THE FUNDAMENTAL GROUP, RATIONAL CONNECTEDNESS AND THE POSITIVITY OF KÄHLER MANIFOLDS

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Abstract. First a conjecture asserting that any compact Kähler manifold $N$ with $\text{Ric}^\perp > 0$ must be simply-connected is confirmed by adapting the comass of $(p,0)$-forms into a maximum principle via the viscosity consideration. Secondly the projectivity and the rational connectedness of a Kähler manifold of complex dimension $n$ under the condition $\text{Ric}_k > 0$ (for some $k \in \{1, \cdots , n\}$) is proved, generalizing the previous result of Campana, and Kollár-Miyaoka-Mori independently, for the Fano manifolds. Thirdly we show that under the assumption of Picard number one a manifold with $\text{Ric}^\perp > 0$ is Fano. Then via a new curvature notion motivated by $\text{Ric}^\perp$, the cohomology vanishing $H^q(N, T^\perp N) = \{0\}$ for any $1 \leq q \leq n$ (as well as a deformation rigidity result) for classical Kähler C-spaces with $b_2 = 1$ is proved, which generalizes the classical result of Calabi-Vesentini. This new curvature (which is quadratic in terms of linear maps from $T^\perp N$ to $T^\perp\perp N$) leads to a related notion of Ricci curvature, $\text{Ric}^\perp$. We also showed that a compact Kähler manifold with $\text{Ric}^\perp > 0$ is projective and simply-connected.

1. Introduction

A result of Kobayashi [15] asserts that a compact Kähler manifold with $\text{Ric} > 0$ must be simply-connected. Same conclusion was proved a bit earlier by Tsukamoto [27] for compact Kähler manifold with positive holomorphic sectional curvature. It was known via examples of Hitchin [13] (see also [1, 23]) that the two conditions are independent for complex dimension greater than one.

In [23], motivated by the Laplace comparison theorem and the holomorphic Hessian comparison theorem, orthogonal Ricci curvature $\text{Ric}^\perp$, which is defined by

$$\text{Ric}^\perp(X, \overline{X}) = \text{Ric}(X, \overline{X}) - R(X, \overline{X}, X, \overline{X})/|X|^2$$

for any type $(1,0)$ tangent vector $X$, was studied. For a compact Kähler manifold $N^n (n = \dim_{\mathbb{C}}(N))$, with $\text{Ric}^\perp > 0$ everywhere, it was shown in [23] that the manifold is always projective, has finite $\pi_1(N)$, and has vanishing Hodge numbers: $h^{p,0} = 0$ for $p = 1, 2, n - 1,$ and $n$. Further studies of compact Kähler manifolds with $\text{Ric}^\perp > 0$ were carried in a recent work [22]. Besides a complete classification for threefolds, a partial classification for fourfolds, a Frankel type result was proved for compact Kähler manifolds with $\text{Ric}^\perp > 0$ in [22]. Many examples were constructed in [23, 22] illustrating that $\text{Ric}^\perp, H,$ and $\text{Ric}$ are completely independent except the trivial relation $\text{Ric}(X, \overline{X}) = \text{Ric}^\perp(X, \overline{X}) + H(X)/|X|^2$. Recall that the holomorphic sectional curvature is defined as $H(x) = R(X, \overline{X}, X, \overline{X})$. Despite these results, the following conjecture (cf. Conjecture 1.6, [23]) remains open except for $n = 2, 3, 4$. 

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Conjecture 1.1. Let $N^n$ $(n \geq 2)$ be a compact Kähler manifold with $\text{Ric}^> 0$ everywhere. Then for any $1 \leq p \leq n$, there is no non-trivial global holomorphic $p$-form, namely, the Hodge number $h^{p,0} = 0$. In particular, $N^n$ is simply-connected.

The “in particular” part, namely the simply-connectedness of compact Kähler manifolds, would follow from Hirzebruch’s Riemann-Roch formula [12] (noting that by Theorem 1.7 of [23] $N$ is algebraic) as follows: First the Euler characteristic number

$$\chi(O_N) = 1 - h^{1,0} + h^{2,0} - \cdots + (-1)^n h^{n,0}$$

where $O_N$ is the structure sheaf, satisfies that $\chi(O_{\tilde{N}}) = g \chi(O_N)$ by the Riemann-Roch-Hirzebruch formula, if $\tilde{N}$ is a finite $g$-sheets covering of $N$. On the other hand, the vanishing of all Hodge numbers $h^{p,0}$ for $1 \leq p \leq n$ (which is the main part of the conjecture) asserts that $\chi(O_N) = 1$ for both $N$ and $\tilde{N}$, if $\tilde{N}$ is compact and of $\text{Ric}^> 0$ (hence projective), which forces $g = 1$. The assertion that $N^n$ must be simply-connected (cf. [15]) follows from that the universal cover $\tilde{N}$ of $N$ is of $\text{Ric}^> \delta > 0$, hence is compact and projective by Theorem 3.2 of [23]. This argument was used in [15] proving the simply-connectedness of a Fano manifold. It was also used in [23] for the special case $n = 2, 3, 4$.

In this paper we prove the above conjecture for all $n \geq 2$. In fact we prove a stronger result which asserts the vanishing of $h^{p,0}$ under a weaker curvature condition related to $p$.

This condition was first introduced in Section 4 of [23], which we recall below.

Motivated by [24], for any $k$-subspace $\Sigma \subset T^*_x N$, we define

$$S_k^\perp(x, \Sigma) \triangleq k \int_{Z \in \Sigma, |Z| = 1} \text{Ric}^,(Z, \overline{Z}) \ d\theta(Z) \quad (1.1)$$

where $\int f(Z) \ d\theta(Z)$ denotes $\int_{\theta_1 = 0}^{2\pi} \int_{\theta_2 = 1}^{2\pi} \cdots \int_{\theta_{k-1} = 1}^{2\pi} f(Z) \ d\theta(Z)$. The \{ $S_k^\perp(x, \Sigma)$ \} interpolate between $\text{Ric}^,(X, \overline{X})$ and $\frac{n-1}{n} S(x)$ (see Lemma 4.1). We say $S_k^\perp(x) > 0$ if for any $k$-subspace $\Sigma \subset T^*_x M$, $S_k^\perp(x, \Sigma) > 0$. It is easy to see that $S_k^\perp > 0$ implies $S_k^\perp > 0$ for $k \geq l$. And it is not hard to prove that

$$S_k^\perp(x, \Sigma) = (\text{Ric}(E_1, \overline{E}_1) + \text{Ric}(E_2, \overline{E}_2) + \cdots + \text{Ric}(E_k, \overline{E}_k)) - \frac{2}{(k + 1)} S_k(x, \Sigma).$$

Here $S_k(x, \Sigma)$ is the $k$-scalar curvature defined in [24] (namely taking the average of holomorphic sectional curvature instead of $\text{Ric}^,$ over the unit sphere of $\Sigma$ in (1.1)). The collection of $k$-scalar curvatures \{ $S_k(x, \Sigma)$ \} for $k = 1, \cdots, n$, interpolates between the holomorphic sectional curvature $H(X)$ and the scalar curvature $S(x)$. The above equation in particular implies that $S_k^\perp(x) = \frac{n-1}{n} S(x)$. The relation (1.1) suggested a question: \textit{whether or not} $S_k^\perp(x) > 0$ \textit{implies} $h^{p,0} = 0$ \textit{for} $p \geq k$. The first theorem of this paper answers this question affirmatively, which implies Conjecture 1.1 since $\text{Ric}^> > 0$ implies $S_k^\perp > 0$ for any $1 \leq k \leq n$.

Theorem 1.2. Let $(N, g)$ be a compact Kähler manifolds such that $S_k^\perp(x) > 0$ for any $x \in N$. Then $h^{p,0} = 0$ for any $p \geq k$. In particular, if $\text{Ric}^> > 0$, $h^{p,0}(N) = 0$ for all $1 \leq p \leq n$, and $N$ is simply-connected.

The case $p = 2$ was proved in Section 4 of [23]. The proof here is motivated by an idea of [20] in proving a new Schwarz Lemma by the author. We recall that idea first before explaining the related idea here. Starting from the work of Ahlfors, the Schwarz Lemma concerns estimating the gradient of a holomorphic map $f$ between two Kähler (or Hermitian)
manifolds $(M^n, h)$ and $(N^n, g)$. For that it is instrumental to study the pull-back $(1, 1)$-form $f^*\omega_0$, where $\omega_0$ is the Kähler form of $(N, g)$. The traditional approach (before the work of [20]) is to compute the Laplacian of the trace of $f^*\omega_0$. But in [20], the author estimated the largest singular value of $df$, equivalently the biggest eigenvalue of $f^*\omega_0$, via the action of the $\partial\bar{\partial}$-operator acting on the maximum eigenvalue via a viscosity consideration. It allows the author to prove another natural generalization of Ahlfors’ result with a sharp estimate on the largest singular value in terms of the holomorphic sectional curvatures of both the domain and target manifolds. This estimate can be viewed as a complex version of Pogorelov’s estimate for solutions of the Monge-Ampère equation. To prove the vanishing of holomorphic $(p, 0)$-forms under the assumption of $\text{Ric}^+ > 0$, the action of $\partial\bar{\partial}$-operator on the comass of holomorphic $(p, 0)$-forms (cf. [9, 28]), through a viscosity consideration with the help of some basic properties of the comass from Whitney’s classic [28], holds the key. This new idea also allows an alternate proof of the main theorem in [24].

Combining this new idea with the work of [24] we study the rational connectedness of compact Kähler manifolds under the condition $\text{Ric}_k > 0$. The notion $\text{Ric}_k$ is a variation of Ricci curvature introduced in [20] to prove that any Kähler manifold with $\text{Ric}_k < 0$ uniformly must be k-hyperbolic, a concept generalizing the Kobayashi hyperbolicity (which amounts to 1-hyperbolic). Simply put $\text{Ric}_k$ is the Ricci curvature of the curvature operator $R$ of $(N, h)$ restricted to $k$-dimensional subspaces of the holomorphic tangent space $T^*_N N$. The condition $\text{Ric}_k > 0$ for $k = 1$ is equivalent to that the holomorphic sectional curvature $H > 0$. For $k = n$, $\text{Ric}_k$ is the Ricci curvature. By [13, 1] $\text{Ric}_k > 0$ is independent from $\text{Ric}_\ell > 0$ for $k \neq \ell$ (cf. also [31, 23] for more examples), and that the class of manifolds with $\text{Ric}_k > 0$ for $k \neq n$ contains non-Fano manifolds. However, we prove the following result.

**Theorem 1.3.** Let $(N^n, h)$ be a compact Kähler manifold with $\text{Ric}_k > 0$, for some $1 \leq k \leq n$. Then $N$ is projective and rationally connected. In particular, $\pi_1(N) = \{0\}$.

The projectivity is proved by a vanishing theorem similar to Theorem 1.2. See Theorem 3.1. Namely we show that $h^{p, 0} = 0$ for any $1 \leq p \leq n$ under the assumption that $\text{Ric}_k > 0$ for some $1 \leq k \leq n$. The rational connectedness is proved by showing another vanishing theorem, whose validity is a criterion of the rational connectedness proved in [5]. Both the techniques of [24] and the one utilizing the comass for $(p, 0)$-forms introduced in Section 2 of this paper are crucial in proving these two vanishing theorems. The result above generalizes both the result for Fano manifolds [4, 18] (the case $k = n$, namely the Fano case of Campana, Kollár-Miyaoka-Mori), and the more recent result for the compact Kähler manifolds with positive holomorphic sectional curvature [11] by Heier-Wong (cf. also [32] for the projectivity for the case $k = 1$), since $\text{Ric}_1 > 0$ amounts to $H > 0$ and $\text{Ric}_n = \text{Ric}$. It seems that $\text{Ric}_k > 0$ has nothing to do with that Ricci curvature is $k$-positive in general. At least when $k = 1$, Hitchin’s examples show that they are independent. However it is related to the notion of $q$-Ricci studied in Riemannian geometry [29]. In particular, if the $2k-1$-Ricci is positive in the sense of $H$, then $\text{Ric}_k > 0$. The positivity of the $2k-1$-Ricci is a much stronger condition than $\text{Ric}_k > 0$ since it puts the strict positivity requirement on all $2k$-dimensional subspaces of the tangent space at $x$, most of which are neither invariant under the almost complex structure, nor a subspace of $T^* N$. Moreover $\text{Ric}_k \geq 0$ does not imply $\text{Ric}_{k+1} \geq 0$, unlike the $q$-Ricci conditions.

In Section 4 of the paper addresses the question when compact Kähler manifolds with $\text{Ric}^+ > 0$ are Fano. This question was raised in [23]. We give an affirmative answer for a special case.
Let \( (N, h) \) be a compact Kähler manifold of complex dimension \( n \). Then (i) if \( \text{Ric}^<<0 \) and the Picard number \( \rho(N) = 1 \), then \( N \) must be Fano; (ii) if \( \text{Ric}^<<0 \) and \( h^{1,1}(N) = 1 \), \( N \) must be projective with ample canonical line bundle \( K_N \). In particular in the case (i) \( N \) admits a Kähler metric with positive Ricci, and in the case of (ii) \( N \) admits a Kähler-Einstein metric with negative Einstein constant.

Since it was proved in [23] that \( N \) is projective and \( h^{1,0}(N) = h^{2,0}(N) = 0 = h^{0,2}(N) = h^{1,1}(N) \) under the assumption that \( \text{Ric}^<<0 \), the assumption of \( \rho(N) = 1 \) for case (i) is equivalent to the assumption that the second Betti number \( b_2 = 1 \). In [22], it has been shown that for all Kähler \( C \)-spaces of classical type with \( b_2 = 1 \) the canonical Kähler-Einstein metric satisfies \( \text{Ric}^<<0 \).

To put Theorem 1.4 into perspectives it is proper to recall some earlier works. First related to \( \text{Ric}^<<0 \) there exists a stronger condition called the nonnegative quadratic orthogonal bisectional sectional curvature, studied by various people including authors of [30] and [6], etc. Quadratic bisectional curvature (abbreviated as \( \text{QB} \)), is defined for any real vector \( \overline{a} = (a_1, \ldots, a_n)^r \) and any unitary frame \( \{E_i\} \) of \( T^*N \), \( \text{QB}(\overline{a}) = \sum_{i,j} R_{i,j}^1(a_i - a_j)^2 \). Invariantly it can be formulated as a quadratic form (in terms of a curvature operator \( R \)) acting on Hermitian symmetric tensors \( \{A\} \) (at any given point on the manifold) as

\[
\text{QB}_R(A) \doteq \langle R, A^2 \rangle - AA^A.
\]

Interested readers can refer to [21] for the notations involved.\(^1\) Its nonnegativity, abbreviated as (NQOB), is equivalent to that \( \text{QB}(\overline{a}) \geq 0 \) for any \( \overline{a} \) and any unitary frame \( \{E_i\} \). This curvature condition was formally introduced in [30] (perhaps appeared implicitly in the work of Bishop-Goldberg in 1960s).\(^2\) In [6] the following was proved by Chau-Tam (cf. [6], Theorem 4.1):

**Theorem 1.5.** Let \( (N, h) \) be a compact Kähler manifold with \( \text{NQOB} \) with \( h^{1,1}(N) = 1 \). Assume further that \( N \) is locally irreducible then \( c_1(M) > 0 \).

In this regard, Theorem 1.4 has the following corollary

**Corollary 1.6.** Let \( (N, h) \) be a compact Kähler manifold of complex dimension \( n \) with \( \text{Ric}^<<0 \). Assume further that \( h^{1,1}(N) = 1 \) and \( N \) is locally irreducible. Then \( c_1(N) > 0 \), namely \( N \) is Fano. A similar result holds under the assumption \( \text{Ric}^<<0 \leq 0 \).

Since (NQOB) implies that \( \text{Ric}^<<0 \geq 0 \) (cf. [6, 23]), one can view the above corollary as a generalization of Theorem 1.5 of Chau-Tam. There are certainly compact Kähler manifolds with \( b_2 > 1 \) (cf. construction in [22] via projectivized bundles) and \( \text{Ric}^<<0 > 0 \). It remains an interesting question whether or not the same conclusion of Theorem 1.4 (i) holds without the assumption \( h^{1,1} = 1 \). Since \( \text{QB} > 0 \) implies that \( h^{1,1} = 1 \), as a consequence we have that any compact Kähler manifold with \( \text{QB} > 0 \) must be Fano. It similarly remains an

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\(^1\)Another formulation of \( \text{QB} \) curvature is to view it as a Hermitian quadratic form defined for Hermitian symmetric tensors \( A: T^*N \to T^*N \), defined as

\[
\text{QB}_R(A) = \sum_{\alpha, \beta = 1}^n R(A(E_{\alpha}), \overline{A}(E_{\alpha}), E_{\beta}, E_{\beta}) - R(E_{\alpha}, E_{\beta}, A(E_{\beta}), \overline{A}(E_{\alpha}))
\]

for any unitary orthogonal frame \( \{E_{\alpha}\} \) of \( T^*N \).

\(^2\)In Section 3 of this paper we formally introduce another (quadratic form type) curvature, which is also related to \( \text{Ric}^<<0 \), motivated by the work of Calabi-Vesentini in 1960s.
interesting question whether or not the same conclusion of part (ii) holds without assuming that $h^{1,1} = 1$.

Even though the above result toward $N$ being Fano (assuming $\text{Ric}^{\perp} > 0$) is with a simple proof, and far from the final one, the investigation of the relation between $QB$ and $\text{Ric}^{\perp}$ naturally leads to some new results concerning the cohomology vanishing theorem of $(0,1)$-forms valued in the holomorphic tangent bundle. By combining the techniques and results of [3], [7], [14], and [22] we obtain the deformation rigidity for classical Kähler C-spaces, as a consequence of the criterion of Frölicher and Nijenhuis [10, 17].

**Theorem 1.7.** Let $N^n$ be a classical Kähler C-space with $n \geq 2$ and $b_2 = 1$. Then $H^q(N, T^*N) = \{0\}$, for $1 \leq q \leq n$, and $N$ is deformation rigid in the sense that it does not admit nontrivial infinitesimal holomorphic deformation.

The result was proved via a Kodaira-Bochner formula (cf. [3]) and the role of a curvature notion $^dCQB$ (dual-cross quadratic bisectional curvature) played in such a Kodaira-Bochner formula. This particular result perhaps is implied by Bott’s earlier result [2]. But it is derived as a general vanishing theorem for manifolds with $^dCQB > 0$. Hence we have a more general result, at least before one can obtain a classification on all Kähler manifolds with $^dCQB > 0$. The dual-cross quadratic bisectional curvature $^dCQB$ is related to the quadratic orthogonal bisectional $QB$. One can refer Sections 4 and 5 for detailed discussion on this new curvature.

The new dual-cross quadratic bisectional curvature $^dCQB$ naturally induces a Ricci type curvature (in a similar fashion as $QB$ is related to $\text{Ric}^{\perp}$, which is explained in Section 4). It is denoted by $\text{Ric}^{\perp}$, and is defined, for any $X \in T_x^*N$, as

$$\text{Ric}^{\perp}(X, \overline{X}) = \text{Ric}(X, \overline{X}) + H(X)/|X|^2.$$  

This notion of Ricci curvature is not as natural as $\text{Ric}^{\perp}$. However for the compact Kähler manifolds with $\text{Ric}^{\perp} > 0$ we have the following result similar to the $\text{Ric}^{\perp} > 0$ case.

**Theorem 1.8.** Let $(N, h)$ be a compact Kähler manifold with $\text{Ric}^{\perp} > 0$. Then $h^{p,0} = 0$ for all $n \geq p \geq 1$. In particular, $N$ is simply-connected and $N$ is projective.

This implies that the manifolds with a uniform positive lower bound on $^dCQB$ must be compact, projective, and simply-connected. The proof of the above result makes use of the comass and viscosity ideas introduced in Section 2 and follows a similar line of argument as the proof of Theorem 1.2. In Section 5 we also prove a diameter estimate and a result similar to Corollary 1.6 for $\text{Ric}^{\perp}$.

The cross quadratic bisectional curvature (abbreviated as $CQB$, introduced in Section 4) and its dual $^dCQB$ (studied in Section 5) are all positive on the compact classical Kähler C-spaces with $b_2 = 1$ and on some exceptional ones. Since $CQB > 0$ (as $QB > 0$) implies $\text{Ric}^{\perp} > 0$, this generalizes the result of [22]. On the other hand it was shown by Chau-Tam [7] that $QB > 0$ fails to hold for all Kähler C-spaces with $b_2 = 1$, and it was shown in [22] that there exists a non-homogenous example compact Kähler manifold with $\text{Ric}^{\perp} > 0$. Hence it is perhaps reasonable to expect that one of these two new curvature notions possibly provides a curvature characterization of the compact Kähler C-spaces with $b_2 = 1$. Towards this direction we can prove (in Theorem 4.3) that a compact Kähler manifold with $CQB > 0$ must be rationally connected. It is also Fano if $b_2 = 1$, under either the assumption $CQB > 0$ or $^dCQB > 0$. More ambitious expectation is that they perhaps shed some lights on the generalized Hartshorne conjecture concerning the Fano manifolds with a nef tangent bundle.
In Sections 6, we study the gap between $QB > 0$ and $\text{Ric}^+ > 0$. Most results in this paper can be adapted to Hermitian manifolds without much difficulty, if the notions of involved curvatures are properly extended.

2. Comass and the proof of Theorem 1.2

Let $V$ be a Euclidean space. For a $r$-covector $\omega$ the comass is defined in [28] as

$$\|\omega\|_0 = \sup\{\|\omega(a)\| \colon a \text{ is a simple } r\text{-vector, } \|a\| = 1\}.$$ 

Here the norm $\|\cdot\|$ is the norm (a $L^2$-norm in some sense) induced by the inner product defined for simple vectors $a = x_1 \wedge \cdots \wedge x_r$, $b = y_1 \wedge \cdots \wedge y_r$, with $x_i, y_j \in V$, as

$$\langle a, b \rangle = \det(\langle x_i, y_j \rangle)$$

and then extended bi-linearly to all $r$-covectors $a$ and $b$ which are linear combination of simple vectors. The following results concerning the comass are well-known. The interested readers can find their proof in Whitney’s classics [28] or Federer’s [9].

**Proposition 2.1.** (i) $\|\omega\|_0 = \sup\{\|\omega(a)\| \colon \|a\|_0 = 1\}$, where $\|a\|_0$ is the mass of $a$ defined as

$$\|a\|_0 = \inf\{\sum \|a_i\| \colon a = \sum a_i, \text{ the } a_i \text{ simple}\}.$$

(ii) For each $\omega$ there exists a $r$-vector $b$ such that $\|\omega\|_0 = |\omega(b)|$, $b$ is simple, and $\|b\| = 1$.

(iii) If $\omega$ is simple, $\|\omega\|_0 = |\omega|$.

(iv) $\|\omega\| \geq \|\omega\|_0 \geq \frac{k!(n-k)!}{n!}\|\omega\|$.

We shall prove the theorem via an argument by contradiction. Assume that $\phi$ is a harmonic $(p,0)$-form which is not zero. It is well-known that it is holomorphic. Let $|\phi|_0(x)$ be its comass at $x$. Then its maximum (nonzero) must be attained somewhere at $x_0 \in N$. We shall exam $\phi$ more closely in a coordinate chart (to be specified later) of $x_0$. By the above proposition, at $x_0$, there exits a simple $p$-vector $b$ with $\|b\| = 1$, which we may assume to be $\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_p}$ for a unitary frame $\{\frac{\partial}{\partial z_k}\}_{k=1,\ldots,n}$ at $x_0$, such that $\max_{x \in N} |\phi|(x) = |\phi|_0(x_0) = |\phi(b)|$. If we denote $\phi = \frac{1}{p!} \sum I_p a_{I_p} dz^{i_1} \wedge \cdots \wedge dz^{i_p}$, where $I_p = (i_1, \ldots, i_p)$ runs all $p$-tuples with $i_t \neq i_s$ if $s \neq t$,

$$|\phi|_0(x_0) = |a_{12\ldots p}|(x_0).$$

Extends the frame to a normal complex coordinate chart $U$ centered at $x_0$. This means that at $x_0$, the metric tensor $g_{\alpha\beta}$ satisfies (cf. [26])

$$g_{\alpha\beta} = \delta_{\alpha\beta}, \quad dg_{\alpha\beta} = 0, \quad \frac{\partial^2 g_{\alpha\beta}}{\partial z_\gamma \partial z_\delta} = 0.$$

Now $\phi = \frac{1}{p!} \sum I_p a_{I_p} dz^{i_1} \wedge \cdots \wedge dz^{i_p}$ for $x \in U$ with $a_{I_p}(x)$ being holomorphic.

Let $\tilde{\phi}(x) = a_{12\ldots p}(x) dz^{i_1} \wedge \cdots \wedge dz^{i_p}$ locally. Clearly it is also holomorphic in $U$. Let $b(x)$ be the extended $p$-vector $\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_p}$ (which not necessarily of norm 1) at $x$. For any $a = v_1 \wedge \cdots \wedge v_p$, we denote $a^T = P(v_1) \wedge \cdots \wedge P(v_p)$ with $P$ being the unitary projection to the $p$-dimensional subspace spanned by $\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_p}\}$ (namely $b$).

**Lemma 2.1.** For $x \in U$, $|\tilde{\phi}|_0(x) \leq |\phi|_0(x)$. 
Proof. Pick a simple $p$-vector $a$ so that $a^T \neq 0$. Then at $x$ we have
\[
\frac{|\hat{\phi}(a)|}{|a|} = \frac{|\hat{\phi}(a^T)|}{|a|} \leq \frac{|\phi(a^T)|}{|a^T|} = \frac{|\phi(a^T)|}{|a^T|} \leq \|\phi\|_0(x).
\]
This proves that $||\hat{\phi}||/0(x) \leq ||\phi||_0(x)$. \( \square \)

As a consequence since $||\hat{\phi}||/0(x) \leq ||\phi||_0(x_0)$, we have that
\[
||\hat{\phi}(x)\| = ||\hat{\phi}||/0(x) \leq ||\phi||/0(x) = ||\hat{\phi}||/0(x_0) = ||\phi||/0(x).
\]
In summary, we have constructed a simple holomorphic $(p,0)$-form $\hat{\phi}(x)$ in the neighborhood of $x_0$ such that its norm attains its maximum value at $x_0$. Now we recall that the $\partial\overline{\partial}$-Bochner formula (cf. [16]) for a holomorphic $(p,0)$-form $\phi = \frac{1}{p!}\sum I_p a^i dz^1 \wedge \cdots \wedge dz^p$ yields for any $v \in T_{x_0}^* N$
\[
0 \geq (\sqrt{-1} \partial\overline{\partial} ||\phi||^2, \frac{1}{\sqrt{-1}} v^\wedge \overline{v}) = (\nabla_v \overline{\phi}, \nabla_{\overline{v}} \phi) + \frac{1}{p!} \sum I_p \sum_{k=1}^p \sum_{l=1}^n \langle R_{ve_{ikj}}, \overline{a}_{ikj}, a_{ijkl} \rangle. \tag{2.1}
\]
Given that $\hat{\phi}$ is simply, namely only nonzero $a_{lk}$ is the one with $I_p = (1,2, \cdots, p)$ or its permutations, then the above implies that at $x_0$
\[
0 \geq \sum_{j=1}^p R_{ve_{ij}}. \tag{2.2}
\]
Now we are essentially at the same position of the proof in [23]. For the sake of the completeness we include the argument below. Let $\Sigma = \text{span}\{ \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^p} \}$. It is easy to see from (2.2) that $S_p(x_0, \Sigma) \leq 0$, where $S_p(x_0, \Sigma)$ denotes the scalar curvature of the curvature $R$ restricted to $\Sigma$. In fact $S(x_0, \Sigma) = \sum_{i,j=1}^p R_{ij}$. On the other hand as in [23]
\[
\frac{1}{p} S_p^+ (x_0, \Sigma) = \int_{Z \in \Sigma, ||Z||=1} \text{Ric}^+(Z, \overline{Z}) \ d\theta(Z) = \int_{Z \in \Sigma, ||Z||=1} \left( \text{Ric}(Z, \overline{Z}) - H(Z) \right) \ d\theta(Z) = \int \frac{1}{Vol(S^{2n-1})} \int_{S^{2n-1}} \left( \int_{S^{2n-1}} \left( \text{Ric}(Z, \overline{Z}, W, \overline{W}) - H(Z) \right) \ d\theta(W) \right) \ d\theta(Z) = \frac{1}{p} \left( \text{Ric}_{11} + \text{Ric}_{22} + \cdots + \text{Ric}_{pp} \right) - \frac{2}{p(p+1)} S_p(x_0, \Sigma). \tag{2.3}
\]
Applying (2.2) to $v = \frac{\partial}{\partial x^i}$ for $i = p+1, \cdots, n$, and summing the obtained inequalities we have that
\[
\text{Ric}_{11} + \text{Ric}_{22} + \cdots + \text{Ric}_{pp} = S_p(x_0, \Sigma) + \sum_{i=p+1}^n \sum_{j=1}^p R_{ij} \leq S_p(x_0, \Sigma). \tag{2.4}
\]
Combining (2.3) and (2.4) we have that
\[
0 < S_p^+(x_0) = S_p^+ (x_0, \Sigma) \leq S_p(x_0, \Sigma) \leq \frac{2}{p+1} S_p(x_0, \Sigma) = \frac{p-1}{p+1} S_p(x_0, \Sigma). \tag{2.5}
\]
This, for $p \geq 2$, implies $S_p(x_0, \Sigma) > 0$, a contradiction, since we have shown that a consequence of (2.2) is $S_p(x_0, \Sigma) \leq 0$. 

From the definition of $S^k_+$ it is easy to see that $\text{Ric}^+ > 0$ implies that $S^k_+ > 0$ for all $k \in \{1, \cdots , n\}$. Hence $h^{p,0} = 0$ for all $p \geq 2$ by the above under the assumption $\text{Ric}^+ > 0$. On the other hand $\tau_1$ is finite by the result of [23]. This in particular implies that $b_1 = 2h^{1,0} = 0$. The simply-connectedness claimed in Theorem 1.2 follows from the argument of [15] illustrated in the introduction.

Remark 2.1. The argument here also provides a more direct proof of the vanishing theorem in [24]. It is clear that the Kählerity is not used essentially except in the estimation on $S^k(x, \Sigma)$. Hence one easily formulate a corresponding result for Hermitian manifolds. We leave this to interested readers. The concepts of $S_k(x, \Sigma)$ and $S^k_+(x_0, \Sigma)$ were conceived in [23, 24]. Moreover the argument of [24] can be adapted to prove more general vanishing theorem related to the rational connectedness. See the next section for more details.

3. Rational connectedness and $\text{Ric}_k$

A complex manifold $N$ is called rationally connected if any two points of $N$ can be joined by a chain of rational curves. Various criterion on the rational connectedness have been established by various authors. In particular the following was prove in [5]:

Proposition 3.1. Let $N$ be a projective algebraic manifold of complex dimension $n$. Then $N$ being rationally connected if and only if for any ample line bundle $L$, there exist $C(L)$ such that

$$H^0(N, ((T'N)^*) \otimes L^{\otimes \ell}) = \{0\}$$

(3.1)

for any $p \geq C(L)\ell$, with $\ell$ being any positive integer.

It was proved in [11] that a compact projective manifold with positive holomorphic sectional curvature must be rationally connected. The projectivity was proved in [32] afterwards (an alternate proof of the rational connectedness was also given there). In [20], the concept $\text{Ric}_k$ was introduced, which interpolates between the holomorphic sectional curvature and the Ricci curvature. Precisely for any $k$ dimensional subspace $\Sigma \subset T^*_p N$, $\text{Ric}_k(x, \Sigma)$ is the Ricci curvature of $R|_\Sigma$. Under $\text{Ric}_k < 0$, the $k$-hyperbolicity was proved in [20].

We say $\text{Ric}_k(x) > \lambda(x)$ if $\text{Ric}_k(x, \Sigma)(v, v) > \lambda |v|^2$, for any $v \in \Sigma$ and for every $k$-dimensional subspace $\Sigma$. Similarly $\text{Ric}_k > 0$ means that $\text{Ric}_k(x) > 0$ everywhere. The condition $\text{Ric}_k > 0$ does not become weaker as $k$ increases since more $v$ needs to be tested. In fact Hitchin [13] illustrated examples of Kähler metrics (on surfaces) with $\text{Ric}_1 > 0$, but does not have $\text{Ric}_2 > 0$. More examples can be found in [1, 23]. But it is easy to see that $S_k > 0$ does follows from $\text{Ric}_k > 0$, and $S_k > 0$ becomes weaker as $k$ increases with $S_1$ being the same as the holomorphic sectional curvature and $S_n$ being the scalar curvature. The following result is obvious by the vanishing theorem of [24].

Lemma 3.1. For any $\lambda$, $\text{Ric}_k(x) \geq \lambda$ implies that $S_k \geq k\lambda$. In particular, for a compact Kähler manifold with $\text{Ric}_k > 0$, $h^{p,0} = 0$ for $p \geq k$.

Hence if $\text{Ric}_2 > 0$, $N$ is also projective by the result of [24]. Naturally one would ask whether or not a compact Kähler manifold with $\text{Ric}_k > 0$ for some $k \in \{3, \cdots , n-1\}$ is projective since the projectivity has been known for the case for $k = 1$ and the case $k = n$. We first provide an affirmative answer to this question.

Theorem 3.1. Let $(N^n, h)$ be a compact Kähler manifold with $\text{Ric}_k > 0$ for some $1 \leq k \leq n$. Then $h^{p,0} = 0$. In particular, $N$ must be projective.
Proof. By the above lemma we have that $h^{p,0} = 0$ for $p \geq k$. Hence we only need to focus on the case $p < k$. The first part of proof of Theorem 1.2 asserts that if $\phi$ is a holomorphic $(p,0)$-form, which is non-trivial, then (2.2) holds. Namely there exists $x_0 \in N$, and a unitary normal coordinate centered at $x_0$ such that at $x_0$:

$$\sum_{j=1}^{p} R_{v e j j} \leq 0$$  \hspace{1cm} (3.2)

for any $v \in T_{x_0}N$.

Now we pick a $k$-subspace $\Sigma \subset T_{x_0}N$ such that it contains the $p$-dimensional subspace spanned by $\{ \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^p} \}$. Then by the assumption $Ric_k > 0$,

$$\int_{v \in S_{2k-1} \subset \Sigma} R_{v e j j} \ d\theta(v) = \frac{1}{k} Ric_k \left( \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^j} \right) > 0$$

for every $j \in \{1, \ldots, p\}$. Thus we have that

$$\int_{v \in S_{2k-1} \subset \Sigma} \sum_{j=1}^{p} R_{v e j j} \ d\theta(v) > 0.$$

This is a contradiction to (3.2). The contradiction proves that $h^{p,0} = 0$ for $p < k$. The projectivity follows from $h^{2,0} = 0$ and a theorem of Kodaira (cf. [19], Theorem 8.3 of Chapter 3).

For $k = 1, 2, n$, the result for these cases are previously known except the case that $k = 2, p = 1$.

The argument above proves a bit more. We call the curvature $R$ is $BC$-$p$ positive at $x_0$ ($BC$ stands for the bisectional curvature) if for any $p$-unitary orthogonal vectors $\{E_1, \ldots, E_p\}$, there exists a $v \in T_{x_0}N$ such that

$$\sum_{i=1}^{p} R_{v \in E_i \Sigma_i} > 0.$$  \hspace{1cm} (3.3)

We say that $(N, h)$ is $BC$-$p$ positive if it holds all $x_0 \in N$. Hence $BC$-$1$ positivity is the same as $RC$-positivity for the tangent bundle defined in [32]. For general $p$, $BC$-$p$ positivity is considerably weaker than $RC$-positivity. The $BC$-$p$ positivity can be easily adapted to Hermitian bundles over Hermitian manifolds. So is the following result. For example if we endow a compact complex manifold $N^n$ with a Hermitian metric. Let $R$ be its curvature, which can be viewed as section of $\bigwedge^{1,1}(\text{End}(T_xN))$. The condition (3.3) still makes perfect sense. Modifying the proof above we have the following results.

**Corollary 3.2.** If the curvature of a Hermitian manifold $(N^n, h)$ satisfies that $BC$-$p$ positive for some $1 \leq p \leq n$, then $h^{p,0} = 0$. Hence any Kähler manifold with $BC$-$2$ positive curvature must be projective. In particular, the $2$-positivity of $Ric_k$ (of course for some $k \geq 2$ to be meaningful) is sufficient for the projectivity of $N$.

**Proof.** By the above proof of theorem, we only need to focus on the last statement. The $2$-positivity of $Ric_k$ implies that for any $k$-dimensional $\Sigma \subset T_{x_0}N$ and any two unitary orthogonal $E_1, E_2 \in \Sigma$

$$Ric_k(x_0, \Sigma)(E_1, \overline{E_1}) + Ric_k(x_0, \Sigma)(E_2, \overline{E_2}) > 0.$$
This clearly implies BC-2 positivity since for any given unitary orthogonal \( \{E_1, E_2\} \) one can apply the above to some \( \Sigma \) containing \( \{E_i\}_{i=1,2} \).

**Proposition 3.2.** For a Kähler manifold \((N, h)\), \(S_k(x_0) > 0\) implies BC-p positive for any \(p \geq k\), and \(\text{Ric}_k(x_0) > 0\) implies BC-p positive for any \(1 \leq p \leq n\).

**Proof.** The first claim follows from the simple observation that \(S_k > 0\) implies that \(S_p > 0\) for any \(p \geq k\). The second one follows from the proof of the theorem above. \(\Box\)

The Corollary 3.2 provides a generalization of the projective embedding theorem proved in \([24]\). Towards the rational connectedness we prove the following result.

**Theorem 3.3.** Let \((N^n, h)\) be a compact projective manifold with \(\text{Ric}_k > 0\) for some \(k \in \{1, \cdots, n\}\). Then (3.1) holds, and \(N\) must be rationally connected.

**Proof.** Before the general case, we start with a proof for the special case \(k = 1\) by proving the above criterion in Proposition 3.1 directly via the \(\partial\overline{\partial}\)-Bochner formula. Let \(s\) be a holomorphic section in \(H^0(N, ((T^*N)^*)^{\otimes p} \otimes L^{\otimes \ell})\). Locally it can be expressed as

\[
s = \sum_{I_p} a_{I_p, \ell} dz^{i_1} \otimes \cdots \otimes dz^{i_p} \otimes e^\ell
\]

with \(I_p = (i_1, \cdots, i_p) \in \mathbb{N}^p\), and \(e\) being a local holomorphic section of \(L\) and \(e^\ell = e \otimes \cdots \otimes e\) being the \(\ell\) power of \(e\). Equip \(L\) with a Hermitian metric \(a\) and let \(C_a\) be the corresponding curvature form. The point-wise norm \(|s|^2\) is with respect to the induced metric of \(((T^*N)^*)^{\otimes p}\) and \(L^{\otimes \ell}\). The \(\partial\overline{\partial}\)-Bochner formula implies that for any \(v \in T_xN\):

\[
\partial_v \overline{\partial}_v |s|^2 = |\nabla_v s|^2 + \sum_{I_p} \sum_{t=1}^p \sum_{\alpha=1}^p \langle a_{I_p, \ell} R_{v;v, t} \gamma_{i_{t-1}, \ell} dz^{i_1} \otimes \cdots \otimes dz^{i_{t-1}} \otimes dz^t \otimes \cdots \otimes dz^{i_p} \otimes e^\ell, s \rangle \\
- \sum_{I_p} \langle a_{I_p, \ell} \ell C_a(v, \overline{v}) dz^{i_1} \otimes \cdots \otimes dz^{i_p} \otimes e^\ell, s \rangle. \tag{3.4}
\]

Applying the above equation at the point \(x_0\), where \(|s|^2\) attains its maximum, with respect to a normal coordinate centered at \(x_0\). Pick a unit vector \(v\) such that \(H(v)\) attains its minimum on \(S^{2n-1} \subset T^*_xN\). By the assumption \(H > 0\), there exists a \(\delta > 0\) such that \(H(v) \geq \delta\) for any unit vector and any \(x \in N\). Diagonalize \(R_{v;v, t}(\gamma_{i_{t-1}, \ell})\) by a suitable chosen unitary frame \(\{\overline{\partial}_{\gamma_{i_{t-1}, \ell}}, \cdots, \overline{\partial}_{\gamma_{i_{p-1}, \ell}}\}\). Applying the first and second derivative tests, it shows that if \(v \in S^{2n-1}\), \(H(v)\) attains its minimum, then \(R_{v;v, w} \geq \frac{\delta}{2}\), and \(R_{v;w, w} = 0\), for any \(w\) with \(|w| = 1\), and \(\langle w, \overline{v} \rangle = 0\). This implies that

\[
R_{v;w, a} = |\mu_1|^2 R_{v;w, v} + |\beta_1|^2 R_{v;w, w} \geq \frac{\delta}{2}
\]

where we write \(\overline{\partial}_{\gamma_{i_{t-1}, \ell}} = \mu_1 v + \beta_1 w\) with \(|\mu_1|^2 + |\beta_1|^2 = 1\), \(w \in \{v\}^\perp\) and \(|w| = 1\). (This perhaps goes back to the work of Berger. See also for example [32] or Corollary 2.1 of [23].)

If \(A\) is the upper bound of \(C_a(v, \overline{v})\), we have that

\[
0 \geq \partial_v \overline{\partial}_v |s|^2 \geq \left(\frac{p\delta}{2} - \ell A\right) |s|^2.
\]

This is a contradiction for \(p \geq \frac{3\delta}{4}\) if \(s \neq 0\). Hence we can conclude that for any \(p \geq C(L)\ell\) with \(C(L) = \frac{3A}{\delta}\), \(H^0(N, ((T^*N)^*)^{\otimes p} \otimes L^{\otimes \ell}) = \{0\}\).
For the general case, namely $\text{Ric}_k > 0$ for some $k \in \{1, \ldots, n\}$, we combine the argument above with the proof of the vanishing theorem in [24]. At the point $x_0$ where the maximum of $|s|^2$ is attained, we pick $\Sigma$ such that $S_k(x_0, \Sigma)$ attains its minimum $\delta_1 > 0$. For simplicity of the notations, we denote the average of a function $f(X)$ over the unit sphere $S^{2k-1}$ in $\Sigma$ as $\int f(X)$. The second variation consideration in [24] gives the following useful estimates.

**Proposition 3.3** (Proposition 3.1 of [24]). Let $\{E_1, \ldots, E_m\}$ be a unitary frame at $x_0$ such that $\{E_i\}_{1 \leq i \leq k}$ spans $\Sigma$. Let $I$ be any non-empty subset of $\{1, 2, \ldots, k\}$. Then for any $E \in \Sigma, E' \in \Sigma$, and any $k + 1 \leq p \leq m$, we have

\[
\int R(E, E', Z, Z) d\theta(Z) = \int R(E', E, Z, Z) d\theta(Z) = 0, \quad (3.5)
\]

\[
\int \left( R(E_p, E_p', Z, Z) + \sum_{j \in I} R(E_j, E_j, Z, Z) \right) d\theta(Z) \geq \frac{S_k(x_0, \Sigma)}{k(k+1)}, \quad (3.6)
\]

\[
\int R(E_p, E_p, Z, Z) d\theta(Z) \geq \frac{S_k(x_0, \Sigma)}{k(k+1)}. \quad (3.7)
\]

As [24], we may choose the frame so that $\int R_{\varphi(\Sigma)}$ is diagonal. Integrating (3.4) over the unit sphere $S^{2k-1} \subset \Sigma$ we have that

\[
0 \geq \int \partial_i \bar{\partial}_j |s|^2 d\theta(v) \geq \sum_{l_p} |a_{l_p}|^2 \int \left( \sum_{\alpha=1}^p R_{\varphi_{\alpha} z_\alpha} - \ell C_a(v, \bar{v}) \right) d\theta(v).
\]

Here we have chosen a unitary frame $\{\varphi_{\alpha} z_\alpha, \ldots, \varphi_{\alpha} z_m\}$ so that $\int R_{\varphi(\Sigma)} d\theta(v)$ is diagonal.

As in [24], decompose $\frac{\partial}{\partial s^\alpha}$ into the sum of $\mu_i E_i \in \Sigma$ and $\beta_j E'_j \in \Sigma^\perp$ with $|E_i| = |E'_j| = 1$ and $|\mu_i|^2 + |\beta_j|^2 = 1$. If we denote the lower bound of $\text{Ric}_k$ by $\delta_2 > 0$, by (3.5) and (3.7)

\[
\int R_{\varphi_{\alpha} z_\alpha} d\theta(v) = |\mu_1|^2 \int R_{\varphi_{\alpha} E_1 \bar{E}_1} d\theta(v) + |\beta_1|^2 \int R_{\varphi_{\alpha} E_1 \bar{E}_1} d\theta(v) \geq \frac{|\mu_1|^2}{k} \delta_2 + \frac{|\beta_1|^2}{k(k+1)} \delta_1
\]

\[
\geq \min \left( \delta_1, \delta_2 \right) \frac{k}{k(k+1)}.
\]

The above estimate holds for any $\frac{\partial}{\partial s^\alpha}$ as well. Hence combining two estimates above we have that

\[
0 \geq \int \partial_i \bar{\partial}_j |s|^2 d\theta(v) \geq \left( p \frac{\min (\delta_1, \delta_2)}{k(k+1)} - \ell A \right) |s|^2.
\]

The same argument as the special case $k = 1$ leads to a contradiction, if $p \geq C(L) \ell$ for suitable chosen $C(L)$, provided that $s \neq 0$. This proves the vanishing theorem claimed in (3.1) for $\text{Ric}_k > 0$.

The simply-connectedness part of Theorem 1.3 follows from Theorem 3.1 and the argument of [15] (recalled in the introduction) via Hirzebruch’s Riemann-Roch theorem. It can also be inferred from the rational connectedness and Corollary 4.29 of [8]. It is expected that the construction via the projectivization in [22, 31] would give more examples of Kähler manifolds with $\text{Ric}_k > 0$. \qed
Since the boundedness of smooth Fano varieties (namely there are finitely many deformation types) was also proved in [18], it is natural to ask whether or not the family of Kähler manifolds with $\text{Ric}^+ > 0$ (for some $k$, particularly for $n$ large and $n-k \neq 0$ small) is bounded. The result fails for $H > 0$ given Hirzebruch’s examples (cf. also [1]). Before one proves that every Kähler manifold with $\text{Ric}^+ > 0$ is Fano, it remains an interesting future project to investigate the rational connectedness of compact Kähler manifolds with $\text{Ric}^+ > 0$.

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Corollary 3.4. Any compact Kähler manifold $(N, h)$ with $QB > 0$, or more generally with $\text{Ric}^+ > 0$ and $\rho(N) = 1$, must be rationally connected.

The same conclusion holds if $\text{Ric}^+ \geq 0$, $(N^n, h)$ is locally irreducible and $\rho(N) = 1$.

4. COMPACT KÄHLER MANIFOLDS WITH $h^{1,1} = 1$ AND $CQB$

Recall the following result from [23], which is a consequence of a formula of Berger.

Lemma 4.1. Let $(N^n, h)$ be a Kähler manifold of complex dimension $n$. At any point $p \in N$,

$$\frac{n-1}{n(n+1)} S(p) = \frac{1}{\text{Vol}(S^{2n-1})} \int_{|Z|=1, Z \in T_p^* N} \text{Ric}^+ (Z, \overline{Z}) \, d\theta(Z)$$

(4.1)

where $S(p) = \sum_{i=1}^n \text{Ric}(E_i, E_i)$ (with respect to any unitary frame $\{E_i\}$) denotes the scalar curvature at $p$.

Note that the first Chern form $c_1(N) = \frac{i}{2\pi} r_{ij} dz^i \wedge d\overline{z}^j$, with $r_{ij} = \text{Ric} \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^j} \right)$. Let $\omega_h = \frac{i}{2\pi} h_{ij} dz^i \wedge d\overline{z}^j$ be the Kähler form (the normalization is to make the Kähler and Riemannian settings coincide). A direct computation via a unitary frame gives

$$c_1(N)(y) \wedge \omega_h^{n-1}(y) = \frac{1}{n} S(y) \omega_h^n(y).$$

(4.2)

We also let $V(N) = \int_N \omega_h^n$. The normalization above makes sure that the volume of an algebraic subvariety has its volume being an integer.

Recall that for any line bundle $L$ its degree $d(L)$ is defined as

$$d(L) = \int_N c_1(L) \wedge \omega_h^{n-1}.$$  

(4.3)

When $h^{1,1}(N) = 1$, it implies that $[c_1(N)] = \ell [\omega_h]$ for some constant $\ell$. Hence we have that $d(K_N^{-1}) = \ell V(N)$.

Under the assumption (i) of Theorem 1.4, we know that $S(y) > 0$ point-wisely by Lemma 4.1, which then implies that $d(K_N^{-1}) > 0$, hence $\ell > 0$. This shows that $[c_1(N)] > 0$. Now Yau’s solution to the Calabi’s conjecture [33, 26] implies that $N$ admits a Kähler metric such that its Ricci curvature is $\ell \omega_h > 0$.

The proof for statement (ii) is similar. The existence of negative Kähler-Einstein metric follows from the Aubin-Yau theorem [33, 26].

To prove Corollary 1.6 we observe that if $\ell = 0$ in the above argument, it implies that $S(y) \equiv 0$. Hence by Lemma 4.1 we have that $\text{Ric}^+ \equiv 0$. By Theorem 6.1 of [22] it implies
that $N$ is flat for $n \geq 3$, or $n = 2$ and $N$ is either flat or locally a product. This contradicts to the assumption of local irreducibility.

Note that the same argument can be applied to conclude the same result for holomorphic sectional curvature.

**Proposition 4.1.** Let $(N, h)$ be a compact Kähler manifold of complex dimension $n$. Assume further that $h^{1,1}(N) = 1$. Then (i) if $H > 0$, then $N$ must be Fano; (ii) if $H < 0$, $N$ must be projective with ample canonical line bundle $K_N$. In particular in the case (i) $N$ admits a Kähler metric with positive Ricci, and in the case of (ii) $N$ admits a Kähler-Einstein metric with negative Einstein constant.

Before introducing the other curvatures related to $QB$ we first observe that in (1.2) if we replace $A$ by its traceless part $\tilde{A} = A - \lambda \text{id}$ with $\lambda = \frac{\text{trace}(A)}{n}$, it remains the same. Namely $QB(A) = QB(\tilde{A})$. Hence $QB$ is defined on the quotient space $S^2(\mathbb{C}^n)/\{\text{Id}\}$, with $S^2(\mathbb{C}^n)$ being the space of Hermitian symmetric transformations of $\mathbb{C}^n$. Now $QB > 0$ means that $QB(A) > 0$ for all $A \neq 0$ as an equivalence class. This suggests a refined positivity $QB_k > 0$, for any $1 \leq k \leq n$, defined as $QB(A) > 0$ for any $A \notin \{\text{Id}\}$ of rank not greater than $k$. Clearly for $k < n$, a nonzero Hermitian symmetric matrix with rank no greater than $k$ can not be in $\{\text{Id}\}$. It is easy to see $QB_1 > 0$ is equivalent to $Ric^+ > 0$ and $QB_n > 0$ is equivalent to $QB > 0$. Naturally a possible approach towards the classification of $Ric^+ > 0$ is through the family of Kähler manifolds with $QB > 0$ and $QB_k > 0$.

Now we introduce the first of two associated curvatures. We call the first one the cross quadratic bisectional curvature $CQB$, defined as a Hermitian quadratic form on linear maps $A : T^nN \to T^*N$:

$$ CQB_R(A) = \sum_{\alpha, \beta = 1}^n R(A(E_\alpha), A(E_\alpha), E_\beta, E_\beta) - R(E_\alpha, E_\beta, A(E_\alpha), A(E_\beta)) $$

(4.4)

for any unitary frame $\{E_\alpha\}$ of $T'M$. This is similar to (1.2). But here we allow $A$ to be any linear maps. We say $R$ has $CQB > 0$ if $CQB(A) > 0$ for any $A \neq 0$. For any $X \neq 0$, if we choose $\{E_\alpha\}$ with $E_1 = \frac{X}{|X|}$, and let $A$ be the linear map satisfying $A(E_1) = E_1$ and $A(E_\alpha) = 0$ for any $\alpha \geq 2$, it is easy to see that $CQB_R(A) = Ric^+(X, X) / |X|^2$. Hence $CQB > 0$ implies that $Ric^+ > 0$ as well. However as shown in Theorem 3.3 $CQB > 0$ holds for all classical Kähler $C$-spaces with $b_2 = 1$, unlike $QB$, which fails to be positive on about $20\%$ of Kähler $C$-spaces with $b_2 = 1$. The expression $CQB$ was implicit in the work of Calabi-Vesentini [3] where the authors studied the deformation rigidity of compact quotients of Hermitian symmetric spaces of noncompact type. We can introduce the concept $CQB_k > 0$ (or $CQB_k < 0$), defined as $CQB(A) > 0$ for any $A$ with rank not greater than $k$.

**Proposition 4.2.** (i) The condition $CQB_1 > 0$ implies $Ric^+ > 0$, in particular $N$ satisfies $h^{0,0} = 0$, $\pi_1(N) = \{0\}$, and $N$ is projective.

(ii) If $N$ is compact with $n \geq 2$, and $CQB_2 > 0$, then Ricci curvature is 2-positive.

**Proof.** Part (i) is proved in the paragraph above together with Theorem 1.2. For part (ii), for any unitary frame $\{E_\alpha\}$, let $A$ be the map defined as $A(E_1) = E_2$ and $A(E_2) = -E_1$, and $A(E_\alpha) = 0$ for all $\alpha > 2$. Then the direct checking shows that $CQB > 0$ is equivalent to

$$ Ric(E_1, E_1) + Ric(E_2, E_2) > 0. $$
Since this holds for any unitary frame we have the 2-positivity of the Ricci curvature. □

Tracing the argument in [3], which is essentially based on the Akizuki-Nakano formula, we have the following result.

**Theorem 4.1.** Let \((N, h)\) be a compact Kähler manifold with quasi-negative \(CQB\) (namely \(CQB \leq 0\) and \(\omega < 0\) at least at one point). Then

\[
H^1(N, T^*N) = \{0\}.
\]

In particular, \(N\) is deformation rigid in the sense that it does not admit nontrivial infinitesimal holomorphic deformation.

**Proof.** Let \(\phi = \sum_{i=1}^n \phi_i^\alpha dz^\alpha \otimes E_i\) be a \((0, 1)\)-form taking value in \(T^*N\) with \(\{E_i\}\) being a local holomorphic basis of \(T^*N\). The Akizuki-Nakano formula gives

\[
(\Delta \phi - \Delta \phi^\beta)\phi^\beta = R_{j\beta}^\alpha \phi^\alpha - R_{j\beta}^\alpha \phi^\alpha.
\]  

Under a normal coordinate we have that

\[
(\Delta \phi^\alpha - \Delta \phi^\alpha)\phi^\beta = -\left(\text{Ric}_{ij} \phi^i\phi^j - R_{ij\beta} \phi^i\phi^j\phi^\beta\right).
\]

Hence if \(\Delta \phi^\alpha = 0\), we then have

\[
0 = \int_N |\partial \phi|^2 + \int_N |\partial^* \phi|^2 - \int_N \left(\text{Ric}_{ij} \phi^i\phi^j - R_{ij\beta} \phi^i\phi^j\phi^\beta\right).
\]

Letting \(A(\phi^\beta) = \phi^\beta E_i\), the assumption amounts to that the expression in the third integral above is negative over the open subset where \(CQB < 0\) if \((\phi_i^\beta) \neq 0\). This implies \((\phi_i^\beta) \equiv 0\) over this open subset, hence \(\phi = 0\) by the unique continuation and the harmonic equations. □

It has been proved in [23] that if \(\text{Ric} < 0\), then \(H^0(N, T^*N) = \{0\}\). By Table 1 of [3] and the proof of Theorem 4.4 below, all locally Hermitian symmetric spaces of noncompact type satisfy \(CQB < 0\). Hence the above theorem generalizes Calabi-Vesentini’s result. It is desirable to have new examples beyond the locally Hermitian symmetric ones.

The results above naturally lead to the following questions \((Q1)\): Does \(H^1(N, T^*N) = \{0\}\) hold under the weaker assumption that \(\text{Ric}^+ < 0\)? Do all Kähler \(C\)-spaces (the canonical Kähler metric) with \(b_2 = 1\) satisfy \(CQB > 0\) (below we provide a partial answer to this)? Does \(CQB > 0\) imply that \(b_2 = 1\), hence the manifold is Fano by Theorem 1.4? These remain to be interesting projects for future investigations with the ultimate goal of a classification of compact Kähler manifolds with \(CQB > 0\). We should point out that in [7] it was shown that not all Kähler \(C\)-spaces with \(b_2 = 1\) satisfy \(Q_0 > 0\). By flipping the sign we have the following corollary.

**Corollary 4.2.** Let \((N, h)\) be a compact Kähler manifold with quasi-positive \(CQB\). Then

\[
\mathcal{H}^0_0(N, T^*N) = \mathcal{H}^1_0(N, \Omega) = H^0(N, \Omega^1(\Omega)) = \{0\},
\]

where \(\Omega = (T^*N)^*\). If only \(\text{Ric}^+ > 0\) is assumed, then \(H^0(N, \Omega) = \{0\}\).

In fact we can strengthen the argument to prove the following result.
Theorem 4.3. Assume that \((N, h)\) is a compact Kähler manifold with \(CQB > 0\). Then for any ample line bundle \(L\), there exist \(C(L)\) such that

\[
H^0(N, ((T^*N)^*)^{\otimes p} \otimes L^{\otimes \ell}) = \{0\}
\]

(4.6)

for any \(p \geq C(L)\ell\), with \(\ell\) being any positive integer. In particular \(N\) is rationally connected.

Proof. First observe that a holomorphic section of \(((T^*N)^*)^{\otimes (p+1)} \otimes L^{\otimes \ell}\) can be viewed as a holomorphic \((1, 0)\) form valued in \(((T^*N)^*)^{\otimes p} \otimes L^{\otimes \ell}\). Write it as \(\varphi = \varphi^b \circ dz^\alpha \otimes dz^{1} \otimes \cdots \otimes dz^{p} \otimes e^{\ell}\). Applying the Arizuki-Nakano formula to the \(\delta\)-harmonic \(\varphi\) as above, using the formula for the curvature of the tensor products, and under a normal coordinate, we have that

\[
0 \leq \left(\square_{\partial} \varphi, \varphi\right) \leq \int_M \left(\Omega_{j \gamma}^I \varphi^I_\alpha \varphi^I_\gamma - \Omega_{j \gamma}^I \varphi^I_\alpha \varphi^I_\gamma \right) + \mathcal{A} \ell |\varphi|^2 \leq \int_M (-p \delta |\varphi|^2 + \mathcal{A} |\varphi|^2)
\]

where \(\Omega_{j \gamma}^I \varphi^I_\alpha \varphi^I_\gamma\) is the curvature of \(((T^*N)^*)^{\otimes p}\) and \(\Omega_{j \gamma}^I\) is the corresponding mean curvature, \(\delta > 0\) is the lower bound of \(CQB\), \(A\) is an upper bound of the scalar curvature of \(L\) (equipped with a Hermitian metric of positive curvature). This implies that \(\varphi = 0\) if \(p/\ell\) is sufficiently large, hence the result.

Putting the proofs of [7], [14] and [22] together we have the following result.

Theorem 4.4. Let \(N^n\) be a compact Hermitian symmetric space \((n \geq 2)\), or classical Kähler C-space with \(n \geq 2\) and \(b_2 = 1\). Then the (unique up to constant multiple) Kähler-Einstein metric has \(CQB > 0\).

Proof. If we write \(A(E_\beta) = A^\beta E_i\), it is easy to see if we change to a different unitary frame \(\overline{E_\alpha} = B_\alpha^\beta E_\beta\), the effect on \(A\) is \(BAB^\tau\) with \(B\) being a unitary transformation. Now

\[
CQB(A) = Ric_{ij} A^i_\beta A^j_\beta - R_{jir\beta} A^i_\tau A^j_\tau.
\]

Given that for \(A\) symmetric or skew symmetric one can put it into the corresponding normal form via the unitary frame transformations, it suggests that it is useful to write \(A\) into sum of the symmetric and skew-symmetric parts. For the special case \(Ric = \lambda h\), namely the metric is Kähler-Einstein with \(\lambda > 0\), if we decompose \(A\) into the symmetric part \(A_1\) and the skew-symmetric part \(A_2\), noting that \(R_{jir\beta}\) is symmetric in \(\tau, j, \beta\) we have

\[
CQB(A) = \lambda |A_1|^2 + \lambda |A_2|^2 - R_{jir\beta}(A_1)_\tau(A_1)_\beta \geq \lambda |A_1|^2 - R_{jir\beta}(A_1)_\tau(A_1)_\beta.
\]

Now note that the term \(R_{jir\beta}(A_1)_\tau(A_1)_\beta\) is the Hermitian symmetric action \(Q\) on the symmetric tensor (matrix) \(A\) considered in [14] and [3]. Let \(\nu\) denote the biggest eigenvalue of \(Q\). As in [22], to verify the result we just need to compare \(\lambda\) and \(\nu\). This can be done for all Hermitian symmetric spaces by Table 2 in [3]. Note that \(\lambda\) here is \(\frac{R}{n}\) in Calabi-Vesentini’s paper [3]. For the classical homogeneous examples which are not Hermitian symmetric we can use the comparison done in [22] with the data supplied by [14] and [7]. If we use the notation of [14] and [7], only the three types below need to be checked:

\[
(B_r, \alpha_i)_{r \geq 3, 1 < i < r}; \quad (C_r, \alpha_i)_{r \geq 3, 1 < i < r}; \quad (D_r, \alpha_i)_{r \geq 4, 1 < i < r-1}.
\]

The verification in Section 2 of [22] applies verbatim.

\(\square\)
The above result strengthens the one in [22] since \( CQB > 0 \) implies \( \text{Ric}^+ > 0 \). Note that the result also holds for the exceptional (non-Hermitian symmetric) Kähler C-space (\( (F_4, \alpha_4) \)) since for such a space \( \lambda = 11/2 \) and the biggest eigenvalue of \( Q \) is 1. A natural project afterwards is to classify all the compact Kähler manifolds with \( CQB > 0 \) hoping a curvature characterization of the Kähler C-spaces, after which one perhaps can attempt the \( \text{Ric}^+ > 0 \) classification through \( CQB_k > 0 \). Given the example in [22], there certainly are compact Kähler manifolds with \( \text{Ric}^+ > 0 \), but not homogeneous.

The second related curvature is a dual version of \( CQB \), which appeared implicitly when considering the compact dual of the noncompact Hermitian symmetric spaces in [3]. We denote it by \( dCQB \). It is defined as a quadratic Hermitian form of maps \( A : T'N \to T''N \), defined as

\[
dCQB_R(A) \equiv R(\overline{A(E_i)}, A(E_i), E_k, \overline{E}_k) + R(E_i, \overline{E}_k, \overline{A(E_i)}, A(E_k)).
\]

Similarly we can introduce the concept \( dCQB_k > 0 \). The analogy of \( \text{Ric}^+ \) is

\[
\text{Ric}^+(X, X) \equiv \text{Ric}(X, X) + H(X)/|X|^2.
\]

Once fixing a unitary frame of \( T'N \) (hence its dual) one can decompose the \( dCQB(A) \) into the sum of \( dCQB(A_1) + dCQB(A_2) \) with \( A_1 \) be the symmetric part and \( A_2 \) being the skew-symmetric part of \( A \). We say \( dCQB_k > 0 \) defined as \( dCQB(A) > 0 \) for any \( A \neq 0 \) with rank no greater than \( k \). It is easy to see that \( dCQB_1 > 0 \) implies that \( \text{Ric}^+ > 0 \) if we let \( A \) be the map satisfying \( A(E_i) = \overline{E}_1 \) and \( A(E_i) = 0 \) for all \( i \geq 2 \). We discuss geometric implications of these two curvature notions in details in the next section.

5. Positive \( dCQB \), deformation rigidity of Kähler C-spaces and \( \text{Ric}^+ \)

Properly formulated, results proved in [23] for manifolds with \( \text{Ric}^\pm \) can be extended to \( \text{Ric}^+ \). The argument via the second variational formulae in the proof of Bonnet-Meyer theorem proves the compactness of the Kähler manifolds if the \( \text{Ric}^+ \) is uniformly bounded from below by a positive constant.

**Theorem 5.1.** Let \( (N^n, h) \) be a Kähler manifold with \( \text{Ric}^+(X, X) \geq (n+3)\lambda|X|^2 \) with \( \lambda > 0 \). Then \( N \) is compact with diameter bounded from the above by \( \sqrt{\frac{2n}{(n+3)^2}} \pi \). Moreover, for any geodesic \( \gamma(\eta) : [0, \ell] \to N \) with length \( \ell > \sqrt{\frac{2n}{(n+3)^2}} \pi \), the index \( i(\gamma) \geq 1 \).

Note that the result is slightly better than \( \sqrt{\frac{2n}{n+1}} \lambda \pi \), the one predicted by the Bonnet-Meyer estimate assuming \( \text{Ric}(X, X) \geq (n+1)\lambda|X|^2 \) for \( n \geq 2 \). But it is roughly about \( \sqrt{2} \) times the one predicted by the Tsukamoto’s theorem in terms of the lower bound of the holomorphic sectional curvature. Let \( N = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \), namely the product of \( n \) copies of \( \mathbb{P}^1 \), its diameter is \( \sqrt{2} \pi \). An easy computation shows that it has \( \text{Ric} = 2 \) and \( H \geq \frac{2}{n} \). This shows that the upper bound provided by Tsukamoto’s theorem holds equality on both \( \mathbb{P}^n \) and \( N = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \). The product of \( n \)-copies of \( \mathbb{P}^1 \) also illustrates a compact Kähler manifold (after proper scaling) with \( \text{Ric} = n + 1 \), but its diameter is roughly about \( \sqrt{2} \) times of that of \( \mathbb{P}^n \). The product example and \( \mathbb{P}^n \) indicate that the above estimate on the diameter is far from being sharp.
We prove Theorem 1.8 via a vanishing theorem with weaker assumptions. For that we introduce the scalar curvatures $S^+_k(x, \Sigma)$ which is defined as

$$S^+_k(x, \Sigma) = k \int_{Z \in \Sigma, |Z| = 1} \text{Ric}^+(Z, \overline{Z}) \, d\theta(Z)$$

for any $k$-dimensional subspace $\Sigma \subset T_x N$. Similarly we say $S^+_k > 0$ if $S^+_k(x, \Sigma) > 0$ for any $x$ and $\Sigma$.

**Theorem 5.2.** Assume that $S^+_k > 0$, then $h^{p,0} = 0$ for $k \leq p \leq n$.

**Proof.** The first part of proof follows similarly as in that of Theorem 1.2. Assuming the existence of a nonzero holomorphic $(p,0)$-form $\phi$ leads to the conclusion that at the point $x_0$ where the maximum of the comass $\|\phi\|_0$ is attained we have that

$$0 \geq \sum_{j=1}^p R_{v_{e_j}}$$

for any $v \in T_{x_0} N$, for a particularly chosen frame $\{ \frac{\partial}{\partial z_i} \}_{i=1, \ldots, n}$ with $\Sigma = \text{span}\{ \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_p} \}$.

This implies that $S^+_p(x_0, \Sigma) \leq 0$, by applying the above to $v = \{ \frac{\partial}{\partial z_i} \}_{1 \leq i \leq p}$.

Now a similar calculation as that of Section 2 shows that

$$\frac{1}{p} S^+_p(x_0, \Sigma) = \int_{Z \in \Sigma, |Z| = 1} \text{Ric}^+(Z, \overline{Z}) \, d\theta(Z) = \int_{Z \in \Sigma, |Z| = 1} (\text{Ric}(Z, \overline{Z}) + H(Z)) \, d\theta(Z)$$

$$= \int \frac{1}{Vol(S^{2n-1})} \left( \int_{S^{2n-1}} (nR(Z, \overline{Z}, W, \overline{W}) + H(Z)) \, d\theta(W) \right) \, d\theta(Z)$$

$$= \int \frac{1}{Vol(S^{2n-1})} \int_{S^{2n-1}} \left( \int (nR(Z, \overline{Z}, W, \overline{W}) + H(Z)) \, d\theta(Z) \right) \, d\theta(W)$$

$$= \frac{1}{p} (\text{Ric}_{11} + \text{Ric}_{22} + \cdots + \text{Ric}_{pp}) + \frac{2}{p(p+1)} S^+_p(x_0, \Sigma). \quad (5.1)$$

Using the estimate (2.2) similarly as in Section 2 (cf. (2.4)) we have that

$$\text{Ric}_{11} + \text{Ric}_{22} + \cdots + \text{Ric}_{pp} \leq S^+_p(x_0, \Sigma).$$

Thus together with (5.1) it implies that

$$0 < S^+_k(x_0) \leq S^+_p(x_0, \Sigma) \leq \frac{p+3}{p+1} S^+_p(x_0, \Sigma).$$

This is a contradiction. \qed

Theorem 1.8 follows from the above theorem since $\text{Ric}^+ > 0$ implies that $S^+_p > 0$ for all $1 \leq p \leq n$. Applying the similar argument as that of the last section we also have the following result.

**Proposition 5.1.** Let $(N^n, h)$ be a compact Kähler manifold of complex dimension $n$ with $\text{Ric}^+ > 0$. Assume further that $h^{1,1}(N) = 1$ (or $\rho(N) = 1$). Then $c_1(N) > 0$, namely $N$ is Fano.

This follows from the lemma below, which is the analogue of Lemma 4.1, and the proof in last section verbatim.
Lemma 5.1. Let \((N^n, h)\) be a Kähler manifold of complex dimension \(n\). At any point \(p \in N\),

\[
\frac{n+3}{n(n+1)} S(p) = \frac{1}{\text{Vol}(S^n)} \int_{|z|=1, Z \in T_z N} \text{Ric}^+(Z, Z) \, d\theta(Z)
\]

(5.2)

where \(S(p) = \sum_{i=1}^{n} \text{Ric}(E_i, E_i)\) (with respect to any unitary frame \(\{E_i\}\)) denotes the scalar curvature at \(p\).

Following the argument in the Appendix of [22] we also have that a \(\text{Ric}^+\)-Einstein Kähler metric must be of constant curvature. In particular, the one with zero scalar curvature must be flat. Hence we have the same result as Corollary 1.6 if we replace \(\text{Ric}^+\) by \(\text{Ric}^+\).

Corollary 5.3. Let \((N, h)\) be a compact Kähler manifold of complex dimension \(n\) with \(\text{Ric}^+ \geq 0\). Assume further that \(h^{1,1}(N) = 1\) and \(N\) is locally irreducible. Then \(c_1(N) > 0\), namely \(N\) is Fano. Similar result holds under the assumption \(\text{Ric}^+ \leq 0\).

The result similar to Corollary 3.4 holds for \(\text{Ric}^+ > 0\) and \(\rho(N) = 1\), in view of Theorem 1.8, Proposition 5.1 and Corollary 5.3.

Corollary 5.4. Any compact Kähler manifold \((N, h)\) with \(\text{Ric}^+ > 0\) and \(\rho(N) = 1\), must be rationally connected.

The same holds if \(\text{Ric}^+ > 0\) is replaced with \(\text{Ric}^+ \geq 0\) and \((N^n, h)\) is locally irreducible. For compact Kähler manifolds with \(\text{Ric}^+ < 0\), we have the result below.

Proposition 5.2. Let \((N, h)\) be a compact Kähler manifold with \(\text{Ric}^+ < 0\). Then \(N\) does not admits any nonzero holomorphic vector field.

The proof is the same as that of [23]. A dual version of Theorem 4.1 is the following result.

Theorem 5.5. (i) For \((N, h)\) a compact Kähler manifold with quasi-positive \(4\text{CQB}\),

\[
H^1(N, T'N) = \{0\}.
\]

In particular, \(N\) is deformation rigid in the sense that it does not admit nontrivial infinitesimal holomorphic deformation.

(ii) If compact Kähler manifold \((N, h)\) has \(\text{CQB}_2 > 0\), then its Ricci curvature is \(2\)-positive.

(iii) If \((N, h)\) is compact with \(\text{CQB}_1 > 0\), then \(N\) is projective and simply-connected.

Proof. For (i) one may use the conjugate operator \# : \(A^{0,1}(T'N) \to A^{1,0}(T'N)^\ast\) which is defined for \(\phi = \phi_a dz^a \otimes E_i\), with \(\{E_i\}\) being a unitary frame of \(T'N\), as

\[
\# \phi = \overline{\phi_a^i} dz^a \otimes \overline{E_i}.
\]

Since \(\#(\partial \phi) = \overline{\partial}(\#(\phi))\), it implies that \(\overline{\partial}^*(\#(\phi)) = \#(\overline{\partial}^* \phi)\). Together \# induces an isomorphism between \(\mathcal{H}_\partial^{p,0}(N, T'N)\) and \(\mathcal{H}_{\overline{\partial}}^{2p}(N, (T'N)^\ast)\). To prove the result, it suffices to show that any \(\psi \in \mathcal{H}_\partial^{1,0}(N, (T'N)^\ast)\), \(\psi = 0\). Now we apply the Kodaira-Bochner formula for \(\Delta_\partial\) operator, and get for \(\psi = \psi_a^i \, dz^a \otimes \overline{E_i}\)

\[
(\Delta_\partial \psi)_a^i = -h^{a\beta} \nabla_{\overline{\beta}} \psi_a^i + R^j_{\overline{a} \overline{\beta}} \psi_a^j + (\text{Ric})^\sigma_a^i \psi_a^i.
\]

(5.3)

Taking product with \(\overline{\psi}_a\), as before under the unitary frame, if \(\Delta_\partial \psi = 0\) we have that

\[
0 = \int_N |\nabla \psi|^2 + \int_N \left[ (\text{Ric})_{\alpha \sigma} \psi_{\sigma}^i \overline{\psi}_{\alpha}^i + R_{i j \sigma \alpha} \psi_{\sigma}^i \overline{\psi}_{\alpha}^i \right].
\]
The claimed result follows in the similar way as in the proof of Theorem 4.1.

For part (ii), for any unitary frame \( \{E_i\} \), let \( A \) be the rank 2 skew-symmetric transformation: \( A(E_1) = E_2, A(E_2) = -E_1, \) and \( A(E_k) = 0 \) for all \( k \geq 3 \). Then as in the \( CQB > 0 \) case, the second part in the expression of \( ^dCQB \) vanishes and the first part yields \( \text{Ric}(E_1, E_1) + \text{Ric}(E_2, E_2) \).

Part (iii) follows from that \( ^dCQB_1 > 0 \) is the same as \( \text{Ric}^+ > 0 \) and Theorem 1.8. \( \square \)

By a similar argument (comparing the Einstein constant with the smallest eigenvalue of the symmetric curvature \( Q \) obtained in tables of [14]) as in the proof of Theorem 4.4 we also have the following corollary concerning Kähler C-spaces.

**Corollary 5.6.** Let \( N^n \) be a classical Kähler C-space with \( n \geq 2 \) and \( b_2 = 1 \), or a compact exceptional Hermitian symmetric space with \( n \geq 2 \). Then the (unique up to constant multiple) Kähler-Einstein metric has \( ^dCQB > 0 \). In particular, for a classical Kähler C-space \( N \) with \( b_2 = 1 \), \( H^2(N, T^*N) = \{0\} \) with \( 1 \leq q \leq n \), and \( N \) is deformation rigid in the sense that it does not admit nontrivial infinitesimal holomorphic deformation.

**Proof.** To check \( ^dCQB > 0 \), writing \( A(E_i) = A_i^T E_i \), we apply the similar argument as the case of \( CQB \) to see that if we decompose \( A \) into \( A_1 + A_2 \), symmetric and skew-symmetric parts, then

\[
^dCQB(A) \geq \lambda |A_1|^2 + R_{\text{idem}}(A_1)_{ij}^T (A_1)_{ji}^k.
\]

Here \( \lambda \) being the Einstein constant of the canonical metric. Then we reduce the problem to check that \( \lambda + \nu_1 > 0 \) with \( \nu_1 \) being the smallest eigenvalue of \( Q \). Recall that \( Q \) is the self-adjoint linear operator defined as (via extension)

\[
Q(X \cdot Y, Z \cdot W) = R_{XZ}YW
\]

for \( X \cdot Y = \frac{1}{2}(X \otimes Y + Y \otimes X) \). This quadratic curvature was considered previously in [3, 14]. We apply their results below. The Hermitian symmetric case again follows from Table 2 of [3]. For the nonsymmetric classical Kähler C-spaces, we check them as follows. Note that in [7] and [14] the same normalization for the canonical metric was used. For \( (B_r, \alpha_i)_{r \geq 1} \), \( \lambda = 2r - i \). According to Table 4 of [14] \( \nu_1 = -2(r - i) + 1 \) or \( -2r \geq 2(r - i) + 2 \), clearly \( 2(r - i) > 2r - i > 2 \). Also \( 2r - i - 2r + 2i + 1 = i + 1 > 0 \). These verify the result for both cases of \( \nu_1 \).

For \( (C_r, \alpha_i)_{r \geq 1} \), \( \lambda = 2r - i + 1 \). According to Table 7 of [14], \( \nu_1 = -2(r - i + 1) \). Hence \( \lambda + \nu_1 = i - 1 > 0 \) for \( i \geq 2 \). This verifies the result.

For \( (D_r, \alpha_i)_{r \geq 1} \), \( \lambda = 2r - i - 1 \). According to Table 10 of [14] \( \nu_1 = -2(r - i) + 2 \) or \( -2r \geq 2(r - i) + 2 \), clearly \( 2(r - i) > 2(r - i) + 2 \), this also verifies the result.

This proved the \( H^1(N, T^*N) = \{0\} \). For \( q > 1 \), the argument of [3] implies that one only needs to check that \( \lambda + \frac{q + 1}{q} \nu_1 > 0 \). This is a consequence of the \( q = 1 \) case above. \( \square \)

For the exceptional space \( (F_4, \alpha_4) \) since \( \lambda = 11/2 \) and \( \nu_1 = -5 \), the above result also holds. Hence it should not be surprising that the result in the corollary holds for the rest (22 of them total) exceptional Kähler C-spaces. The deformation rigidity result above only holds infinitesimally. It does not implies that for any deformation each fiber is biholomorphic to the central fiber as the main theorem of [25].
Assume that $\Omega$ is a nonzero element in $\mathbb{C}^B$ with rank bounded from above. Let $QB > 0$ symmetric. Within $\Omega$ of a Hermitian symmetric one with $p$ (cf. Theorem 5.4 in Chapter 3 of [19]). This shows that $\Omega$ can be decomposed into the sum $\Omega = \Omega_{p}^{1,1} + \Omega_{B}^{1,1}$ with $p$ positive for $QB$. To the condition $Ric$ one where $\Omega$ is harmonic, then $\partial \Omega = \bar{\partial} \Omega = 0$. It can be verified that $\Omega_{1}$ and $\Omega_{2}$ are both harmonic (cf. Theorem 5.4 in Chapter 3 of [19]). This shows that $\Omega$ can be decomposed into the sum of a Hermitian symmetric one with $- \sqrt{-1}$ times another Hermitian symmetric one. Namely $\mathcal{H}_{\Omega}^{1,1} = \mathcal{H}_{\Omega}^{1,1} - \sqrt{-1} \mathcal{H}_{\Omega}^{1,1}$, where $\mathcal{H}_{\Omega}^{1,1}$ is the spaces of harmonic $\Omega$ with $(A_{ij})$ being Hermitian symmetric. Within $\mathcal{H}_{\Omega}^{1,1}$ we consider $\mathcal{H}_{\Omega}^{1,1} \setminus \{C\omega\}$ to prove $b_{2} = 1$ under the assumption $QB > 0$, it suffices to show that $\mathcal{H}_{\Omega}^{1,1} \setminus \{C\omega\} = \{0\}$. We can stratify the space into ones with rank bounded from above. Let $\mathcal{H}_{\Omega}^{1,1}$ denote the subspace of $\mathcal{H}_{\Omega}^{1,1}$ which consists of $\Omega = \sqrt{-1} A_{ij} dz^{i} \wedge dz^{j}$ with $(A_{ij})$ being Hermitian symmetric and of rank no greater than $k$ everywhere on $N$. The following result can be shown.

**Theorem 6.1.** Assume that $(N, g)$ is a compact Kähler manifold with quasi-positive $QB_{k}$ with $k < n$. Then $\mathcal{H}_{\Omega, k}^{1,1}(N) = \{0\}$. In particular, $Ric^{+} > 0$ implies that $\mathcal{H}_{\Omega, 1}^{1,1}(N) = \{0\}$.

**Proof.** Assume that $\Omega$ is a nonzero element in $\mathcal{H}_{\Omega, k}^{1,1}(N)$. Applying the $\Delta$ operator to $\|\Omega\|^{2}$, by Kodaira-Bochner formula we have that

$$\frac{1}{2} \left(\nabla_{\gamma} \nabla_{\gamma} + \nabla_{\gamma} \nabla_{\gamma} \right) \|\Omega\|^{2}(x) = \|\nabla_{\gamma} \Omega\|^{2}(x) + \|\nabla_{\gamma} \Omega\|^{2}(x) + 2QB(\Omega)(x).$$

Integrating on $N$ we have that

$$0 = \int_{N} \left[\|\nabla_{\gamma} \Omega\|^{2}(x) + \|\nabla_{\gamma} \Omega\|^{2}(x) \right] d\mu(x) + 2 \int_{N} QB(\Omega)(x) d\mu(x) > 0.$$
The last strictly inequality due to that by the unique continuation we know at a neighborhood $U$ where $QB_k > 0$, $\Omega$ can not be identically zero. The contradiction implies that $\Omega \equiv 0$.

For any holomorphic line bundle $L$ over $N$ with a Hermitian metric $a$, its first Chern form $c_1(L, a) = -\frac{i}{2\pi} \partial \overline{\partial} \log a$ is a Hermitian symmetric $(1,1)$-form. If $\eta$ is the harmonic representative of $c_1(L, a)$, then $\eta$ is Hermitian symmetric by the uniqueness of the Hodge decomposition and Kähler identities (cf. [19], Chapter 3). The following is a simple observation towards possible topological meanings of the rank of $\eta$ (the minimum $k$ such that $\eta \in H_{\delta,k}^{1,1}$, denoted as $rk(L)$).

**Proposition 6.1.** Recall that the numerical dimension of $L$ is defined as

$$nd(L) = \max \{ k = 0, \ldots, n : c_1(L)^k \neq 0 \}.$$  

Then $rk(L) \geq nd(L)$.

The proof of the above theorem also shows that if $QB_k \geq 0$, then any element in $H_{s,k}^{1,1}(N)$ must be parallel. Thus we have the dimension estimate:

$$\dim(H_{s,k}^{1,1}(N)) \leq k^2.$$  

In fact the existence of a non-vanishing $(1, 1)$-form of rank at most $k$ has a strong implication due to the De Rham decomposition.

**Corollary 6.2.** Assume that $QB_k \geq 0$ and $H_{s,k}^{1,1}(N) \neq \{0\}$. Then $N$ must be locally reducible. In particular, if $N$ is locally irreducible and $Ric^+ \geq 0$, then $H_{s,k}^{1,1}(N) = \{0\}$.

**Proof.** By the above, we know that the nonzero $\Omega \in H_{s,k}^{1,1}(N)$ must be parallel. Its null space is invariant under the parallel transport. This provides a nontrivial parallel distribution, hence the local splitting. □

The product example $\mathbb{P}^2 \times \mathbb{P}^2$, which satisfies $Ric^+ > 0$ and supports non-trivial rank 2 harmonic $(1, 1)$-forms, shows that the above result is sharp for $Ric^+ > 0$. Irreducible examples of dimension greater than 4 were constructed via the projectivized bundles in [22].

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