

# Poincaré-Lelong equation via the Hodge Laplace heat equation

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ABSTRACT

In this paper, we develop a method of solving the Poincaré-Lelong equation, mainly via the study of the large time asymptotics of a global solution to the Hodge-Laplace heat equation on  $(1, 1)$ -forms. The method is effective in proving an optimal result when  $M$  has nonnegative bisectional curvature. It also provides an alternate proof of a recent gap theorem of the first author.

## 1. Introduction

Solving the Poincaré-Lelong equation amounts to, for a given real  $(1, 1)$ -form  $\rho$ , finding a smooth function  $u$  such that  $\sqrt{-1}\partial\bar{\partial}u = \rho$ . Motivated by geometric considerations, on a complete noncompact Kähler manifold  $(M, g)$ , this was first studied by Mok et al. [7] under some restrictive conditions including a point-wise quadratic decay on  $\|\rho\|$ , nonnegative bisectional curvature and maximum volume growth on  $M$ . There have been many publications since then (e.g. [9], [14], [2]). Finally in [15], the following result was proved.

**THEOREM 1.1.** *Let  $M^n$  (with  $m = 2n$  being the real dimension) be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Let  $\rho$  be a real  $d$ -closed  $(1, 1)$ -form. Suppose that*

$$\int_0^\infty \left( \int_{B_o(s)} \|\rho\|(y) d\mu(y) \right) ds < \infty, \quad (1.1)$$

and

$$\liminf_{r \rightarrow \infty} \left[ \exp(-\alpha r^2) \cdot \int_{B_o(r)} \|\rho\|^2(y) d\mu(y) \right] < \infty \quad (1.2)$$

for some  $\alpha > 0$ . Then there is a solution  $u$  of the Poincaré-Lelong equation

$$\sqrt{-1}\partial\bar{\partial}u = \rho.$$

Moreover, for any  $0 < \epsilon < 1$ ,  $u$  satisfies

$$\begin{aligned} \alpha_1 r \int_{2r}^\infty k(s) ds + \beta_1 \int_0^{2r} sk(s) ds &\geq u(x) \\ &\geq \beta_3 \int_0^{2r} sk(s) ds - \alpha_2 r \int_{2r}^\infty k(s) ds - \beta_2 \int_0^{\epsilon r} sk(x, s) ds \end{aligned} \quad (1.3)$$

for some positive constants  $\alpha_1(m)$ ,  $\alpha_2(m, \epsilon)$  and  $\beta_i(m)$ ,  $1 \leq i \leq 3$ , where  $r = r(x)$ . Here  $k(x, s) = \int_{B_x(s)} \|\rho\|$  and  $k(s) = k(o, s)$ , where  $o \in M$  is fixed.

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Due to the technical nature of the assumption (1.2), which arises from the parabolic method employed in [15], and is related to the uniqueness of the heat equation, it is desirable to be able to remove it. The purpose of this paper is to prove:

**THEOREM 1.2.** *Let  $(M^n, g)$  be a complete noncompact Kähler manifold (of complex dimension  $n$ ) with nonnegative Ricci curvature and nonnegative quadratic orthogonal bisectional curvature. Suppose that  $\rho$  is a smooth  $d$ -closed real  $(1, 1)$ -form on  $M$  and let  $f = \|\rho\|$  be the norm of  $\rho$ . Suppose that*

$$\int_0^\infty k_f(r) dr < \infty. \tag{1.4}$$

*Then there is a smooth function  $u$  so that  $\rho = \sqrt{-1}\partial\bar{\partial}u$ . Moreover, for any  $0 < \epsilon < 1$ , the estimate (1.3) holds.*

Note that in the theorem we only require  $(M^n, g)$  has nonnegative Ricci curvature and nonnegative quadratic orthogonal bisectional curvature (see Section 3 for the definition), which is weaker than the nonnegativity of the bisectional curvature (see also Section 7 for the discussion on the relations between various curvature conditions). The current result is also more general than those in a previous version of our work and the proof is less involved.

The solution space to a Poincaré-Lelong equation clearly is an affine space consisting  $u_s + u_h$  with  $u_s$  being a special solution and  $u_h$  being any element of the linear space of the pluriharmonic functions. The estimate (1.3) selects the minimum one among them. In the view that on Kähler manifolds with nonnegative Ricci curvature, the sublinear growth is the optimal necessary condition to imply the constancy of a pluriharmonic function, the assumption (1.1) is almost the optimal condition which one can expect to ensure that estimate (1.3) selects the unique (up to a constant) solution.

The use of the Hodge-Laplace heat equation here is motivated by that of [12]. On the other hand here we establish the closedness of the global  $(1, 1)$ -form solution to the Hodge-Laplace heat equation, and in Section 5 obtain an alternate proof to the gap theorem in [12] without appealing to the relative monotonicity.

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## 2. A general method of solving the Poincaré-Lelong equation

Let  $(M^n, g)$  be a complete noncompact Kähler manifold with complex dimension  $n$  with Kähler form. Let  $\rho$  be a real  $(1, 1)$  form. If in a local coordinate  $\rho = \sqrt{-1}\rho_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}$  we denote the trace of  $\rho$ ,  $\sum_{i,j=1}^n g^{i\bar{j}}\rho_{i\bar{j}}$  by  $\text{tr}(\rho)$  and is equal to  $\Lambda\rho$ , where  $\Lambda$  is the adjoint of  $L$  with  $L\sigma = \omega \wedge \sigma$ .

Let  $\Delta = -\Delta_d = -(d\delta + \delta d)$  be the Hodge Laplacian for forms. On Kähler manifolds it is well known that  $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$ .

In this section we shall prove the result below, which reduces the solving of the Poincaré-Lelong equation into the study of two parabolic equations, namely the heat equation and Hodge-Laplace heat equation, and obtaining relevant estimates.

THEOREM 2.1. *Suppose  $M$  has nonnegative Ricci curvature. Suppose further that the following are true.*

(a) *There is an  $(1, 1)$ -form  $\eta(x, t)$  satisfying*

$$\begin{cases} \eta_t(x, t) - \Delta\eta(x, t) = 0, & \text{in } M \times [0, \infty); \\ \eta(x, 0) = \rho(x), & x \in M, \end{cases} \quad (2.1)$$

*such that  $\eta(x, t)$  is closed for all  $t$ , and for some  $p > 0$*

$$\lim_{R \rightarrow \infty} \frac{1}{R^2 V_o(R)} \int_0^T \int_{B_o(R)} \|\eta\|^p(x, t) d\mu(x) dt = 0 \quad (2.2)$$

*for all  $T > 0$ . Moreover,  $\lim_{t \rightarrow \infty} \eta(x, t) = 0$ .*

(b) *There is a function  $u(x)$  solving  $\Delta u(x) = \text{tr}(\rho)(x)$ , and a solution  $v(x, t)$  of*

$$\begin{cases} v_t(x, t) - \Delta v(x, t) = 0, & \text{in } M \times [0, \infty); \\ v(x, 0) = 2u(x), & x \in M, \end{cases} \quad (2.3)$$

*such that for the same  $p > 0$ ,*

$$\lim_{R \rightarrow \infty} \frac{1}{R^2 V_o(R)} \int_0^T \int_{B_o(R)} |v|^p(x, t) + |u(x)|^p d\mu(x) dt = 0 \quad (2.4)$$

*for all  $T > 0$  and  $\lim_{t \rightarrow \infty} \partial\bar{\partial}v(x, t) = 0$ .*

Then  $2\sqrt{-1}\partial\bar{\partial}u = \rho$ .

Before we prove the theorem, let us first recall the following consequence of Kähler identities.

LEMMA 2.1. *For  $(1, 1)$ -form  $\eta$ , we have*

$$\partial\bar{\partial}\Lambda\eta = \sqrt{-1}\Delta_{\bar{\partial}}\eta - \bar{\partial}\Lambda\partial\eta + \partial\Lambda\bar{\partial}\eta - \Lambda\partial\bar{\partial}\eta. \quad (2.5)$$

*Hence if  $\Lambda\partial\eta = \Lambda\bar{\partial}\eta = \Lambda\partial\bar{\partial}\eta = 0$ , then*

$$\partial\bar{\partial}\Lambda\eta = \sqrt{-1}\Delta_{\bar{\partial}}\eta = \sqrt{-1}\Delta_{\partial}\eta. \quad (2.6)$$

*In particular, if  $\eta$  is  $d$ -closed then (2.6) is true.*

*Proof.* Recall the Kähler identities:  $\partial\Lambda - \Lambda\partial = -\sqrt{-1}\bar{\partial}^*$ ,  $\bar{\partial}\Lambda - \Lambda\bar{\partial} = \sqrt{-1}\partial^*$ .  $\Delta_{\partial} = \Delta_{\bar{\partial}}$ , we have that

$$\begin{aligned} \partial\bar{\partial}\Lambda\eta &= \partial\Lambda\bar{\partial}\eta + \sqrt{-1}\partial\partial^*\eta \\ &= \sqrt{-1}\Delta_{\partial}\eta + \partial\Lambda\bar{\partial}\eta - \sqrt{-1}\partial^*\eta \\ &= \sqrt{-1}\Delta_{\partial}\eta + \partial\Lambda\bar{\partial}\eta - (\bar{\partial}\Lambda - \Lambda\bar{\partial})\partial\eta. \end{aligned}$$

From this (2.5) follows.  $\square$

We also need the following maximum principle for solutions of the heat equation.

LEMMA 2.2. *Suppose  $(M^m, g)$  is a complete noncompact Riemannian manifold with nonnegative Ricci curvature and  $u$  is a smooth nonnegative subsolution of the heat equation on  $M \times [0, T]$  such that there exists  $R_i \rightarrow \infty$  and  $p > 0$  such that*

$$\lim_{i \rightarrow \infty} \frac{1}{R_i^2 V_o(R_i)} \int_0^T \int_{B_o(R_i)} u^p(y, s) d\mu(y) ds = 0. \quad (2.7)$$

*Then  $u(x, t) \leq \sup_{y \in M} u(y, 0)$ . In particular, if  $u_1$  and  $u_2$  are two solutions of the heat equation such that  $|u_1|$  and  $|u_2|$  satisfy the decay conditions (2.7) and if  $u_1 = u_2$  at  $t = 0$ , then  $u_1 \equiv u_2$ .*

*Proof.* By [5, Theorem 1.2], for any  $\epsilon > 0$ , there is a constant  $C > 0$  independent of  $R$  such that

$$\sup_{B_o(\frac{1}{2}R) \times [0, T]} u^p \leq \frac{C}{R^2 V_o(R)} \int_0^T \int_{B_o(R)} u^p(y, s) d\mu(y) ds + (1 + \epsilon) \left( \sup_{y \in M} u(y, 0) \right)^p$$

if  $R^2 \geq 4T$ . Let  $R \rightarrow 0$  and then let  $\epsilon \rightarrow 0$ , the first assertion follows.

To prove the second assertion, apply the above argument to  $(|u_1 - u_2|^2 + \epsilon)^{\frac{1}{2}}$  for  $\epsilon > 0$  and let  $\epsilon \rightarrow 0$ .  $\square$

*Proof of Theorem 2.1.* Let  $\eta$  be as in (a). Let  $\phi = \text{tr}(\eta)$ , namely  $\Lambda(\eta)$ . Then  $\phi$  satisfies the heat equation in  $M \times [0, \infty)$  with initial value  $\text{tr}(\rho)$ . Let

$$w(x, t) = -2 \int_0^t \phi(x, s) ds.$$

Then

$$w_t - \Delta w = -2\text{tr}(\rho), \quad w(x, 0) = 0.$$

Hence  $\tilde{v}(x, t) = 2u(x) - w(x, t)$  satisfies

$$\tilde{v}_t - \Delta \tilde{v} = 0, \quad \tilde{v}(x, 0) = 2u(x).$$

By (2.2), (2.4) and Lemma 2.2, we conclude that  $v = 2u - w$ .

On the other hand, by Lemma 2.1 and the fact that  $\eta$  is closed, we have

$$\begin{aligned} \frac{d}{dt} (\eta + \sqrt{-1} \partial \bar{\partial} w) &= -2\Delta_{\bar{\partial}} \eta + \sqrt{-1} \partial \bar{\partial} w_t \\ &= -2\Delta_{\bar{\partial}} \eta - 2\sqrt{-1} \partial \bar{\partial} \Lambda \eta \\ &= 0. \end{aligned}$$

At the same time, at  $t = 0$ ,  $\eta + \sqrt{-1} \partial \bar{\partial} w(\cdot, t) = \rho$ . Hence this equation holds for all  $t$ . That is to say,

$$\eta + 2\sqrt{-1} \partial \bar{\partial} u - \sqrt{-1} \partial \bar{\partial} v = \rho.$$

Since  $\lim_{t \rightarrow \infty} \eta(x, t) = \lim_{t \rightarrow \infty} \sqrt{-1} \partial \bar{\partial} v(x, t) = 0$ , we have  $2\sqrt{-1} \partial \bar{\partial} u = \rho$ .  $\square$

*Remark 2.1.* From the proof it is easy to see that if in (a) we only assume that  $\Lambda \partial \eta = \Lambda \bar{\partial} \eta = \Lambda \partial \bar{\partial} \eta = 0$ , then the conclusion of the theorem is still true. Moreover from the proof we see that  $\eta$  is closed. In fact, one can check if  $\eta$  satisfies the equation  $\frac{\partial}{\partial t} \eta + \Delta \eta = 0$ , then

$$\Phi = \eta(t) + \int_0^t (\bar{\partial}^* \partial + \partial^* \bar{\partial}) \eta - \sqrt{-1} \int_0^t \Lambda \partial \bar{\partial} \eta.$$

is closed, provided that  $\rho(x, 0)$  is closed. Hence in particular,  $\Lambda \partial \eta = \Lambda \bar{\partial} \eta = \Lambda \partial \bar{\partial} \eta = 0$  implies that  $d\eta = 0$ .

Hence in order to solve the Poincaré-Lelong equation, it is sufficient to find  $\eta$  and  $u, v$  as in the theorem.

By the previous work [15, 14], under certain average growth condition on  $\|\rho\|$  we can find  $u$  and  $v$  satisfying (b) of Theorem 2.1. We now list this known result and a useful estimate below.

Let  $(M, g)$  be a complete noncompact Riemannian manifold and let  $o \in M$  be a fixed point. For a smooth function  $f$  on  $M$ , let

$$k_f(r) = \frac{1}{V_o(r)} \int_{B_o(r)} |f|. \quad (2.8)$$

where  $B_x(r)$  is the geodesic ball of radius  $r$  with center at  $x$  and  $V_x(r)$  is the volume of  $B_x(r)$ . First recall the following result from [15, Lemma 1.1]:

LEMMA 2.3. *Let  $(M, g)$  be a complete noncompact Riemannian manifold with nonnegative Ricci curvature and let  $o \in M$  be a fixed point. Let  $h \geq 0$  be a continuous function. Let  $H(x, y, t)$  be the heat kernel and let*

$$v(x, t) = \int_M H(x, y, t)h(y)d\mu(y).$$

Assume that  $v$  is defined on  $M \times [0, T]$  with

$$\liminf_{r \rightarrow \infty} \exp\left(-\frac{r^2}{20T}\right) \int_{B_o(r)} h = 0. \tag{2.9}$$

Then for any  $r^2 \geq t > 0$  and  $p \geq 1$

$$\frac{1}{V_o(r)} \int_{B_o(r)} v^p(x, t)dx \leq C(n, p) \left[ k_{hp}(4r) + t^{-p} \left( \int_{4r}^\infty s \exp\left(-\frac{s^2}{20t}\right) k_h(s)ds \right)^p \right].$$

*Proof.* Note the proof in [15] (page 467-468) can be carried over because only (2.9) is needed for the integration by parts. □

PROPOSITION 2.1. *Let  $(M, g)$  be a complete noncompact Riemannian manifold with nonnegative Ricci curvature and let  $o \in M$  be a fixed point. Let  $f$  be a smooth function on  $M$  such that*

$$\int_0^\infty k_f(r)dr < \infty.$$

Then we can find functions  $u$  and  $v$  with  $\Delta u = f$ ,  $v$  satisfying (2.3) such that (2.4) is true for  $p = 1$ , and  $\lim_{t \rightarrow \infty} \partial \bar{\partial} v(x, t) = 0$ . Moreover  $u$  satisfies (1.3). In fact,  $u$  and  $v$  are given by

$$u(x) = \int_M (G(o, y) - G(x, y)) f(y)d\mu(y), \text{ if } M \text{ is nonparabolic}$$

$$v(x, t) = \int_M H(x, y, t)u(y)d\mu(y).$$

Here  $G(x, y)$  is the minimal positive Green's function of  $M$  (if  $M$  is nonparabolic) and  $H(x, y, t)$  is the heat kernel of  $M$ .

*Proof.* First consider the case that  $M$  is nonparabolic. The existence of  $u$  and  $v$  given by the expressions in the proposition so that  $\lim_{t \rightarrow \infty} \partial \bar{\partial} v(x, t) = 0$  follows from [15, Lemma 6.1]. By the proof of [15, Lemma 6.1], we have

$$k_u(r) = o(r^2).$$

From this and Lemma 2.3, (2.4) is true for  $p = 1$  in this case.

In general, by considering  $\widetilde{M} = M \times \mathbb{R}^4$ , we can find  $\tilde{u}$  as above. By [2, p.458–460],  $\tilde{u}$  is independent of  $y \in \mathbb{R}^4$  for  $(x, y) \in M \times \mathbb{R}^4$ . Let  $u(x) = \tilde{u}(x, y)$ . Then  $\Delta u = f$  and  $k_u(r) = o(r^2)$ . The existence of  $v$  is as in the previous case. □

To apply Theorem 2.1 we also need a  $d$ -closed solution of the Hodge-Laplace heat equation with estimate (2.2). The existence of such solutions is the content of the next two sections.

### 3. Solution of the Hodge-Laplace heat equation: preliminary results

In this section we collect some basic results needed for the later discussions and construct a solution of the Hodge-Laplace heat equation (2.1) via suitable approximation.

First we need the existence of an exhaustion function on complete manifolds with non-negative Ricci curvature with the property that it has bounded complex Hessian if additionally  $(M^n, g)$  is a Kähler manifold with nonnegative quadratic orthogonal bisectional curvature.

Recall that a Kähler manifold  $(M^n, g)$  is said to have *nonnegative quadratic orthogonal bisectional curvature* if at any point and any unitary frame  $\{e_i\}$ ,

$$\sum_{i,j} R_{i\bar{i}j\bar{j}}(a_i - a_j)^2 \geq 0 \quad (3.1)$$

for all real numbers  $a_i$ . Let  $o \in M$  be a fixed point,  $r(o, x)$  be the distance function to  $o$ .

Let

$$\zeta(x, t) = \int_M H(x, y, t) r(o, y) d\mu(y)$$

be the positive solution of the heat equation with  $r(o, x)$  being the initial value. Here  $H(x, y, t)$  is the heat kernel.

**PROPOSITION 3.1.** (i) *Assume that  $(M, g)$  has nonnegative Ricci curvature, then for any  $t > 0$   $\zeta(x, t)$  is a smooth exhaustion function. Moreover  $|\nabla\zeta|(x, t) \leq 1$ .*

(ii) *Assume additionally that  $(M, g)$  has nonnegative quadratic orthogonal bisectional curvature, then for any  $t > 0$ , there exists  $C$  such that  $\|\partial\bar{\partial}\zeta\|(x, t) \leq C$ . Furthermore  $C$  can be chosen with  $C = \frac{1}{\sqrt{2t}}$ .*

*Proof.* The part (i) follows from Corollary 1.1 and Lemma 1.4 of [15]. For part (ii), note that the Bochner formula implies that

$$\left(\Delta - \frac{\partial}{\partial t}\right) |\nabla\zeta|^2 \geq 2\|\nabla^2\zeta\|^2. \quad (3.2)$$

As in the proof of Lemma 3.1 of [15], by multiplying a suitable cut-off function to (3.2) and performing the integration by parts we have that for any  $x \in M$ , and for any  $T, R > 0$ ,

$$\int_0^T \int_{B_x(R)} \|\nabla^2\zeta\|^2 \leq C \left( R^{-2} \int_0^T \int_{B_x(2R)} |\nabla\zeta|^2 + \int_{B_x(2R)} |\nabla r|^2 \right) \leq C(1 + R^{-2}T) \quad (3.3)$$

for some constant  $C$  independent of  $x$  and  $R$ . Now one can apply Theorem 1.1 of [5] to conclude the bound of  $\|\partial\bar{\partial}\zeta\|$ , since by the proof of Lemma 2.1 of [15] (see also [3], Lemma 5.1 of [16]) one only needs the nonnegativity of the quadratic orthogonal bisectional curvature to conclude that  $\|\partial\bar{\partial}\zeta\|(x, t)$  is a sub-solution to the heat equation (see also (3.8) below). More precisely under the curvature assumption of (ii) we have that

$$\left(\Delta - \frac{\partial}{\partial t}\right) \|\partial\bar{\partial}\zeta\|^2(x, t) \geq 2\|\nabla\partial\bar{\partial}\zeta\|^2(x, t). \quad (3.4)$$

Putting (3.3) and (3.4) together we have that

$$\left(\Delta - \frac{\partial}{\partial t}\right) (|\nabla\zeta|^2 + 2t\|\partial\bar{\partial}\zeta\|^2) \geq 0. \quad (3.5)$$

The estimate  $|\nabla\zeta|(x, t) \leq 1$  and (3.4) enable one to apply the maximum principle to  $|\nabla\zeta|^2 +$

$2t\|\partial\bar{\partial}\zeta\|^2$ , and conclude that

$$|\nabla\zeta|^2(x, t) + 2t\|\partial\bar{\partial}\zeta\|^2(x, t) \leq 1.$$

This gives the precise upper bound claimed in part (ii).  $\square$

The Bochner formulae we have applied above are the special cases of the general formulae described below. Let  $\eta$  be a  $(p, q)$ -form valued in a holomorphic Hermitian vector bundle  $E$  with local frame  $\{E_\alpha\}$  and locally  $\eta = \sum \eta^\alpha E_\alpha$ .

$$\eta^\alpha = \frac{1}{p!q!} \sum \eta_{I_p \bar{J}_q}^\alpha dz^{I_p} \wedge dz^{\bar{J}_q}.$$

Here  $I_p = (i_1, \dots, i_p)$ ,  $\bar{J}_q = (\bar{j}_1, \dots, \bar{j}_q)$  and  $dz^{I_p} = dz^{i_1} \wedge \dots \wedge dz^{i_p}$ ,  $dz^{\bar{J}_q} = dz^{\bar{j}_1} \wedge \dots \wedge dz^{\bar{j}_q}$ . The Kodaira-Bochner formulae include the following two identities:

$$\begin{aligned} (\Delta_{\bar{\partial}}\eta)_{I_p \bar{J}_q}^\alpha &= - \sum_{ij} g^{\bar{j}i} \nabla_i \nabla_{\bar{j}} \eta_{I_p \bar{J}_q}^\alpha + \sum_{\nu=1}^q \Omega_{\beta \bar{j}_\nu}^{\alpha \bar{l}} \eta_{I_p \bar{j}_1 \dots (\bar{l})_\nu \dots \bar{j}_q}^\beta \\ &\quad + \sum_{\nu=1}^q R_{\bar{j}_\nu}^{\bar{l}} \eta_{I_p \bar{j}_1 \dots (\bar{l}) \dots \bar{j}_q}^\alpha - \sum_{\mu=1}^p \sum_{\nu=1}^q R_{i_\mu \bar{j}_\nu}^{k \bar{l}} \eta_{i_1 \dots (k)_\mu \dots i_p \bar{j}_1 \dots (\bar{l})_\nu \dots \bar{j}_q}^\alpha \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} (\Delta_{\bar{\partial}}\eta)_{I_p \bar{J}_q}^\alpha &= - \sum_{ij} g^{i\bar{j}} \nabla_{\bar{j}} \nabla_i \eta_{I_p \bar{J}_q}^\alpha + \sum_{\nu=1}^q \Omega_{\beta \bar{j}_\nu}^{\alpha \bar{l}} \eta_{I_p \bar{j}_1 \dots (\bar{l})_\nu \dots \bar{j}_q}^\beta - \sum_{\beta} \Omega_{\beta}^{\alpha} \eta_{I_p \bar{J}_q}^\beta \\ &\quad + \sum_{\mu=1}^p R_{i_\mu}^k \eta_{i_1 \dots (k)_\mu \dots i_p \bar{J}_q}^\alpha - \sum_{\mu=1}^p \sum_{\nu=1}^q R_{i_\mu \bar{j}_\nu}^{k \bar{l}} \eta_{i_1 \dots (k)_\mu \dots i_p \bar{j}_1 \dots (\bar{l})_\nu \dots \bar{j}_q}^\alpha, \end{aligned} \quad (3.7)$$

where  $(k)_\mu$  means that the index in the  $\mu$ -th position is replaced by  $k$ . Here

$$\Theta_\beta^\alpha = \frac{\sqrt{-1}}{2\pi} \sum \Omega_{\beta i \bar{j}}^\alpha dz^i \wedge d\bar{z}^j$$

is the curvature of  $E$  and  $\Omega_\beta^\alpha = \Lambda \Theta_\beta^\alpha$  is the mean curvature.

LEMMA 3.1. *Let  $(M, g)$  be a Kähler manifold.*

- (i) *Suppose that  $M$  has nonnegative quadratic bisectional curvature. Let  $\eta(x, t)$  be a  $(1, 1)$ -form satisfying*

$$\frac{\partial}{\partial t} \eta(x, t) - \Delta \eta(x, t) = \xi(x, t)$$

*with  $\xi(x, t)$  being another  $(1, 1)$ -form. Then*

$$\left( \frac{\partial}{\partial t} - \Delta \right) \|\eta\|^2(x, t) \leq -2\|\nabla\eta\|^2(x, t) + 2\langle \xi, \eta \rangle(x, t). \quad (3.8)$$

*In particular, for any  $\epsilon > 0$ ,*

$$\left( \frac{\partial}{\partial t} - \Delta \right) (\|\eta\|^2(x, t) + \epsilon)^{\frac{1}{2}} \leq \|\xi\|(x, t).$$

- (ii) *Suppose that  $M$  has nonnegative Ricci curvature. Let  $\sigma(x, t)$  be a  $(1, 0)$ -form satisfying*

$$\frac{\partial}{\partial t} \sigma(x, t) - \Delta \sigma(x, t) = 0.$$

*Then*

$$\left( \frac{\partial}{\partial t} - \Delta \right) \|\sigma\|^2(x, t) \leq -2\|\nabla\sigma\|^2(x, t). \quad (3.9)$$

In particular, for any  $\epsilon > 0$ ,

$$\left(\frac{\partial}{\partial t} - \Delta\right) (\|\sigma\|^2(x, t) + \epsilon)^{\frac{1}{2}} \leq 0.$$

*Proof.* Applying (3.6) and (3.7) to  $\eta(x, t), \sigma(x, t)$  and using the curvature assumption, the results follow.  $\square$

Our strategy of constructing a global solution  $\eta$  to the Hodge-Laplace heat equation is by two approximations. First we construct  $\rho^{(i)}$  by multiplying the initial  $d$ -closed  $(1, 1)$ -form  $\rho$  by suitable cut-off functions  $\phi^{(i)}$ , which we describe below.

Let  $\kappa(s)$  be a smooth cut-off function on  $\mathbb{R}$  such that  $\kappa(s) = 0$  for  $|s| > 1$ . For a sequence  $\{R_i\}$  with  $R_i \rightarrow \infty$ , let

$$\phi^{(i)}(x) = \kappa\left(\frac{\zeta(x, 1)}{R_i}\right)$$

where  $\zeta(z, t)$  is the exhaustion function constructed by solving the heat equation with distance function as the initial data. Proposition 3.1 implies that there exists an absolute constant  $C_1 > 0$  such that

$$|\nabla\phi^{(i)}|(x) + \|\partial\bar{\partial}\phi^{(i)}\| \leq \frac{C}{R_i}. \quad (3.10)$$

Let  $\rho^{(i)}(x) = \phi^{(i)}(x)\rho(x)$ . In general one can not expect that  $\rho^{(i)}$  is closed. But we have the following estimate:

$$\|\partial\rho^{(i)}\|(x) + \|\bar{\partial}\rho^{(i)}\|(x) + \|\partial\bar{\partial}\rho^{(i)}\|(x) \leq \frac{C_1}{R_i}\|\rho\|(x). \quad (3.11)$$

A solution  $\eta^{(i)}(x, t)$  of the Hodge-Laplace heat equation with initial data  $\rho^{(i)}(x)$  is constructed via the compact exhaustion detailed as follows. Let  $\{\Omega_\nu\}$  be a sequence of exhaustion domains. Let  $\Phi_\nu$  be the solution of the following initial boundary value problem:

$$\left\{ \begin{array}{l} \Phi_t - \Delta\Phi = 0, \text{ in } \Omega_\nu \times (0, \infty); \\ \lim_{t \rightarrow 0} \Phi(x, t) = \rho^{(i)}(x), \text{ } x \in \Omega_\nu; \\ \mathbf{n}\Phi = 0, \text{ on } \partial\Omega_\nu \times (0, \infty); \\ \mathbf{t}\Phi = 0, \text{ on } \partial\Omega_\nu \times (0, \infty). \end{array} \right. \quad (3.12)$$

Here  $\mathbf{n}\Phi$  and  $\mathbf{t}\Phi$  denote the normal and tangential parts of  $\Phi$  (see [8] for the basic definitions and related properties). The solvability of (3.12) is classical. See for example [1], [8], [4]. For  $\nu$  large, the solution  $\Phi_\nu$  is smooth since the initial and boundary values are compatible. Let

$$v^{(i)}(x, t) \doteq \int_M H(x, y, t) \|\rho^{(i)}\|(y) d\mu(y) \text{ and } v(x, t) \doteq \int_M H(x, y, t) \|\rho\|(y) d\mu(y). \quad (3.13)$$

Clearly  $v^{(i)}(x, t) \leq v(x, t)$ , provided  $v$  is defined. The Bochner formula (3.8) and the maximum principle imply that

$$\|\Phi_\nu\|(x, t) \leq v^{(i)}(x, t).$$

Thus by the Schauder's estimate, after possibly passing to a subsequence,  $\{\Phi_\nu\}$  converges to a limit solution  $\eta^{(i)}(x, t)$ , which is a  $(1, 1)$ -form with the initial value  $\rho^{(i)}(x)$ . Note that if  $\rho$  is real, then  $\eta^{(i)}$  is real by uniqueness. To summarize, we have:

**LEMMA 3.2.** *Let  $\eta^{(i)}$  be as above. Suppose that  $v(x, t)$  in (3.13) is well-defined. Then, after possibly passing to a subsequence,  $\{\eta^{(i)}\}$  converges to a  $(1, 1)$ -form  $\eta$  satisfying the Hodge-Laplacian heat equation on  $M \times [0, \infty)$  with initial value  $\rho$ . Moreover,*

$$\|\eta^{(i)}\|(x, t) \leq v^{(i)}(x, t) \text{ and } \|\eta\|(x, t) \leq v(x, t). \quad (3.14)$$



In particular for each  $i$ ,  $\|\eta^{(i)}\|$  is uniformly bounded.

Next we will prove that under certain conditions,  $\eta$  satisfies the conditions in Theorem 2.1(a).

#### 4. Global solutions of the Hodge-Laplace heat equation

In this section we prove the following result on the global solutions of the Hodge-Laplace heat equation.

**THEOREM 4.1.** *Let  $(M, g)$  be a complete Kähler manifold with nonnegative quadratic orthogonal bisectional curvature and with nonnegative Ricci curvature. Assume that  $\rho$  is a  $d$ -closed  $(1, 1)$ -form such that  $f = \|\rho\|$  satisfies*

$$\limsup_{R \rightarrow \infty} \frac{k_f(R)}{R^2} = 0. \quad (4.1)$$

Then there exists a solution of

$$\begin{cases} \eta_t - \Delta \eta = 0, & \text{in } M \times [0, \infty); \\ \eta(x, 0) = \rho(x), & x \in M. \end{cases} \quad (4.2)$$

such that  $\eta$  is a closed  $(1, 1)$ -form. Furthermore we have that

(a) for any  $T > 0$ ,

$$\lim_{R \rightarrow \infty} \frac{1}{R^2 V_o(R)} \int_0^T \int_{B_o(R)} \|\eta\|(x, t) d\mu(x) dt = 0; \quad (4.3)$$

(b)  $\lim_{t \rightarrow \infty} \eta(x, t) = 0$  for all  $x \in M$  provided that  $\lim_{R \rightarrow \infty} k_f(R) \rightarrow 0$ .

*Proof.* By the decay assumption on  $\|\rho\|$ ,

$$v(x, t) = \int_M H(x, y, t) \|\rho\|(y) d\mu(y)$$

is well-defined by [15, Lemma 2.1]. Let  $\{\eta^{(i)}(x, t)\}$  and  $\eta$  be the solutions of the Hodge-Laplace heat equation obtained in Lemma 3.2, which are real  $(1, 1)$ -form. Then  $\|\eta\|(x, t) \leq v(x, t)$ . Now by (4.1) we have that  $\frac{k_f(4R)}{R^2} \rightarrow 0$  as  $R \rightarrow \infty$ . By Lemma 2.3 and the fact that  $\|\eta\|(x, t) \leq v(x, t)$ , we conclude that (4.3) is true for any  $T > 0$ .

By [5, Theorem 1.1], if additionally assume that  $\lim_{R \rightarrow \infty} k_f(R) = 0$ , we have that  $\lim_{t \rightarrow \infty} v(x, t) = 0$ . Hence  $\lim_{t \rightarrow \infty} \eta(x, t) = 0$ .

It remains to prove  $d\eta = 0$ . By Remark 2.1, it is sufficient to prove that  $\Lambda \partial \eta = \Lambda \bar{\partial} \eta = \Lambda \partial \bar{\partial} \eta = 0$ . These will be established via lemmas below. With them we complete the proof of the theorem.  $\square$

**LEMMA 4.1.** *Let  $\eta^{(i)}$  and  $\eta$  as in the proof of Theorem 4.1. Let  $\sigma_1^{(i)} = \Lambda \partial \eta$  and  $\sigma_2^{(i)} = \Lambda \bar{\partial} \eta^{(i)}$ . Then*

$$\|\sigma_j^{(i)}\|(x, t) \leq C \int_M H(x, y, t) \|\rho_i\|(y) d\mu(y) \quad (4.4)$$

for  $j = 1, 2$ . In particular, for each  $i$ ,  $\|\sigma_j^{(i)}\|$  are uniformly bounded in  $M \times [0, \infty)$ . Moreover,  $\Lambda \partial \eta = \Lambda \bar{\partial} \eta = 0$ .

*Proof.* By Lemma 3.2, for each  $i$ ,  $\|\eta^{(i)}\|$  is uniformly bounded on  $M \times [0, \infty)$ . By Lemma 3.1

$$\left( \frac{\partial}{\partial t} - \Delta \right) \|\eta^{(i)}\|^2 \leq -2 \|\nabla \eta^{(i)}\|^2.$$

As in the proof of Proposition 3.1, for all there is  $C > 0$ , such that for  $T > 0$ ,  $r > 0$ ,

$$\int_0^T \int_{B_o(r)} \|\nabla \eta^{(i)}\|^2 \leq C(1 + r^{-2}T).$$

Hence

$$\int_0^T \int_{B_o(r)} \|\sigma_1^{(i)}\|^2 \leq C(1 + r^{-2}T).$$

On the other hand, by Lemma 3.1 again, for any  $\epsilon > 0$ ,

$$\left( \frac{\partial}{\partial t} - \Delta \right) \left( \|\sigma_1^{(i)}\|^2 + \epsilon \right)^{\frac{1}{2}} \leq 0.$$

Hence one can apply the maximum principle to conclude that

$$\left( \|\sigma_1^{(i)}\|^2(x, t) + \epsilon \right)^{\frac{1}{2}} \leq \int_M H(x, y, t) \left( \|\Lambda \partial \rho^{(i)}\|(y) + \epsilon \right) d\mu(y).$$

Let  $\epsilon \rightarrow 0$ , we have

$$\|\sigma_1^{(i)}\|(x, t) \int_M H(x, y, t) \|\Lambda \partial \rho^{(i)}\|(y) d\mu(y).$$

Similarly, one can prove

$$\|\sigma_2^{(i)}\|(x, t) \int_M H(x, y, t) \|\Lambda \bar{\partial} \rho^{(i)}\|(y) d\mu(y).$$

The last assertion follows from (3.11) and the fact that a subsequence of  $\eta^{(i)}$  converge locally uniformly to  $\eta$ .  $\square$

Next we want to prove that  $\Lambda \partial \bar{\partial} \eta(x, t) = 0$ . For that let

$$\theta = \int_0^t \Lambda \partial \bar{\partial} \eta(x, \tau) d\tau, \quad \text{and correspondingly} \quad \theta^{(i)}(x, t) = \int_0^t \Lambda \partial \bar{\partial} \eta^{(i)}(x, \tau) d\tau.$$

For that we need more lemmas.

LEMMA 4.2. *For any  $T > 0$ , there exists  $C_i > 0$  such that*

$$\int_0^T \int_{B_o(R)} \|\theta^{(i)}\|^2(x, t) d\mu(x) dt \leq C_i \tag{4.5}$$

for all  $R$ .

*Proof.* Let  $w^{(i)}(x, t) = \Lambda \eta^{(i)}(x, t)$ . By Lemma 3.2,  $w^{(i)}(x, t)$  is uniformly bounded. It also satisfies the heat equation. Using the differential equation/inequality

$$\begin{aligned} \left( \Delta - \frac{\partial}{\partial t} \right) \left( w^{(i)} \right)^2(x, t) &= 2|\nabla w^{(i)}|^2(x, t), \\ \left( \Delta - \frac{\partial}{\partial t} \right) |\nabla w^{(i)}|^2(x, t) &\geq 2\|\nabla^2 w^{(i)}\|^2(x, t) \end{aligned}$$

integration by parts as in the proof of Proposition 3.1, for  $R^2 \geq T$

$$\int_0^T \int_{B_o(R)} \left( |\nabla w^{(i)}|^2(x, t) + \|\nabla^2 w^{(i)}\|^2(x, t) \right) d\mu(x) dt \leq C_i. \tag{4.6}$$

for some  $C_i > 0$  independent of  $R$ . Let  $\sigma_1^{(i)}(x, t) = \Lambda \bar{\partial} \eta^{(i)}$  and  $\sigma_2^{(i)}(x, t) = \Lambda \partial \eta^{(i)}$ . By Lemma 2.1,

$$\theta^{(i)}(x, t) = \int_0^t \partial \bar{\partial} w^{(i)}(x, \tau) d\tau + \int_0^t \sqrt{-1} \Delta \bar{\partial} \eta^{(i)}(x, \tau) + \partial \sigma_1^{(i)}(x, \tau) - \bar{\partial} \sigma_2^{(i)}(x, \tau) d\tau. \tag{4.7}$$

Note

$$\left\| \int_0^t \sqrt{-1} \Delta \bar{\partial} \eta^{(i)}(x, \tau) d\tau \right\| = \left\| \int_0^t \sqrt{-1} \eta_\tau^{(i)} d\tau \right\| \leq v^{(i)}(x, t) + \|\rho^{(i)}\|(x). \quad (4.8)$$

On the other hand by Lemma 3.1

$$\left( \Delta - \frac{\partial}{\partial t} \right) \|\sigma_j^{(i)}\|^2(x, t) \geq 2 \|\nabla \sigma_j^{(i)}\|^2(x, t), \text{ for } j = 1, 2, \quad (4.9)$$

and  $\|\sigma_j^{(i)}\|^2$  are uniformly bounded by Lemma 4.1. Integrating by parts after multiplying (4.9) by a cut-off function as before we have

$$\int_0^T \int_{B_o(R)} \|\nabla \sigma_1^{(i)}\|^2(x, t) + \|\nabla \sigma_2^{(i)}\|^2(x, t) d\mu(x) dt \leq C_i. \quad (4.10)$$

The result follows from (4.6), (4.7), (4.8) and (4.10).  $\square$

Lemma 4.2 allows us to apply the maximum principle to  $\|\theta^{(i)}\|(x, t)$  once we can establish that  $\|\theta^{(i)}\|(x, t)$  is a subsolution of a heat type equation. To this end we have the following lemma.

LEMMA 4.3. For any  $\epsilon > 0$ ,

$$\left( \Delta - \frac{\partial}{\partial t} \right) \left( \|\theta^{(i)}\|(x, t) + \epsilon \right)^{\frac{1}{2}} \geq -\|\Lambda \partial \bar{\partial} \rho^{(i)}\|(x). \quad (4.11)$$

*Proof.* First direct calculation shows that

$$\begin{aligned} \frac{\partial}{\partial t} \|\theta^{(i)}\|(x, t) - \Delta \|\theta^{(i)}\|(x, t) &= \Lambda \partial \bar{\partial} \eta^{(i)}(x, t) - \int_0^t \Delta \Lambda \partial \bar{\partial} \eta^{(i)}(x, \tau) d\tau \\ &= \Lambda \partial \bar{\partial} \eta^{(i)}(x, t) - \int_0^t \Lambda \partial \bar{\partial} \Delta \eta^{(i)}(x, \tau) d\tau \\ &= \Lambda \partial \bar{\partial} \eta^{(i)}(x, t) - \int_0^t \Lambda \partial \bar{\partial} (\eta^{(i)})_t(x, \tau) d\tau \\ &= \Lambda \partial \bar{\partial} \rho^{(i)}(x). \end{aligned}$$

The claimed inequality follows from (3.8) and Lemma 3.1.  $\square$

LEMMA 4.4.  $\theta(x, t) = 0$  for all  $x, t$ .

*Proof.* Let

$$z^{(i)}(x, t) = \int_0^t \int_M H(x, y, \tau) \|\Lambda \partial \bar{\partial} \rho^{(i)}\|(y) d\mu(y) d\tau.$$

It is easy to see that

$$\left( \Delta - \frac{\partial}{\partial t} \right) z^{(i)}(x, t) = -\|\Lambda \partial \bar{\partial} \rho^{(i)}\|(x).$$

Note that  $z^{(i)}$  is bounded. By Lemmas 4.2, 4.3 and the maximum principle Lemma 2.2, we conclude that

$$\begin{aligned} \|\theta^{(i)}\|(x, t) &\leq z^{(i)}(x, t) \\ &\leq \frac{C_1}{R_i} \int_0^t \int_M H(x, y, \tau) \|\rho\|(y) d\mu(y) d\tau \\ &\leq \frac{C_1}{R_i} \int_0^t v(x, \tau) d\tau \\ &\rightarrow 0 \end{aligned}$$

as  $i \rightarrow \infty$ . Here we have used (3.11).  $\square$

By the definition of  $\theta$ , we conclude that  $\Lambda\partial\bar{\partial}\eta = 0$ . The proof of Theorem 4.1 is completed.

### 5. An alternate proof of the gap theorem

In this section we shall give an alternate proof of the gap theorem of [12]. The proof here avoids the use of the *relative monotonicity*. It uses a Li-Yau-Hamilton type estimate in [13] together with Lemma 4.1. The proof also avoids to solve the Poincaré-Lelong equation.

**THEOREM 5.1.** *Let  $(M, g)$  be a complete Kähler manifold with nonnegative bisectional curvature. Assume that  $\rho \geq 0$  is a  $d$ -closed  $(1, 1)$ -form. Suppose that*

$$\int_0^r s \int_{B_o(s)} \|\rho\|(y) d\mu(y) ds = o(\log r) \quad (5.1)$$

for some  $o \in M$ . Then  $\rho \equiv 0$ .

*Proof.* By the assumption (5.1), we can apply Theorem 4.1. Let  $\eta^{(i)}$  and  $\eta$  as in the proof of Theorem 4.1 with  $\|\eta^{(i)}\|$  is bounded and

$$\|\eta\|(x, t) \leq \int_M H(x, y, t) \|\rho\|(x) d\mu.$$

Since  $\rho_i$  is nonnegative, by [15, Theorem 2.1], we can conclude that  $\eta^{(i)}$  is nonnegative. Hence  $\eta$  is nonnegative. By [13, Theorem 1.1], we have

$$w_t + \frac{1}{2} (\bar{\partial}^* \Lambda \bar{\partial} + \partial^* \Lambda \partial) \eta + \frac{w}{t} \geq 0 \quad (5.2)$$

where  $w = \Lambda\eta$ . By Lemma 4.1,  $\Lambda\bar{\partial}\eta = \Lambda\partial\eta = 0$ . Let  $u(x)$  be the solution of  $\Delta u = \Lambda\rho$  obtained in [14], which satisfies the estimates (1.3). Let  $v(x, t)$  be the solution of the heat equation with initial data  $2u$ . By the Proposition 2.1 and the estimates of  $u$  and  $\eta$ , we conclude that

$$v(x, t) = 2u(x) + \int_0^t w(x, \tau) d\tau.$$

By Proposition 2.1, through the previous work [14], for any  $x_0 \in M$  we have that

$$u(x) = o(\log(r(x_0, x)))$$

as  $x \rightarrow \infty$ , where  $r(x_0, x)$  denotes the distance function as before. By the moment type estimate (cf. Theorem 3.1 of [10]) we have that  $v(x_0, t) = o(\log t)$  as  $t \rightarrow \infty$ . This fact, together with (5.2) implies that  $\Lambda\rho(x_0, t) = 0$  for any  $t > 0$ . Otherwise assume that  $\Lambda\rho(x_0, t_0) = \delta > 0$  for some  $t_0 > 0$ . Note that we have  $v_t = 2\Lambda\eta = 2w$ . Then by (5.2) we have that for all  $t \geq t_0$ ,

$$v_t(x, t) \geq \frac{2t_0\delta}{t},$$

which in particular implies that

$$v(x_0, t) \geq 2t_0\delta \log\left(\frac{t}{t_0}\right) - v(x_0, t_0).$$

This contradicts  $v(x, t) = o(\log(t))$ . The contradiction implies that  $\Lambda\eta(x_0, t) = 0$  for any  $t \geq 0$ . Hence we have that  $\Lambda\rho(x) \equiv 0$  which implies the claim  $\rho \equiv 0$  by the nonnegativity of  $\rho$ .

□

## 6. Solution of Poincaré-Lelong equation

In this section we shall prove the result generalizing [15, Theorem 6.1]. With the notations of previous sections we can state the main theorem.

**THEOREM 6.1.** *Let  $(M^n, g)$  be a complete noncompact Kähler manifold (of complex dimension  $n$ ) with nonnegative Ricci curvature and nonnegative quadratic orthogonal bisectional curvature. Suppose that  $\rho$  is a smooth  $d$ -closed real  $(1, 1)$ -form on  $M$  and let  $f = \|\rho\|$  be the norm of  $\rho$ . Suppose that*

$$\int_0^\infty k_f(r) dr < \infty. \quad (6.1)$$

*Then there is a smooth function  $u$  so that  $\rho = \sqrt{-1} \partial \bar{\partial} u$ . Moreover, for any  $0 < \epsilon < 1$ , the estimate (1.3) holds.*

*Proof.* Observe that the volume comparison implies  $k_f(r) \leq 2^{2n} k_f(s)$  for all  $s \in (r, 2r)$ . Thus (6.1) implies that  $\lim_{r \rightarrow \infty} k_f(r) r = 0$ . Thus Theorem 6.1 follows from Theorems 2.1, 4.1 and Proposition 2.1. □

Finally we make some comments on the relation of conditions of *nonnegative bisectional curvature* (NB), *nonnegative orthogonal bisectional curvature* (NOB), *nonnegative quadratic orthogonal bisectional curvature* (NQOB) (3.1), and the following curvature condition on  $(2, 1)$ -forms  $\sigma$ :

$$-\langle E(\sigma), \sigma \rangle \geq -a^2 |\sigma|^2 \quad (6.2)$$

for some  $a > 0$ , where

$$(E(\sigma))_{ij\bar{k}} = -R_i^l \sigma_{l\bar{j}\bar{k}} - R_j^l \sigma_{i\bar{l}\bar{k}} - R_{\bar{k}}^{\bar{l}} \sigma_{ij\bar{l}} + 2R_i^l R_{\bar{k}}^{\bar{m}} \sigma_{lj\bar{m}} + 2R_j^l R_{\bar{k}}^{\bar{m}} \sigma_{il\bar{m}}. \quad (6.3)$$

For condition (6.2) with  $a = 0$ , we abbreviate as (NCF) (nonnegativity associated with certain 3-forms), as well as an invariant representation of them. This condition on  $(2, 1)$ -forms arises from the Bochner formula on  $(2, 1)$ -forms, which is related to a previous method of constructing global  $d$ -closed solutions to Hodge-Laplace heat equation. Algebraically (NB) is stronger than (NOB), which in turn is stronger than (NQOB). We introduce some notations for convenience. First the curvature operator of a Kähler manifold can be viewed as bilinear form on  $\mathfrak{gl}(n, \mathbb{C})$  (which can be identified with  $\wedge^{1,1}(\mathbb{C}^n)$  via the metric) in the sense that for any  $X \wedge \bar{Y}$  and  $Z \wedge \bar{W}$

$$\langle \text{Rm}(X \wedge \bar{Y}), \overline{Z \wedge \bar{W}} \rangle \doteq \text{Rm}(X \wedge \bar{Y}, \overline{Z \wedge \bar{W}}) = R_{X\bar{Y}W\bar{Z}}.$$

Hence for any  $\Omega = (\Omega^{i\bar{j}})$ , it is easy to check that  $\langle \text{Rm}(\Omega), \bar{\Omega} \rangle = R_{i\bar{j}k\bar{l}} \Omega^{i\bar{j}} \bar{\Omega}^{k\bar{l}} \in \mathbb{R}$ . Hence one can identify (cf. [16]) the condition (NB) as

$$\{\text{Rm} \mid \langle \text{Rm}(\Omega), \bar{\Omega} \rangle \geq 0, \text{ for any } \Omega, \text{rank}(\Omega) = 1\}. \quad (6.4)$$

Similarly, condition (NOB) is equivalent to

$$\{\text{Rm} \mid \langle \text{Rm}(\Omega), \bar{\Omega} \rangle \geq 0, \text{ for any } \Omega, \text{rank}(\Omega) = 1, \Omega^2 = 0\}. \quad (6.5)$$

For the other two conditions we recall two operators from the study of the Ricci flow. The first one is the  $\bar{\Delta}$  operator on  $A, B$ , any two Hermitian symmetric transformations on  $T'M$ ,

defined by

$$\begin{aligned} (A\bar{\wedge}B)_{i\bar{j}k\bar{l}} &= A_{i\bar{j}}B_{k\bar{l}} + B_{i\bar{j}}A_{k\bar{l}} + A_{i\bar{l}}B_{k\bar{j}} + B_{i\bar{l}}A_{k\bar{j}} \\ &= 2\langle(A \wedge \bar{B} + B \wedge \bar{A})(e_i \wedge e_{\bar{j}}), \overline{e_l \wedge e_{\bar{k}}}\rangle \\ &\quad + 2\langle(A \wedge \bar{B} + B \wedge \bar{A})(e_k \wedge e_{\bar{j}}), \overline{e_l \wedge e_{\bar{i}}}\rangle. \end{aligned}$$

The resulting operator so defined is also a Kähler curvature operator which, in particular, satisfies the first Bianchi identity. Here  $(A \wedge B)(X \wedge Y) = \frac{1}{2}(A(X) \wedge B(Y) + B(X) \wedge A(Y))$  as defined in [16]. This operator is the one involved in the  $U(n)$ -invariant irreducible decomposition of the space of the Kähler curvature operators. Now the condition (NQOB) is equivalent to

$$\{\text{Rm} \mid \langle \text{Rm}, A^2 \bar{\wedge} \text{id} - A \bar{\wedge} A \rangle \geq 0, \text{ for all Hermitian symmetric } A\}. \quad (6.6)$$

For (NCF) it can be identified with that

$$\text{Ric} \wedge \text{id} \wedge \text{id} - 2 \text{Rm} \wedge \text{id} \geq 0 \quad (6.7)$$

on the space of  $(2, 1)$ -forms. Here for  $X \wedge Y \wedge Z$ ,  $\text{Rm} \wedge \text{id}(X \wedge Y \wedge Z) = \text{Rm}(X \wedge Y) \wedge Z - \text{Rm}(X \wedge Z) \wedge Y + X \wedge \text{Rm}(Y \wedge Z)$ ,  $\text{Ric} \wedge \text{id} \wedge \text{id}$  is defined as  $(\text{Ric} \wedge \text{id}) \wedge \text{id}$ .

It is known that (NB), and (NOB) are Ricci flow invariant conditions [16]. It would be interesting to find out about (NQOB) and (NCF). Our speculation is that condition (NCF) follows from (NQOB) and  $\text{Ric} \geq 0$ . If this speculation holds (whose validity can be easily checked when  $n = 2$ ) one can have an alternate proof of the result in Section 4.

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