 = r$, and $\Delta_\Phi^+ = \Delta^+$. We will choose g to be the Kähler-Einstein metric. In this case, all $c_j = 1$, and $g_{\alpha_{ik}} = k - i$. It is easy to see that the Ricci curvature is constantly equal to $\mu = 2$.

For any symmetric $n \times n$ matrix A, the quadratic form

$$\langle Q(A), \overline{A} \rangle = \sum_{a,b,c,d=1}^{n} R_{a\overline{b}c\overline{d}} A_{ac} \overline{A_{bd}}$$

is equal to

$$\begin{aligned} & \sum_\alpha R_{\alpha\overline{\alpha}\alpha\overline{\alpha}} |A_{\alpha\alpha}|^2 + \sum_{\alpha<\gamma} 4R_{\alpha\overline{\alpha}\gamma\overline{\gamma}} |A_{\alpha\gamma}|^2 + \sum_{\alpha<\beta<\delta<\gamma} 8\Re\{R_{\alpha\overline{\beta}\gamma\overline{\delta}} A_{\alpha\gamma}\overline{A_{\beta\delta}}\} \\ = \ & \sum_\alpha \frac{2}{g_\alpha}|A_{\alpha\alpha}|^2 + \sum_{i<j<k} \frac{4}{g_{ik}} \left(|A_{ij,ik}|^2 + |A_{jk,ik}|^2 - |A_{jk,ij}|^2\right) + \\ & + \sum_{i<p<q<k} 8\Re\{-\frac{\sqrt{g_{ip}g_{qk}}}{g_{ik}\sqrt{g_{iq}g_{pk}}} A_{qk,iq}\overline{A_{pk,ip}} + \frac{\sqrt{g_{pq}}}{\sqrt{g_{ik}g_{iq}g_{pk}}} A_{pq,ik}\overline{A_{pk,iq}}\} \end{aligned} \tag{4.7}$$

Let us denote by X and Y the two terms in the last line above. We have

$$\mathrm{CQB}_{\tilde{E}}(A) = \mu||A||^2 - \langle Q(A), \overline{A}\rangle, \quad {}^d\mathrm{CQB}_{\tilde{E}}(A) = \mu||A||^2 + \langle Q(A), \overline{A}\rangle.$$

In order to check that CQB ≥ 0 and ^{d}CQB > 0 for $(SU(r+1)/\mathbb{T}, g)$, the flag manifold of type A with Einstein metric, it suffices to take care of the two crossing terms X and Y. For Y, the square root part of the coefficient is less than $\frac{1}{2}$, so we have

$$|Y| \leq \sum_{i<p<q<k} 4|A_{pq,ik}\overline{A_{pk,iq}}| \leq \sum_{i<p<q<k} 2|A_{pq,ik}|^2 + 2|A_{pk,iq}|^2$$

Note that in $2||A||^2 = 2\langle A, \overline{A}\rangle = \sum_\alpha |A_{\alpha\alpha}|^2 + 4\sum_{\alpha<\gamma} |A_{\alpha\gamma}|^2$, each $|A_{pq,ik}|^2$ term or $|A_{pk,iq}|^2$ term will appear 4 times, so the Y term will be dominated by $\mu||A||^2$ from above or below. For the X term, let us fix $i < k$ with $k - i = t + 1 \geq 2$. Write $A_{ip,pk} = Z_p$, and write $p' = p - i$. Since the square root part of the coefficient of X is less than 1, we have

$$|X| \leq \sum_{i<k} \sum_{1\leq p'<q'\leq t} \frac{4}{t+1}(|Z_p|^2 + |Z_q|^2) = \sum_{i<p<k} \frac{4(t-1)}{t+1}|Z_p|^2.$$

Again since for each $i < j < k$, the term $|A_{ij,jk}|^2 = |Z_j|^2$ will appear 4 times in $\mu||A||^2$, the X term will be dominated by $\mu||A||^2$ from above and below. Note that for the lower bound part, the term $|Z_p|^2$ will also emerge from the bisectional curvature terms, with coefficient $-\frac{4}{t+1}$. We have $-\frac{4(t-1)}{t+1} - \frac{4}{t+1} = -\frac{4t}{t+1} > -4$, so ^{d}CQB will be nonnegative, and actually positive since its vanishing would imply $A = 0$. We have thus proved Theorem 1.6 stated in the introduction.

Note that if A has only non-trivial entries along the diagonal line for the simple roots, then $\langle Q(A), \overline{A}\rangle = 2||A||^2$, so CQB is only nonnegative and not positive.

Next let us give a non-symmetric example of irreducible Kähler C-space with $b_2 > 1$ that has positive CQB. The smallest dimensional Type A space which is non-symmetric and not a $\mathbb{P}^1$ bundle would be $M^{12} = SU(6)/S(U(2) \times U(2) \times U(2))$, or equivalently, $(A_5, \Phi) = (\mathfrak{sl}_6, \Phi)$ where $\Phi = \{\alpha_2, \alpha_4\}$. It has $n = 12$ and $b_2 = 2$. We have

$$\Delta^+_\Phi = \{\alpha_{kl} \mid 1 \leq k < l \leq 6\} \setminus \{\alpha_{12}, \alpha_{34}, \alpha_{56}\}.$$

Up to a scaling, the Kähler-Einstein metric g has $g_{kl} = g_{\alpha_{kl}} = \sum_{\beta\in\Delta^+_\Phi} B(\alpha_{kl}, \beta)$, so we have

$$g_{13} = g_{14} = g_{23} = g_{24} = 2,$$
$$g_{35} = g_{36} = g_{45} = g_{46} = 2,$$
$$g_{15} = g_{16} = g_{25} = g_{26} = 4.$$

Let us denote by $\Delta_1 = \{\alpha_{15}, \alpha_{16}, \alpha_{25}, \alpha_{26}\}$ and $\Delta_2 = \Delta^+_\Phi \setminus \Delta_1$.

So the curvature components are $R_{\alpha\overline{\alpha}\alpha\overline{\alpha}} = \frac{2}{g_\alpha}$, which is $\frac{1}{2}$ for $\alpha \in \Delta_1$ and 1 for $\alpha \in \Delta_2$. While $R_{\alpha\overline{\alpha}\gamma\overline{\gamma}}$ are given by (4.5). It is easy to see that the Ricci curvature is constantly $\mu = 2$ in this case. The crossing terms $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$ are given by (4.6). We have

$$\mu||A||^2 = \sum_\alpha 2|A_{\alpha\alpha}|^2 + \sum_{\alpha<\gamma} 4|A_{\alpha\gamma}|^2.$$

Now consider the quadratic form $\langle Q(A), \overline{A}\rangle$ given by (4.7). Let us examine the two terms X and Y in the last line of (4.7). For the term Y, note that $i < p < q < k$ could be from $(1, 2, 3, 4)$, $(3, 4, 5, 6)$, in which case the square root part of the coefficient is $\frac{1}{2}$, or from $(1, 2, 3, 5)$, $(1, 2, 3, 6)$, $(1, 2, 4, 5)$, $(1, 2, 4, 6)$, $(1, 2, 5, 6)$, $(1, 3, 5, 6)$, $(1, 4, 5, 6)$, $(2, 3, 5, 6)$, or $(2,4,5,6)$. In each of these last 9 cases the square root part of the coefficient for the Y terms is $\frac{1}{4}$. So we have

$$|Y| \leq 2 \sum_{i<p<q<k} |A_{ik,pq}\overline{A_{iq,pk}}| \leq \sum_{i<p<q<k} (|A_{ik,pq}|^2 + |A_{iq,pk}|^2).$$

Here in the sum we are skipping those terms with $(p,q) = (3,4)$. Note that in $\mu||A||^2$, each of the terms $|A_{iq,pk}|^2$ appears with coefficient 4, so $|Y|$ is strictly dominated by $\mu||A||^2$ from above and below. Next let us consider the X terms. For each of $\alpha = \alpha_{qk}$, $\beta = \alpha_{pk}$, $\gamma = \alpha_{iq}$, $\delta = \alpha_{ip}$ to be in Δ^+_Φ, the indices $i < p < q < k$ could only take the following four cases: $(1,3,4,5)$, $(1,3,4,6)$, $(2,3,4,5)$, $(2,3,4,6)$. In each case, the square root part of the coefficient is 1, while $g_{ik} = 4$, so we have

$$|X| \leq \sum_{i=1}^{2} \sum_{k=5}^{6} |A_{i3,3k}|^2 + |A_{i4,4k}|^2.$$

So each of these $|A_{i3,3k}|^2$ or $|A_{i4,4k}|^2$ term in the quadratic form will be strictly dominated by that from $\mu||A||^2$ from both sides. The other terms are clearly strictly dominated by $\mu||A||^2$ from both above and below. So (M^{12}, g) has positive CQB and positive ^{d}CQB, and we have completed the proof of Theorem 1.8 stated in the introduction.

5. Non-positive cases

One may also consider Kähler manifolds with non-positive CQB or ^{d}CQB. Similar to the nonnegative cases, we have the following results:

Theorem 5.1. *Let (M,g) be a Kähler manifold with CQB$_1 \leq 0$. Then for any $X, Y \in T'_xM$*

$$|X|^2 \operatorname{Ric}(Y,\overline{Y}) - R(X,\overline{X},Y,\overline{Y}) \leq 0. \tag{5.1}$$

The above holds as strict inequality (for nonzero X, Y) if CQB$_1 < 0$. In particular

$$\operatorname{Ric}(Y,\overline{Y}) \leq \frac{1}{n-1} \operatorname{Ric}^\perp(Y,\overline{Y}) \leq 0.$$

Similarly, if (M,g) is Kähler with ^{d}CQB$_1 \leq 0$, then for any $X, Y \in T'_xM$

$$|X|^2 \operatorname{Ric}(Y,\overline{Y}) + R(X,\overline{X},Y,\overline{Y}) \leq 0, \tag{5.2}$$

and the inequality is strict (for nonzero X, Y) when ^{d}CQB$_1 < 0$. In particular, it holds that $\operatorname{Ric}(Y,\overline{Y}) \leq \frac{1}{n+1} \operatorname{Ric}^+(Y,\overline{Y}) \leq 0$.

A product Kähler manifold $M = M_1 \times M_2$ has CQB < 0 (or ≤ 0, or ^{d}CQB < 0, or ^{d}CQB ≤ 0) if and only if each factor is so. For any positive integer k, M has CQB$_k$ (or ^{d}CQB$_k$) < 0 or ≤ 0 if and only if each factor is so.

Proof. The proof is exactly the same as that of Theorem 2.1. $\square$

Theorem 5.2. *Assume that (M, g) be a compact Kähler manifold with $CQB_1 \leq 0$ (or ${}^dCQB_1 \leq 0$). Assume that the universal cover $\tilde{M}$ does not have a flat de Rham factor. Then M must admit a metric with* Ric < 0. *In fact the Kähler-Ricci flow evolves the metric g into a Kähler metric $g(t)_{t\in(0,\epsilon)}$ with negative Ricci curvature for some ϵ.*

Proof. We can prove the result by following the same argument and flipping the sign when needed in the proof of Theorem 3.1. □

Next construct examples of compact Kähler manifolds with negative (non-positive) CQB and ^{d}CQB. First of all, if M^n is a compact quotient of a Hermitian symmetric space $\widetilde{M}$ of non-compact type, then by [3], we see that M always has ^{d}CQB < 0 and CQB ≤ 0, and it will have CQB < 0 when and only when $\widetilde{M}$ does not have the unit disc as an irreducible factor.

For non-locally Hermitian symmetric examples, we adapt the construction of strongly negatively curved manifolds by Mostow and Siu [20] and by the second named author [34], [35]. To state the result, let us recall the notion of *good coverings*.

A finite branched cover $f : M^n \to N^n$ between two compact complex manifolds is called a good cover, if for any $p \in M$, there exists locally holomorphic coordinates $(z_1, \ldots, z_n)$ centered at p and $(w_1, \ldots, w_n)$ centered at $f(p)$, such that f is given by $w_i = z_i^{m_i}$, $1 \leq i \leq n$, where m_i are positive integers. Note that the branching locus B and ramification locus R are necessarily normal crossing divisors in this case.

In [20], Mostow and Siu computed the curvature for the Bergman metric of the Thullen domain $\{|z_1|^{2m} + |z_2|^2 < 1\}$, and used it to construct examples of strongly negatively curved surfaces which is not covered by ball. In [34], the second named author generalized this to higher dimensions, and also at the quotient space level using the Poincaré distance, and showed that (see Theorem 1 of [34]) if N is a compact smooth quotient of the ball, and $B \subset N$ a smooth totally geodesic divisor (possibly disconnected), then for any good cover $f : M \to N$ branched along B, M admits a Kähler metric with negative complex curvature operator. We will use this computation to claim the following:

Theorem 5.3. *Let N^n $(n \geq 2)$ be a smooth compact quotient of the ball, equipped with the complex hyperbolic metric, and let $B \subset N$ be a smooth totally geodesic divisor (possibly disconnected). If $f : M \to N$ is a good cover branched along B, then M admits a Kähler metric g which has negative CQB and negative ^{d}CQB.*

Remark: Such a manifold M is not homotopy equivalent to any locally Hermitian symmetric space, and it is strongly rigid in the sense of Siu, namely, any compact Kähler manifold homotopy equivalent to M must be (anti)biholomorphic to M.

Proof. The construction of the Kähler metrics ω_ε is exactly the same as in the proof of Theorem 1 of [34]. Notice that at the point p in a tubular neighborhood V of the ramification locus R, there exists tangent frame e at p such that $e_i \perp e_j$ whenever $i \neq j$, and under e the only non-zero curvature components of ω_ε are $R_{i\bar{i}j\bar{j}}$, with

$$-R_{1\bar{1}1\bar{1}} = b, \quad -R_{1\bar{1}i\bar{i}} = c, \quad -R_{i\bar{i}i\bar{i}} = 2e, \quad -R_{i\bar{i}j\bar{j}} = e$$

for any $2 \leq i < j$. It was shown that $b > 0$, $c > 0$, $e > 0$, and $nbe > (n-1)^2c^2$.

Note that if we normalize e, namely, replace e_k by $\frac{e_k}{|e_k|}$ for each k, then the above inequalities on b, c, and e still holds. So let us assume that e is unitary at p. For any non-trivial $n \times n$ matrix A, we have $-\mathrm{CQB}_e(A) = P - Q$, and $-{}^d\mathrm{CQB}_e(A) = P + Q$, where

$$
\begin{aligned}
P &= -\sum_{i,j,k,\ell} R_{i\bar{j}k\bar{k}} A_{\ell i}\overline{A_{\ell j}} = -\sum_{i,k,\ell} R_{i\bar{i}k\bar{k}} |A_{\ell i}|^2 \\
&= (b+(n-1)c)\sum_{\ell} |A_{\ell 1}|^2 + (c+ne)\sum_{i>1,\ell} |A_{\ell i}|^2 \\
Q &= -\sum_{i,j,k,\ell} R_{i\bar{j}k\bar{\ell}} A_{ik}\overline{A_{j\ell}} = -\sum_{i} R_{i\bar{i}i\bar{i}} |A_{ii}|^2 - \sum_{i<k} R_{i\bar{i}k\bar{k}} |A_{ik}+A_{ki}|^2 \\
&= b|A_{11}|^2 + 2e\sum_{i>1}|A_{ii}|^2 + c\sum_{i>1}|A_{1i}+A_{i1}|^2 + e\sum_{1<i<k}|A_{ik}+A_{ki}|^2
\end{aligned}
$$

Clearly, $P + Q > 0$ for all $A \neq 0$, and if we write $t_{ij} = |A_{ij}|^2$, we have

$$
\begin{aligned}
P - Q \;=\;& (n-1)c\, t_{11} \;+\; (c+(n-2)e)\sum_{i,k>1} t_{ik} \;+ \\
&+\sum_{i>1}\Big((b+(n-2)c)\, t_{i1} + ne\, t_{1i} - 2c\, \Re(A_{i1}\overline{A_{1i}})\Big),
\end{aligned}
$$

which is positive as $nbe > (n-1)c^2 > c^2$. So the metric ω_ε has $\mathrm{CQB} < 0$ and ${}^d\mathrm{CQB} < 0$ in V, for *any* $\varepsilon > 0$. By choosing ε sufficiently small, one see that CQB and ^{d}CQB will be negative everywhere in M. □

By [34] and [35], we see that there are many examples of such M in $n = 2$. An example in $n = 3$ was constructed by M. Deraux in [9], and we are not aware of any higher dimensional such constructions, even though it has been widely believed that there should be plenty in all dimensions.

Appendix

Proposition 5.1 (Wilking). *Let M^n, g) be a Riemannian manifold of dimension $n \geq 3$. Then $CQB^{\mathcal{R}}{}_1 \geq 0$ (> 0) is the same as the $(n-2)$-nonnegativity (positivity) of curvature. Dually, $CQB^{\mathcal{R}}{}_1 \leq 0$ (< 0) is the same as the $(n-2)$-nonpositivity (negativity) of curvature.*

Proof. Recall that the $n-2$-nonnegativity of the curvature is defined to be for any $n-1$ orthonormal vectors $\{e_0, \cdots, e_{n-2}\}$, $\sum_{j=1}^{n-2} R(e_0, e_j, e_j, e_0) \geq 0$. To verify this under $\mathrm{CQB}^{\mathcal{R}}{}_1 \geq 0$ we pick $X = e_0$ and $Y = e_{n-1}$ with e_{n-1} being the unit vector perpendicular to $\{e_0, \cdots, e_{n-2}\}$. Then $|Y|^2\, \mathrm{Ric}(X, X) - R(X, Y, Y, X) \geq 0$ immediately implies the result.

To prove the other direction we first observe that $(n-2)$-nonnegativity of the curvature implies that $\mathrm{Ric} \geq 0$. This can be seen from that, for any fixed $1 \leq k \leq n-1$

$$\sum_{j=1}^{n-1} R(e_0, e_j, e_j, e_0) - R(e_0, e_k, e_k, e_0) \geq 0.$$

Summing them together we have that

$$(n-2)\,\mathrm{Ric}(e_0, e_0) = (n-1)\,\mathrm{Ric} - \mathrm{Ric} \geq 0.$$

Since it holds for any $(n-1)$-orthonormal vectors $\mathrm{Ric} \geq 0$.

Given $X, Y \neq 0$, let $X = e_0$. And let e_n be the unit maximum eigen-direction of symmetric bilinear form $B(Y, Z) \doteqdot R(X, Y, Z, X)$. Clearly $R(e_0, e_n, e_n, e_0) \geq 0$, otherwise $R(e_0, e_j, e_j, e_0) \leq R(e_0, e_n, e_n, e_0) < 0$ which implies that $\mathrm{Ric}(e_0, e_0) < 0$, a contradiction.

We may assume that $e_0 \perp e_n$. Otherwise we may split e_n as $e_n = e_n^{\perp} + e_n^T$ into the perpendicular and tangent parts with $e_n^T \neq 0$ with $|e_n^{\perp}| < 1$. We may assume that $|e_n^{\perp}| \neq 0$ otherwise we have that $R(e_0, Y, Y, e_0) \leq R(e_0, e_n, e_n, e_0) = 0$. This together with $\mathrm{Ric} \geq 0$ implies that $|Y|^2 \,\mathrm{Ric}(X, X) - R(X, Y, Y, X) \geq 0$. Under the condition $1 > |e_n^{\perp}| > 0$

$$R\left(e_0, \frac{e_n^{\perp}}{|e_n^{\perp}|}, \frac{e_n^{\perp}}{|e_n^{\perp}|}, e_0\right) = \frac{1}{|e_n^{\perp}|^2} R(e_0, e_n, e_n, e_0) > R(e_0, e_n, e_n, e_0)$$

unless we have $R(e_0, e_n, e_n, e_0) = 0$, which implies $|Y|^2 \,\mathrm{Ric}(X, X) - R(X, Y, Y, X) \geq 0$ by the above argument and $\mathrm{Ric} \geq 0$.

Together, the above shows that either $e_0 \perp e_n$ or $|Y|^2 \,\mathrm{Ric}(X, X) - R(X, Y, Y, X) \geq 0$. Under the assumption $e_0 \perp e_n$ we can choose the other vectors $e_1, \cdots, e_{n-2}$ in the subspace perpendicular to $\mathrm{span}\{e_0, e_n\}$ so that we can apply the $(n-2)$-nonnegativity of the curvature to conclude that

$$\mathrm{Ric}(X, X) - R(X, e_n, e_n, e_0) = \sum_{j=1}^{n-2} R(e_0, e_j, e_j, e_0) \geq 0.$$

But for any Y with $|Y| = 1$ we have that

$$|Y|^2 \,\mathrm{Ric}(X, X) - R(X, Y, Y, X) \geq \mathrm{Ric}(X, X) - R(e_0, e_n, e_n, e_0).$$

This proves the other direction as well. The positivity case is the same. $\square$

A restatement of the $\mathrm{CQB}^{\mathcal{R}}{}_1$ part of Theorem 3.2 is

Theorem 5.4. *Assume that (M, g) be a compact Riemannian manifold with nonnegative $(n-2)$-Ricci curvature. Assume that the universal cover $\tilde{M}$ does not have a flat de Rham factor. Then M admits a metric with positive Ricci. In particular its fundamental group is finite. In fact the flow evolves the metric g into a metric $g(t)_{t \in (0, \epsilon)}$ with positive Ricci curvature for some ϵ.*

It is an interesting to investigate if a result similar to the above holds for k-nonnegative curvature with $k < n-2$. Namely if $g(t)$ has positive $k+1$-curvature assuming $g(0)$ has k-nonnegative curvature and $(M, g(t))$ does not split locally.

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