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The fundamental group, rational connectedness and the positivity of Kähler manifolds

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Abstract. Firstly, we confirm a conjecture asserting that any compact Kähler manifold N with $\mathrm{Ric}^{\perp} > 0$ must be simply-connected by applying a new viscosity consideration to Whitney's comass of $(p, 0)$-forms. Secondly we prove the projectivity and the rational connectedness of a Kähler manifold of complex dimension n under the condition $\mathrm{Ric}_k > 0$ (for some $k \in \{1, \dots, n\}$, with Ric_n being the Ricci curvature), generalizing a well-known result of Campana, and independently of Kollár, Miyaoka and Mori, for the Fano manifolds. The proof utilizes both the above comass consideration and a second variation consideration of [40]. Thirdly, motivated by $\mathrm{Ric}^{\perp}$ and the classical work of Calabi and Vesentini [6], we propose two new curvature notions. The cohomology vanishing $H^q(N, T'N) = \{0\}$ for any $1 \le q \le n$ and a deformation rigidity result are obtained under these new curvature conditions. In particular, they are verified for all classical Kähler C-spaces with $b_2 = 1$. The new conditions provide viable candidates for a curvature characterization of homogeneous Kähler manifolds related to a generalized Hartshone conjecture.

1. Introduction

Kähler manifolds bridge the Riemannian manifolds, complex manifolds and complex algebraic manifolds. It avails analytic and geometric techniques in the study of algebraic manifolds via the GAGA principle. The first general result on the *projectivity* of a high dimensional Kähler manifold (N, h), i.e. being able to be realized as a holomorphic submanifold in some projective space $\mathbb{P}^K$, was obtained by Kodaira [22]. Kodaira proved that the projectivity is equivalent to the existence of an *integral* Kähler form ω_h in $H^2(N, \mathbb{Z})$. It was also shown that this cohomological condition is equivalent to the existence of a *positive line bundle* L. A line bundle is positive means that there exists a Hermitian metric h on L such that the Chern form of (L, h) is positive. From the Riemannian geometric point of view the most natural way of associating a line bundle to N is via its *canonical line bundle* ($K_N = \det((T'N)^*)$, the determinant bundle of the holomorphic tangent bundle $(T'N)^*$) and *the anti-canonical line*

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bundle (K_N^{-1}). The associated intrinsic curvature (i.e. $c_1(M)$, the Chern form of K_N^{-1}) is the *Ricci* curvature of (N, h). Compact Kähler manifolds with positive Ricci curvature form a special class of smooth projective/algebraic varieties, i.e. the *Fano manifolds*. Its study and the extension to varieties with various singularities have been one of active focuses of the algebraic geometry during last decades. In this paper we study a family of *intrinsic* curvature conditions (generalizing the Ricci curvature), whose positivity implies *the projectivity and the rational connectedness* of a compact Kähler manifold.

Rational connectedness is an important/useful property for algebraic manifolds [12]. For compact Kähler manifolds with positive Ricci curvature this property was established by Campana [7], Kollár, Miyaoka and Mori [24]. In this paper we show:

Theorem 1.1. *Let (N^n, h) be a compact Kähler manifold with $\mathrm{Ric}_k > 0$ for some $1 \le k \le n$. Then N is projective and rationally connected. In particular, $\pi_1(N) = \{0\}$.*

The Ric_k is defined as the Ricci curvature of the k-dimensional holomorphic subspaces of the holomorphic tangent bundle $T'N$. Hence it coincides with the holomorphic sectional curvature $H(X)$ when $k = 1$, and with the Ricci curvature when $k = n = \dim_{\mathbb{C}}(N)$. The condition $\mathrm{Ric}_k > 0$ is significantly different from its Riemannian analogue, i.e. the so-called q-Ricci (see next section for details), since it exams only the holomorphic subspaces in $T'N$, thus unlike its Riemannian analogue, $\mathrm{Ric}_k > 0$ does not imply $\mathrm{Ric}_{k+1} > 0$; $\mathrm{Ric}_k > 0$ means that every infinitesimal k-dimensional holomorphic subvariety is Fano. The notion of Ric_k was initiated in a recent study of the k-hyperbolicity of a compact Kähler manifold by the author [34]. It is closely related to the degeneracy of holomorphic mappings from $\mathbb{C}^k$ into concerned manifolds (cf. [34, Theorem 1.3]). The condition $\mathrm{Ric}_k > 0$ allows some negativity of (holomorphic) sectional curvature if $k > 1$. Note that all Hirzebruch surfaces (and generalized Hirzebruch manifolds) admit Kähler metric with $\mathrm{Ric}_1 > 0$. This contracts sharply with the Fano condition of $\mathrm{Ric} > 0$. The class of manifolds with $\mathrm{Ric}_k > 0$ (for $k < n$) contains many non-Fano examples. It remains interesting to find for what k the class $\mathrm{Ric}_k > 0$ has finite deformation connected components (cf. [24, 31] for the Fano case).

The proof of Theorem 1.1 is completely different from that of [7, 16, 24]. Here it is built upon recent techniques of applying the (partial) maximum principle via the viscosity consideration developed by the author in [33, 34]. It is in the proof of the projectivity, Whitney's comass (amounts to an operator norm) is employed to localize the problem. The proof of rational connectedness also needs a second variation consideration of author (with F. Zheng [40]) to obtain some desired estimates in the fiber direction of the flags in $T'N$. The proof also uses a recent result of [8].

The literature on the fundamental group $\pi_1(N)$ of a Kähler manifold N is big (cf. [2, 33] and references there). A result of Kobayashi [20] asserts that a compact Kähler manifold with $\mathrm{Ric} > 0$ must be simply-connected. Same conclusion was proved by Tsukamoto [45] for compact Kähler manifold with $H > 0$. The next result of this paper provides an analogue of Kobayashi's and Tsukamoto's theorems for Kähler manifolds with $\mathrm{Ric}^{\perp} > 0$.

Theorem 1.2. *A compact Kähler manifold N with $\mathrm{Ric}^{\perp} > 0$ must be simply-connected.*

The result was conjectured in the paper [37]. In [37], the authors – motivated by the Laplace comparison theorem and the holomorphic Hessian comparison theorem – studied the *orthogonal Ricci curvature*

$$\mathrm{Ric}^{\perp}(X, \overline{X}) \doteqdot \mathrm{Ric}(X, \overline{X}) - \frac{R(X, \overline{X}, X, \overline{X})}{|X|^2}$$

for any type $(1,0)$ tangent vector X. Here $R(X, \overline{X}, X, \overline{X})$ is the *holomorphic sectional curvature* (denoted as $H(X)$), which is the Gauss curvature of the infinitesimal curve tangent to X. For a compact Kähler manifold N^n ($n = \dim_{\mathbb{C}}(N)$), with $\mathrm{Ric}^{\perp} > 0$ everywhere, its projectivity was shown in [37], via a uncommon unitary congruence normal form for $(2,0)$-forms.[1)] It was also proved in [37] that $|\pi_1(N)| < \infty$ in general, and $\pi_1(N) = \{0\}$ for $n = 2, 3, 4$.

Unlike Ric, $\mathrm{Ric}^{\perp}(X, \overline{X})$ does not come from a Hermitian symmetric sesquilinear form. But it can be viewed as the holomorphic sectional curvature of a Bochner curvature operator (namely the curvature operator which arises in the standard Bochner formula computing the Laplacian of the square of the norm of two forms, cf. [36, 37]). Despite this close connection with the holomorphic sectional curvature, the proof of Theorem 1.2 follows the scheme of [20] (for $\mathrm{Ric} > 0$) via a Riemann–Roch–Hirzebruch formula and a vanishing theorem on Hodge numbers $h^{p,0}$. The vanishing of $h^{p,0}$ for all $1 \le p \le n$ needed in [20] is known by Kodaira's vanishing theorem. However, the proof of $h^{p,0} = 0$ for all $1 \le p \le n$ under $\mathrm{Ric}^{\perp} > 0$ (cf. Theorem 2.2) requires a completely new idea which involves a novel use of a viscosity consideration of Whitney's comass. The effective method also plays an important role in the proof of the above rational connectedness result. Note that our proof applies to the case of $H > 0$ (implying Tsukamoto's result). It provides a unified argument for all cases of $\{\mathrm{Ric}_k > 0\}$, $k \in \{1, \dots, n\}$, and $\mathrm{Ric}^{\perp} > 0$ with additional information $h^{p,0} = 0$ for all $1 \le p \le n$.

The study of $\mathrm{Ric}^{\perp} > 0$ in the papers [36, 39] is also motivated by the so-called generalized Hartshorne conjecture (cf. [9, Conjectures 11.1 and 11.2] and [53, Conjecture 8.23]): *A Fano manifold has nef tangent bundle if and only if it is a Kähler C-space.* The first curvature notion one naturally would like to associate with the nefness condition is the so-called almost nonnegativity of bisectional curvature. However, it has been proved recently that the almost nonnegativity of the bisectional curvature [3] implies that the manifolds are diffeomorphic to compact quotients of Hermitian symmetric spaces, provided the volume is non-collapsing. A recent work of the author with X. Li [27] extends this to the (weaker) almost nonnegativity of the orthogonal bisectional curvature. In [47] a curvature positivity notion, namely the quadratic orthogonal bisectional curvature QB (cf. (2.2) for its definition) was proposed (by Wu, Yau and Zheng) for the purpose of a curvature characterization of the Kähler C-spaces. Unfortunately, as shown in [11], it is a bit off the target since only for about eighty percentage of classical Kähler C-spaces with 2nd Betti number $b_2 = 1$ (with the canonical Kähler–Einstein metric) have $\mathrm{QB} > 0$, while for the rest twenty percent manifolds $\mathrm{QB} < 0$ somewhere. As a step back, the positivity of $\mathrm{Ric}^{\perp}$ was studied in [39] for the purpose of the curvature characterization of C-spaces, since on all classical C-spaces with $b_2 = 1$ the canonical Kähler–Einstein metrics satisfy $\mathrm{Ric}^{\perp} > 0$ (see [36]). Further studies of compact Kähler manifolds with $\mathrm{Ric}^{\perp} > 0$ were carried in a recent work [36]: A complete classification for threefolds, a partial classification for fourfolds, and a Frankel-type result were obtained for compact Kähler manifolds with $\mathrm{Ric}^{\perp} > 0$ in [36]. Many examples were also constructed in [36, 37] illustrating

1) Due to E. Cartan implicitly (L.-G. Hua explicitly in a paper of 1945), cf. [37, p. 151]. This partially contributes to the gap between the first appearance of $\mathrm{Ric}^{\perp}$ [28] and meaningful results in [36, 37, 39].

that $\mathrm{Ric}^{\perp}$, H, and Ric are completely independent except the trivial relation

$$\mathrm{Ric}(X, \overline{X}) = \mathrm{Ric}^{\perp}(X, \overline{X}) + \frac{H(X)}{|X|^2}.$$

On the other hand, except for dimension $n = 2, 3$ (and $n = 4$ if one is optimistic) [36], $\mathrm{Ric}^{\perp} > 0$ appears not enough to imply that the manifold is a Kähler C-space. This is partially reflected by the flexible constructions of metrics with $\mathrm{Ric}^{\perp} > 0$ on fiber bundles over a positively curved Kähler manifold in [36].

At the same time, motivated by the local rigidity theorem of Calabi and Vesentini [6], the above relation between $\mathrm{Ric}^{\perp}$ and the generalized Hartshone conjecture, we introduce two stronger (than $\mathrm{Ric}^{\perp}$) notions of intrinsic curvature positivity, namely the *cross quadratic bisectional curvature* (abbreviated as $\mathrm{CQB} > 0$) and its dual ${}^{d}\mathrm{CQB}$ (cf. (2.3) and (2.4)) in this paper. The defining expressions appear similar to the quadratic orthogonal bisectional curvature. However, a sharp contrast is that the positivity of CQB and its dual can be verified for all classical Kähler C-spaces with $b_2 = 1$ (cf. Theorems 5.4 and 6.7). Results as initial studies of these two notions of curvature in this paper includes:

(1) $\mathrm{CQB} > 0$ implies $\mathrm{Ric}^{\perp} > 0$.

(2) The projectivity and simply-connectedness of manifolds with $\mathrm{CQB} > 0$ or ${}^{d}\mathrm{CQB} > 0$ (cf. Theorem 2.7).

(3) A deformation rigidity result for manifolds with quasi-positive ${}^{d}\mathrm{CQB}$ (or quasi-negative CQB).

Since there are non-locally Hermitian symmetric manifolds with $\mathrm{CQB} < 0$, it follows that (3) generalizes the result of Calabi and Vesentini [6]. Utilizing the Kähler–Ricci flow, the Fanoness was proved under the assumptions of $\mathrm{CQB} \geq 0$ (or ${}^{d}\mathrm{CQB} \geq 0$) and the finiteness of $\pi_1(N)$ recently in [38] joint with F. Zheng. In particular, $\mathrm{CQB} > 0$ (or ${}^{d}\mathrm{CQB} > 0$) implies that N is Fano. Hence there is a good chance that one of these two curvature notions can provide the curvature characterization of the Kähler C-spaces. Tracing ${}^{d}\mathrm{CQB}$ (CQB, respectively) leads to a related notion of Ricci curvature, namely Ric^{+} ($\mathrm{Ric}^{\perp}$, respectively). We also show that a compact Kähler manifold with $\mathrm{Ric}^{+} > 0$ is projective and simply-connected. It is also proved in this paper that any compact Kähler manifolds with quasi-positive $\mathrm{Ric}^{\perp}$ and of Picard number one must be Fano. However, the rational connectedness of manifolds with $\mathrm{Ric}^{\perp} > 0$ remains unknown. Since the condition $\mathrm{Ric}^{\perp} > 0$ allows arbitrarily large b_2 (cf. [38] for examples of Type A Kähler C-spaces with $\mathrm{CQB} \geq 0$, $\mathrm{Ric}^{\perp} > 0$, ${}^{d}\mathrm{CQB} > 0$, and with arbitrarily large b_2), the implications of $\mathrm{Ric}^{\perp} > 0$ on the dimension of certain harmonic $(1,1)$-forms is included in the appendix.

The well-known curvature notions for Kähler manifolds include the *sectional curvature*, the *bisectional curvature* $B(X, Y) \doteqdot R(X, \overline{X}, Y, \overline{Y})$, and the *holomorphic sectional curvature* $H(X)$ mentioned above. For Hermitian manifolds there is Griffiths' positivity [15]. Restricted to Kähler manifolds, it is the same as $B > 0$. Various positivity notions in algebraic geometry are discussed in the excellent books of Lazarsfeld [25, 26]. However, the positivity (even the nonnegativity) of bisectional curvature is rather restrictive for compact Kähler manifolds since Mori's solution of the Hartshorne conjecture [29] asserts that if $T'N$ is ample, the complex manifold $N = \mathbb{P}^n$, the complex projective space. In particular, since $B > 0$ implies that $T'N$ is ample (cf. [26, Theorem 6.1.25]), Mori's result implies that the only compact Kähler manifold with $B > 0$ is $\mathbb{P}^n$ (cf. [43], for an independent Kähler geometric proof of Siu and Yau).

Related to results of this paper, the projectivity of compact Kähler manifolds with $S_2 > 0$ was recently proved in [40], generalizing an earlier result of [50] under the stronger assumption $H > 0$. A general (maybe the most general possible) result on a weaker criterion of the projectivity can be found in Corollary 4.4 (i.e. any Kähler manifold of BC-2 positive curvature is projective). The vanishing of $h^{p,0}$ in Theorems 1.1 and 1.2 is also extended to broader cases in Theorem 6.3. For algebraic manifolds with $H > 0$ the rational connectedness was proved in [16] (cf. also [51]). Recent work [33] also contains results on the fundamental groups of compact Kähler manifolds with $\mathrm{Ric}_k \geq 0$, in particular $H \geq 0$. We hope that this paper serves an introduction to relatively new notions of positivity concerning the intrinsic metrics, namely Ric_k, $\mathrm{Ric}_k^\perp$, CQB_k, as well as their dual Ric_k^+, ${}^d\mathrm{CQB}_k$, for Kähler manifolds. One can find many questions/open problems, and examples, in later sections of this paper.

2. Definitions and statements of results

We start with the following conjecture proposed in [37, Conjecture 1.6].

Conjecture 2.1. Let N^n ($n \geq 2$) be a compact Kähler manifold with $\mathrm{Ric}^\perp > 0$ everywhere. Then for any $1 \leq p \leq n$, there is no nontrivial global holomorphic p-form, i.e. the Hodge number $h^{p,0} = 0$. In particular, N^n is simply-connected.

The conjecture was confirmed for $n = 2, 3, 4$ in [37] following a general scheme of Kobayashi. As illustrated in [37], the "in particular" part, namely the simply-connectedness of compact Kähler manifolds, would follow from Hirzebruch's Riemann–Roch formula [17] as follows: Letting $\mathcal{O}_N$ be the structure sheaf, the Euler characteristic number

$$\chi(\mathcal{O}_N) \doteqdot 1 - h^{1,0} + h^{2,0} - \cdots + (-1)^n h^{n,0}$$

satisfies that

$$\chi(\mathcal{O}_{\widetilde{N}}) = \nu \cdot \chi(\mathcal{O}_N)$$

by the Riemann–Roch–Hirzebruch formula if $\widetilde{N}$ is a finite ν-sheets covering of N. On the other hand, the vanishing of all Hodge numbers $h^{p,0}$ for $1 \leq p \leq n$ (which is the main part of the conjecture) asserts that $\chi(\mathcal{O}_N) = 1$ for both N and $\widetilde{N}$ if $\widetilde{N}$ is compact and of $\mathrm{Ric}^\perp > 0$ (hence projective). This forces $\nu = 1$, hence $\pi_1(N) = \{0\}$. Note that the universal cover $\widetilde{N}$ satisfies $\mathrm{Ric}^\perp > \delta > 0$. Hence $\widetilde{N}$ is compact and projective by [37, Theorem 3.2]. This argument was the one used in [20, 37] proving the simply-connectedness of a Fano manifold, and for $n = 2, 3, 4$ with $\mathrm{Ric}^\perp > 0$.

In this paper we prove Conjecture 2.1 for all $n \geq 2$ by a stronger result, namely the vanishing of $h^{p,0}$ under a weaker curvature condition related to p. First we recall this condition (cf. [37, Section 4]). Let Σ be a k-subspace $\Sigma \subset T'_x N$. Let

$$⨍ f(Z)\, d\theta(Z) \doteqdot \frac{1}{\mathrm{Vol}(\mathbb{S}^{2k-1})} \int_{\mathbb{S}^{2k-1}} f(Z)\, d\theta(Z),$$

where $\mathbb{S}^{2k-1}$ is the unit sphere in Σ. Define

$$S_k^\perp(x, \Sigma) \doteqdot k ⨍_{Z \in \Sigma, |Z|=1} \mathrm{Ric}^\perp(Z, \overline{Z})\, d\theta(Z). \tag{2.1}$$

Similarly, $S_k(x,\Sigma)$, the k-scalar curvature of Σ, can be defined by replacing $\mathrm{Ric}^\perp(Z,\overline{Z})$ with $\mathrm{Ric}(Z,\overline{Z})$ in (2.1). Let $S_k^\perp(x) \doteqdot \inf_\Sigma S_k^\perp(x,\Sigma)$. Thus $S_k^\perp(x)>0$ if and only if for any k-subspace $\Sigma \subset T_x'M$, $S_k^\perp(x,\Sigma)>0$. The condition $S_k^\perp(x)>0$, $k\in\{1,\dots,n\}$, interpolate between $\mathrm{Ric}^\perp(X,\overline{X})$ and $\frac{n-1}{n+1}S(x)$ (see Lemma 5.1). It is easy to see that $S_\ell^\perp>0$ implies $S_k^\perp>0$ for $k\geq \ell$. And it is not hard to prove that (cf. (3.2))

$$S_k^\perp(x,\Sigma) = \big(\mathrm{Ric}(E_1,\overline{E}_1)+\mathrm{Ric}(E_2,\overline{E}_2)+\cdots+\mathrm{Ric}(E_k,\overline{E}_k)\big) - \frac{2}{(k+1)}S_k(x,\Sigma).$$

Note that the corresponding collection of k-scalar curvatures $\{S_k(x): k=1,\dots,n\}$ interpolates between the holomorphic sectional curvature $H(X)$ and the scalar curvature $S(x)$. Equation (2.1) in particular implies that $S_n^\perp(x)=\frac{n-1}{n+1}S(x)$. The first theorem of this paper proves that $h^{p,0}=0$ for all $p\geq k$ if $S_k^\perp>0$ or $S_k>0$. Conjecture 2.1 follows since $\mathrm{Ric}^\perp>0$ implies that $S_k^\perp>0$ for all $k\in\{1,\dots,n\}$.

Theorem 2.2. *Let (N,g) be a compact Kähler manifolds such that $S_k^\perp(x)>0$ for any $x\in N$. Then $h^{p,0}=0$ for any $p\geq k$. The same result holds if $S_k>0$. In particular, if* $\mathrm{Ric}^\perp>0$ *(or $H>0$), then $h^{p,0}(N)=0$ for all $1\leq p\leq n$, and N is simply-connected.*

The part $h^{2,0}=0$ was proved in [37,40] under the assumption $S_2(x)>0$. The argument there is limited to $p=2$ since one has to use a normal form for $(2,0)$-forms. The proof here uses a different idea which is developed recently in [34] to prove a new Schwarz Lemma by the author. We recall that idea first before explaining the related details. Starting from the work of Ahlfors, the Schwarz Lemma concerns estimating the gradient of a holomorphic map f between two Kähler (or Hermitian) manifolds (M^m,h) and (N^n,g). For that it is instrumental to study the pull-back $(1,1)$-form $f^*\omega_g$, where ω_g is the Kähler form of (N,g). The traditional approach (before the work of [34]) is to compute the Laplacian of the trace of $f^*\omega_g$. But in [34], the author estimated the largest singular value of df, equivalently the biggest eigenvalue of $f^*\omega_g$, by applying the $\partial\overline{\partial}$-operator to the maximum eigenvalue of $f^*\omega_g$ (which is only continuous in general) via a viscosity consideration. It allows the author to prove another *natural* generalization of Ahlfors' result with a sharp estimate on the largest singular value of df in terms of the holomorphic sectional curvatures of both the domain and target manifolds. This estimate can be viewed as a complex version of Pogorelov's estimate for solutions of the Monge–Ampère equation [41]. To prove the vanishing of holomorphic $(p,0)$-forms under the assumption of $\mathrm{Ric}^\perp>0$, in Section 3 we apply the $\partial\overline{\partial}$-operator on the *comass* of holomorphic $(p,0)$-forms (cf. [13,46]), through a similar viscosity consideration. The comass of a $(p,0)$-form generalizes the biggest singular value of df in some sense since one can view df as a vector valued $(1,0)$-form. It can be done thanks to some basic properties of the comass established by Whitney [46]. This new idea also allows us to prove a generalization of the main theorem in [40] (cf. Corollary 4.4).

By combining this new idea with the work of [40], in Section 4, we prove the projectivity and the rational connectedness of compact Kähler manifolds under the condition $\mathrm{Ric}_k>0$, i.e. Theorem 1.1. The notion Ric_k was introduced in [34] to prove that *any Kähler manifold with* $\mathrm{Ric}_k<0$ *uniformly must be k-hyperbolic*, a concept generalizing the Kobayashi hyperbolicity (which amounts to 1-hyperbolic). Let $\mathrm{Ric}_{k,-}(x)\doteqdot \inf_{\Sigma,v\in\Sigma,|v|=1}\mathrm{Ric}_k(x,\Sigma)(v,\overline{v})$ and similarly $\mathrm{Ric}_{k,+}(x)\doteqdot \sup_{\Sigma,v\in\Sigma,|v|=1}\mathrm{Ric}_k(x,\Sigma)(v,\overline{v})$. Here $\mathrm{Ric}_k(x,\Sigma)$ is the Ricci curvature of the curvature tensor R restricted to $\Sigma\subset T_xN$. Thus $\mathrm{Ric}_{k,-}(x)>\lambda(x)$ if and only if

$\mathrm{Ric}_k(x, \Sigma)(v, \bar{v}) > \lambda |v|^2$, for any $v \in \Sigma$ and for every k-dimensional subspace Σ. We say N has positive Ric_k if $\mathrm{Ric}_{k,-}(x) > 0$ for all $x \in N$. The condition $\mathrm{Ric}_1 > 0$ is equivalent to that the holomorphic sectional curvature $H > 0$. For $k = n$, Ric_k is the Ricci curvature. By [1, 18] $\mathrm{Ric}_k > 0$ is independent from $\mathrm{Ric}_\ell > 0$ for $k \neq \ell$ (cf. also [37, 49] for more examples). The known examples of manifolds with $\mathrm{Ric}_k > 0$ for $k \neq n$ contain mostly *non-Fano* manifolds. It remains interesting to find out for what k and n the deformation types of manifolds with $\mathrm{Ric}_k > 0$ is finite.

As in [37, 40] the projectivity only needs $h^{2,0} = 0$. In Theorem 4.2 we show a stronger vanishing result: $h^{p,0} = 0$ *for any* $1 \leq p \leq n$ *under the assumption that* $\mathrm{Ric}_k > 0$ *for some* $1 \leq k \leq n$. (In Theorem 6.3 this result is extended to $\mathrm{Ric}_k^\perp > 0$ and $\mathrm{Ric}_k^+ > 0$). The rational connectedness is proved in Section 4 by another vanishing theorem, whose validity is a criterion of the rational connectedness, thanks to [8]. Both the second variation estimate from [40] and the one utilizing the comass for $(p, 0)$-forms introduced in Section 3 of this paper are crucial in proving these two vanishing theorems. Theorem 1.1 generalizes both the result for Fano manifolds [7, 24] (the case $k = n$, namely the Fano case of Campana, and Kollár, Miyaoka and Mori), and the more recent result for the compact Kähler manifolds with positive holomorphic sectional curvature [16] by Heier and Wong (cf. also [50] for the projectivity for the case $k = 1$), since $\mathrm{Ric}_1 > 0$ amounts to $H > 0$ and $\mathrm{Ric}_n = \mathrm{Ric}$. It is not clear if $\mathrm{Ric}_k > 0$ has anything to do with that Ricci curvature is k-positive in general. When $k = 1$, Hitchin's examples show that they are independent. However Ric_k is related to the notion of q-Ricci studied in Riemannian geometry which interpolates the Ricci and the sectional curvature. In particular, it is the complex analogue of q-Ricci and if the $(2k - 1)$-Ricci is positive in the sense of Bishop and Wu [4, 48], then $\mathrm{Ric}_k > 0$. The positivity of the $(2k - 1)$-Ricci is a much stronger condition than $\mathrm{Ric}_k > 0$ since it requires the Ricci being positive on all $2k$-dimensional subspaces of the (complexified) tangent space $T_x N$ for all $x \in N$. This makes that the positivity of the p-Ricci implies the positivity of q-Ricci if $q \geq p$. On the other hand since most of $2k$-dimensional (real) subspaces of $T_x N \otimes \mathbb{C}$ are neither invariant under the almost complex structure, nor subspaces of $T'N$, $\mathrm{Ric}_k > 0$ is a lot weaker than $(2k - 1)$-Ricci being positive. A major difference is that $\mathrm{Ric}_k > 0$ does not imply $\mathrm{Ric}_{k+1} > 0$, unlike the q-Ricci positivity condition.

In Section 5 we study the question when a compact Kähler manifold with $\mathrm{Ric}^\perp > 0$ is Fano, a question raised in [37]. We give an affirmative answer under an extra assumption.

Theorem 2.3. *Let* (N, h) *be a compact Kähler manifold of complex dimension* n. *Then:*

(i) *If* $\mathrm{Ric}^\perp$ *is quasi-positive (namely* $\mathrm{Ric}^\perp \geq 0$ *everywhere and* $\mathrm{Ric}^\perp > 0$ *somewhere) and the Picard number* $\rho(N) = 1$, *then* N *must be Fano.*

(ii) *If* $\mathrm{Ric}^\perp$ *is quasi-negative and* $h^{1,1}(N) = 1$, *then* N *must be projective with ample canonical line bundle* K_N.

In particular, in case (i) N *admits a Kähler metric with positive Ricci, and case* (ii) N *admits a Kähler–Einstein metric with negative Einstein constant.*

Since it was proved in [37] that N is projective and

$$h^{1,0}(N) = h^{2,0}(N) = 0 = h^{0,2}(N) = h^{0,1}(N)$$

under the assumption that $\mathrm{Ric}^\perp > 0$, the assumption of $\rho(N) = 1$ for case (i) is equivalent to the assumption that the second Betti number $b_2 = 1$. We should mention that in [36], it has been shown that for all Kähler C-spaces of classical type with $b_2 = 1$ the canonical Kähler–Einstein metric satisfies $\mathrm{Ric}^{\perp} > 0$.

To put Theorem 2.3 into perspectives, it is appropriate to recall some earlier works. First related to $\mathrm{Ric}^{\perp} \geq 0$ there exists a stronger condition called the *nonnegative quadratic orthogonal bisectional sectional curvature*, studied by various people including authors of [47] and [10], etc. The *quadratic orthogonal bisectional curvature* (abbreviated as QB), is defined for any real vector $\vec{\mathbf{a}} = (a_1, \dots, a_n)^{\mathrm{tr}}$ and any unitary frame $\{E_i\}$ of $T'_x N$,

$$\mathrm{QB}(\vec{\mathbf{a}}) = \sum_{i,j} R_{i\bar{i}j\bar{j}}(a_i - a_j)^2.$$

In [35] it was formulated invariantly as a quadratic form on the space of Hermitian symmetric tensors. Precisely for symmetric tensor A,

$$\mathrm{QB}_R(A) \doteqdot \langle R, A^2 \barwedge \mathrm{id} - A \barwedge A \rangle.$$

Interested readers can refer to [35] for the notations involved. For any unitary orthogonal frame of $T'N$, $\{E_\alpha\}$, we adapt

$$\mathrm{QB}_R(A) \doteqdot \sum_{\alpha,\beta=1}^{n} R(A(E_\alpha), \overline{A(E_\alpha)}, E_\beta, \overline{E}_\beta) - R(E_\alpha, \overline{E}_\beta, A(E_\beta), \overline{A(E_\alpha)}). \tag{2.2}$$

Clearly, it is independent of the choice of the unitary frame. Its nonnegativity, abbreviated as NQOB, is equivalent to that $\mathrm{QB}(\vec{\mathbf{a}}) \geq 0$ for any $\vec{\mathbf{a}}$ with respect to any unitary frame $\{E_i\}$. NQOB was formally introduced in [47] (appeared implicitly in the work of Bishop and Goldberg in 1960s). It is easy to see that $\mathrm{QB} > 0$ implies $\mathrm{Ric}^{\perp} > 0$.[2)] In [10] the following was proved by Chau and Tam in [10, Theorem 4.1]:

Theorem 2.4 (Chau–Tam). *Let (N, h) be a compact Kähler manifold with NQOB and $h^{1,1}(N) = 1$. Assume further that N is locally irreducible then $c_1(M) > 0$.*

Theorem 2.3 has the following corollary, which extends the above result.

Corollary 2.5. *Let (N, h) be a compact Kähler manifold of complex dimension n with* $\mathrm{Ric}^{\perp} \geq 0$. *Assume further that $h^{1,1}(N) = 1$ and N is locally irreducible. Then $c_1(N) > 0$, namely N is Fano. A similar result holds under the assumption* $\mathrm{Ric}^{\perp} \leq 0$.

There exists compact Kähler manifolds with $b_2 > 1$ (cf. construction in [36] via projectivized bundles) and $\mathrm{Ric}^{\perp} > 0$. Hence it remains an interesting question whether or not the same conclusion of Theorem 2.3 (i) holds without the assumption $h^{1,1} = 1$. Since the quasi-positivity of QB implies that $h^{1,1} = 1$, as a consequence we have that *any compact Kähler manifold with quasi-positive QB must be Fano*. Whether or not the same conclusion of part (ii) of Theorem 2.3 holds without assuming that $h^{1,1} = 1$ remains open.

As mentioned in Section 1, motivated by the relation of the condition $\mathrm{Ric}^{\perp} > 0$ with the generalized Hartshone conjecture and the work of Calabi and Vesentini we introduce the *cross quadratic bisectional curvature* (abbreviated as CQB) as a Hermitian quadratic form on the

2) Motivated by the work of Calabi and Vesentini [6], we introduce the so-called *cross quadratic bisectional curvature* (abbreviated as CQB), another (quadratic form-type) curvature, whose positivity also implies $\mathrm{Ric}^{\perp} > 0$.

space of linear maps $A : T''_x N \to T'_x N$:

$$\text{(2.3)}\quad \mathrm{CQB}_R(A) \doteqdot \sum_{\alpha,\beta=1}^{n} R(A(\overline{E}_\alpha), \overline{A(\overline{E}_\alpha)}, E_\beta, \overline{E}_\beta) - R(E_\alpha, \overline{E}_\beta, A(\overline{E}_\alpha), \overline{A(\overline{E}_\beta)}).$$

Here $\{E_\alpha\}$ is a unitary frame of $T'_x N$. It is easy to see that $\mathrm{CQB}(A)$ is independent of the choice of the unitary frame. Dually, the *dual cross quadratic bisectional curvature* (${}^d\mathrm{CQB}$) is defined as a Hermitian quadratic form on linear maps $A : T'N \to T''N$:

$$\text{(2.4)}\quad {}^d\mathrm{CQB}_R(A) \doteqdot \sum_{\alpha,\beta=1}^{n} R(\overline{A(E_\alpha)}, A(E_\alpha), E_\beta, \overline{E}_\beta) + R(E_\alpha, \overline{E}_\beta, \overline{A(E_\alpha)}, A(E_\beta)).$$

The advantage of these two curvature notions over QB is demonstrated by

Theorem 2.6. *The following statements hold.*

(i) *Let N^n be a classical Kähler C-space with $n \geq 2$ and $b_2 = 1$. Then the canonical metric satisfies* $\mathrm{CQB} > 0$ *and* ${}^d\mathrm{CQB} > 0$.

(ii) *For a compact Kähler manifolds with quasi-positive* ${}^d\mathrm{CQB}$ *(or quasi-negative CQB), $H^q(N, T'N) = \{0\}$, for $1 \leq q \leq n$, and N is deformation rigid in the sense that it does not admit nontrivial infinitesimal holomorphic deformation. In particular, the deformation rigidity holds for all classical Kähler C-spaces with $b_2 = 1$.*

The proof uses the results of [6,11,19,36]. We also use the criterion of Frölicher and Nijenhuis [14,23] for the deformation rigidity statement. The key is a Kodaira–Bochner formula (cf. [6]) and the role of a curvature notion ${}^d\mathrm{CQB}$ (*dual-cross quadratic bisectional curvature*) played in the Kodaira–Bochner formulae. The rigidity result on Kähler C-spaces can possibly be implied by a result of Bott [5]. Here it follows from a general vanishing theorem for manifolds with ${}^d\mathrm{CQB} > 0$. Hence before a complete classification of Kähler manifolds with ${}^d\mathrm{CQB} > 0$ (cf. [38] for a precise conjecture related to this) the rigidity result above for N with ${}^d\mathrm{CQB} > 0$ is a more general statement. One can refer Sections 5 and 6 for further motivations and detailed discussions on these two new curvatures.

The new dual-cross quadratic bisectional curvature ${}^d\mathrm{CQB}$ naturally induces a Ricci-type curvature (in a similar manner as QB and CQB induces $\mathrm{Ric}^\perp$.) It is denoted by Ric^+, and is defined, for any $X \in T'_x N$, as

$$\mathrm{Ric}^+(X, \overline{X}) = \mathrm{Ric}(X, \overline{X}) - H(X)/|X|^2$$

In Section 6, for the Kähler manifolds with $\mathrm{Ric}^+ > 0$ we have the following result similar to the $\mathrm{Ric}^\perp > 0$ case.

Theorem 2.7. *Let (N, h) be a complete Kähler manifold with* $\mathrm{Ric}^+ \geq \delta > 0$ *(or replaced with any one of* $\{\mathrm{Ric}_k \geq \delta, \mathrm{Ric}_k^\perp \geq \delta, \mathrm{Ric}_k^+ \geq \delta\}$*). Then:*

(i) *N is compact,*

(ii) *$h^{p,0} = 0$ for all $n \geq p \geq 1$.*

In particular, N is simply-connected and N is projective. Since ${}^d\mathrm{CQB} > 0$ *implies* $\mathrm{Ric}^+ > 0$*, this applies to compact manifolds with* ${}^d\mathrm{CQB} > 0$.

The proof of the above result again makes use of the method via a viscosity consideration on the comass (introduced in Section 3) and follows a similar line of argument as the proof of Theorem 2.2. In Section 6 we also prove a diameter estimate and a result similar to Corollary 2.5 for Ric^+.

The *cross quadratic bisectional curvature* and its dual ${}^d\mathrm{CQB}$ are shown positive on some *exceptional Kähler C-spaces* too. Since $\mathrm{CQB} > 0$ (as $\mathrm{QB} > 0$) implies $\mathrm{Ric}^\perp > 0$, Theorem 2.6 generalizes the result of [36]. On the other hand it was shown by Chau and Tam [11] that $\mathrm{QB} > 0$ fails to hold for all Kähler C-spaces with $b_2 = 1$, and it was shown in [36] that there exists a non-homogeneous compact Kähler manifold with $\mathrm{Ric}^\perp > 0$. Hence one of these two new curvature notions will more likely give a curvature characterization of the compact Kähler C-spaces with $b_2 = 1$. Towards this direction we prove (in Theorem 5.3) that *a compact Kähler manifold with* $\mathrm{CQB} > 0$ *must be rationally connected.* This also follows from [7, 24] and the statement that $\mathrm{CQB} > 0$ (or ${}^d\mathrm{CQB} > 0$) implies that N is Fano (cf. [38]). More ambitious project is to apply these curvatures to tackle the generalized Hartshorne conjecture concerning the Fano manifolds with a nef tangent bundle (cf. conjectures formulated in [38]). We also calibrate QB, CQB and ${}^d\mathrm{CQB}$ into QB_k, CQB_k or ${}^d\mathrm{CQB}_k$ with $k \in \{1, \dots, n\}$ to bridge them with $\mathrm{Ric}^\perp$ and Ric^+.

In the appendix, we study the gap in terms of vanishing theorems between $\mathrm{QB} > 0$ and $\mathrm{Ric}^\perp > 0$. Most results in this paper can be adapted to Hermitian manifolds without much difficulty, if the notions of involved curvatures are properly extended.

3. Comass and the proof of Theorem 2.2

In [34] and [33] we developed a viscosity technique to apply a maximum principle to the operator norm of the differential of a holomorphic map. Here we extend the idea to differential forms. The *comass* introduced by Whitney fits our need quite well. We start with a brief summary of its properties. Let V be a Euclidean space. An r-(multi-)vector $\mathbf{a}$ is an element of $\bigwedge_r V$, namely the space of r-multi-linear skew symmetric forms on V^* (the dual of V). Here we identify V and V^* via the inner product when needed. A vector $\mathbf{a}$ is called simple if there exist $v_1, \dots, v_r \in V$ such that

$$\mathbf{a} = v_1 \wedge \cdots \wedge v_r.$$

This can be defined for r-covector ω similarly. For an r-covector ω the comass is defined in [46] as

$$\|\omega\|_0 \doteqdot \sup\{|\omega(\mathbf{a})| : \mathbf{a} \text{ is a simple } r\text{-vector}, \|\mathbf{a}\| = 1\}.$$

Here the norm $\|\cdot\|$ is the norm (an L^2-norm in some sense) induced by the inner product defined for simple vectors $\mathbf{a} = x_1 \wedge \cdots \wedge x_r$, $\mathbf{b} = y_1 \wedge \cdots \wedge y_r$, with $x_i, y_j \in V$, as

$$\langle \mathbf{a}, \mathbf{b} \rangle \doteqdot \det(\langle x_i, y_j \rangle)$$

and then extended bi-linearly to all r-covectors $\mathbf{a}$ and $\mathbf{b}$ which are linear combination of simple vectors. The following results concerning the comass are well known. The interested readers can find their proof in Whitney's classics [46, pp. 52–55, Theorem 13A, Lemma 13a] or Federer's [13, Section 1.8].

Theorem 3.1 (Whitney). *The following statements hold.*

(i) $\|\omega\|_0$ *is a norm and* $\|\omega\|_0 = \sup\{|\omega(\mathbf{a})| : \|\mathbf{a}\|_0 = 1\}$, *where* $\|\mathbf{a}\|_0$ *is the mass of* $\mathbf{a}$ *defined as*

$$\|\mathbf{a}\|_0 \doteqdot \inf\Big\{\sum \|\mathbf{a}_i\| : \mathbf{a} = \sum \mathbf{a}_i, \text{ the } \mathbf{a}_i \text{ simple}\Big\}.$$

(ii) *For any* r*-vector* $\mathbf{a}$, $\|\mathbf{a}\|_0 \geq \|\mathbf{a}\|$, *with equality if and only if* $\mathbf{a}$ *is simple.*

(iii) *For each* ω *there exists an* r*-vector* $\mathbf{b}$ *such that* $\|\omega\|_0 = |\omega(\mathbf{b})|$, $\mathbf{b}$ *is simple, and* $\|\mathbf{b}\| = 1$.

(iv) *If* ω *is simple, then* $\|\omega\|_0 = \|\omega\|$.

(v) *One has* $\|\omega\| \geq \|\omega\|_0 \geq (\frac{r!(n-r)!}{n!})^{\frac{1}{2}}\|\omega\|$, *with the first inequality holds equality if and only if* ω *is simple.*

We shall prove the theorem via an argument by contradiction. Assume that $S_k^{\perp} > 0$ and there exists a $\phi \neq 0$ which is a harmonic $(p, 0)$-form with $p \geq k$. It is well known that it is holomorphic. Let $\|\phi\|_0(x)$ be its comass at x. Then its maximum (nonzero) must be attained somewhere at $x_0 \in N$. We shall exam ϕ more closely in a coordinate chart (to be specified later) of x_0. By the above proposition, at x_0, there exists a simple p-vector $\mathbf{b}$ with $\|\mathbf{b}\| = 1$, which we may assume to be $\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_p}$ for a unitary frame $\{\frac{\partial}{\partial z_k}\}_{k=1,\dots,n}$ at x_0, such that

$$\max_{x\in N} \|\phi\|_0(x) = \|\phi\|_0(x_0) = |\phi(\mathbf{b})|.$$

If we denote

$$\phi = \frac{1}{p!}\sum_{I_p} a_{I_p} dz^{i_1} \wedge \cdots \wedge dz^{i_p},$$

where $I_p = (i_1, \dots, i_p)$ runs all p-tuples with $i_s \neq i_t$ if $s \neq t$, we deduce

$$\|\phi\|_0(x_0) = |a_{12\cdots p}|(x_0).$$

Extend the frame to have a normal complex coordinate chart U centered at x_0. This means that at x_0, the metric tensor $g_{\alpha\overline{\beta}}$ satisfies (cf. [44])

$$g_{\alpha\overline{\beta}} = \delta_{\alpha\beta}, \quad dg_{\alpha\overline{\beta}} = 0, \quad \frac{\partial^2 g_{\alpha\overline{\beta}}}{\partial z_\gamma \partial z_\delta} = 0.$$

With respect to this coordinate

$$\phi(x) = \frac{1}{p!}\sum_{I_p} a_{I_p}(x) dz^{i_1} \wedge \cdots \wedge dz^{i_p}$$

for $x \in U$ with $a_{I_p}(x)$ being holomorphic. Let

$$\widetilde{\phi}(x) \doteqdot \frac{1}{\sqrt{\det(g_{\alpha\overline{\beta}})_{1\leq\alpha,\beta\leq p}}\sqrt{\det(g^{\alpha\overline{\beta}})_{1\leq\alpha,\beta\leq p}}} a_{12\cdots p}(x) dz^1 \wedge \cdots \wedge dz^p.$$

This is defined in U. Since $\widetilde{\phi}$ is simple, we have

$$\|\widetilde{\phi}\|_0^2(x) = \|\widetilde{\phi}\|^2(x) = \frac{|a_{1\cdots p}|^2(x)}{\det(g_{\alpha\overline{\beta}})_{1\leq\alpha,\beta\leq p}}.$$

In particular, $\|\widetilde{\phi}\|_0(x_0) = \|\phi\|_0(x_0)$. On the other hand let $\mathbf{a} = \frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^p}$. Then by the definition of the comass $\|\cdot\|_0$, namely $\|\phi\|_0 = \sup \frac{|\phi(\mathbf{a})|}{\|\mathbf{a}\|}$, taking among all simple nonzero $\mathbf{a}$,

$$\|\phi\|_0^2(x) \geq \frac{|\phi(\mathbf{a})|^2}{\|\mathbf{a}\|^2} = \frac{|a_{1\cdots p}|^2(x)}{\det(g_{\alpha\overline{\beta}})_{1\leq\alpha,\beta\leq p}} \geq \|\widetilde{\phi}\|_0^2(x).$$

Since $\|\phi\|_0(x) \leq \|\phi\|_0(x_0)$, as a consequence we have that

$$\|\widetilde{\phi}\|(x) = \|\widetilde{\phi}\|_0(x) \leq \|\phi\|_0(x) \leq \|\phi\|_0(x_0) = |a_{1\cdots p}(x_0)| = \|\widetilde{\phi}\|_0(x_0) = \|\widetilde{\phi}\|(x_0).$$

In summary, we have constructed a simple $(p,0)$-form $\widetilde{\phi}(x)$ in the neighborhood of x_0 such that its L^2-norm attains its maximum value at x_0.

Now apply $\partial_v\partial_{\overline{v}}$ to $\log\|\widetilde{\phi}\|^2$ at x_0. If $v = \frac{\partial}{\partial z^\gamma}$, we have that at point x_0 that

$$0 \geq -\sum_{\alpha,\beta=1}^{p} g^{\alpha\overline{\beta}} \frac{\partial^2}{\partial z^\gamma \partial \overline{z}^\gamma} g_{\alpha\overline{\beta}} = \sum_{\alpha=1}^{p} R_{\alpha\overline{\alpha}\gamma\overline{\gamma}}.$$

Namely, we have arrived that at x_0,

$$0 \geq \sum_{j=1}^{p} R_{v\overline{v}j\overline{j}}. \tag{3.1}$$

Now we are essentially at the same position of the proof in [37]. For the sake of the completeness we include the argument below. Let $\Sigma = \mathrm{span}\{\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_p}\}$. It is easy to see from (3.1) that $S_p(x_0,\Sigma) \leq 0$, where $S_p(x_0,\Sigma)$ denotes the scalar curvature of the curvature R restricted to Σ. In fact, $S_p(x_0,\Sigma) = \sum_{i,j=1}^{p} R_{i\overline{i}j\overline{j}}$.

On the other hand as in [37],

$$\begin{aligned}
\frac{1}{p} S_p^{\perp}(x_0,\Sigma)
&= \fint_{Z\in\Sigma,\,|Z|=1} \mathrm{Ric}^{\perp}(Z,\overline{Z})\,d\theta(Z) \\
&= \fint_{Z\in\Sigma,\,|Z|=1} (\mathrm{Ric}(Z,\overline{Z}) - H(Z))\,d\theta(Z) \\
&= \fint \frac{1}{\mathrm{Vol}(\mathbb{S}^{2n-1})}\left(\int_{\mathbb{S}^{2n-1}} (nR(Z,\overline{Z},W,\overline{W}) - H(Z))\,d\theta(W)\right) d\theta(Z) \\
&= \frac{1}{\mathrm{Vol}(\mathbb{S}^{2n-1})}\int_{\mathbb{S}^{2n-1}}\left(\fint (nR(Z,\overline{Z},W,\overline{W}) - H(Z))\,d\theta(Z)\right) d\theta(W) \\
&= \frac{1}{p}(\mathrm{Ric}_{1\overline{1}} + \mathrm{Ric}_{2\overline{2}} + \cdots + \mathrm{Ric}_{p\overline{p}}) - \frac{2}{p(p+1)} S_p(x_0,\Sigma).
\end{aligned} \tag{3.2}$$

Applying (3.1) to $v = \frac{\partial}{\partial z_i}$ for $i = p+1,\dots,n$, and summing the obtained inequalities we have that

$$\mathrm{Ric}_{1\overline{1}} + \mathrm{Ric}_{2\overline{2}} + \cdots + \mathrm{Ric}_{p\overline{p}} = S_p(x_0,\Sigma) + \sum_{\ell=p+1}^{n}\sum_{j=1}^{p} R_{\ell\overline{\ell}j\overline{j}} \leq S_p(x_0,\Sigma). \tag{3.3}$$

Combining (3.2) and (3.3), we have that

$$0 < S_k^{\perp}(x_0) \leq S_p^{\perp}(x_0,\Sigma) \leq S_p(x_0,\Sigma) - \frac{2}{p+1} S_p(x_0,\Sigma) = \frac{p-1}{p+1} S_p(x_0,\Sigma).$$

This implies $S_p(x_0,\Sigma) > 0$, a contradiction, since we have shown that a consequence of (3.1) is $S_p(x_0,\Sigma) \leq 0$.

From the definition of $S_k^{\perp}$ it is easy to see that $\mathrm{Ric}^{\perp} > 0$ implies that $S_k^{\perp} > 0$ for all $k \in \{1, \dots, n\}$. Hence $h^{p,0} = 0$ for all $p \geq 1$ by the above under the assumption $\mathrm{Ric}^{\perp} > 0$.

The simply-connectedness claimed in Theorem 2.2 follows from the argument of [20] illustrated in the introduction. The proof under the assumption $S_k > 0$ is similar, but easier in view of (3.1).

Note that under $\mathrm{Ric}^{\perp} > 0$, $\pi_1(N)$ is finite by a result of [37]. This in particular implies that $b_1 = 2h^{1,0} = 0$. The argument here provides an alternate proof of this.

Remark 3.2. The argument here also provides an alternate proof of the main theorem of [40]. It is clear that the Kählerity is not absolutely needed. Hence one can easily formulate a corresponding result for Hermitian manifolds. We leave this to interested readers. The concepts of $S_k(x, \Sigma)$ and $S_k^{\perp}(x_0, \Sigma)$ were introduced in [33, 34, 37, 40].

4. Rational connectedness and Ric_k

A complex manifold N is called rationally connected if any two points of N can be joined by a chain of rational curves. Various criterion on the rational connectedness have been established by various authors. In particular, the following was prove in [8]:

Theorem 4.1 (Campana–Demailly–Peternell). *Let N be a projective algebraic manifold of complex dimension n. Then N is rationally connected if and only if for any ample line bundle L, there exist $C(L)$ such that*

$$H^0(N, ((T'N)^*)^{\otimes p} \otimes L^{\otimes \ell}) = \{0\} \tag{4.1}$$

for any $p \geq C(L)\ell$, with ℓ being any positive integer.

It was proved in [16] that a compact projective manifold with positive holomorphic sectional curvature must be rationally connected. The projectivity was proved in [50] afterwards (an alternate proof of the rational connectedness was also given there). In [34], the concept Ric_k was introduced, which interpolates between the holomorphic sectional curvature and the Ricci curvature. Precisely for any k dimensional subspace $\Sigma \subset T'_x N$, $\mathrm{Ric}_k(x, \Sigma)$ is the Ricci curvature of $R|_{\Sigma}$. Under $\mathrm{Ric}_k < 0$, the k-hyperbolicity was proved in [34].

We say $\mathrm{Ric}_{k,-}(x) > \lambda(x)$ if for any $v \in \Sigma$ and for every k-dimensional subspace Σ,

$$\mathrm{Ric}_k(x, \Sigma)(v, \overline{v}) > \lambda |v|^2.$$

Similarly, $\mathrm{Ric}_k > 0$ means that $\mathrm{Ric}_{k,-}(x) > 0$ everywhere. The condition $\mathrm{Ric}_k > 0$ does not become weaker as k increases since more v needs to be tested. In fact, Hitchin [18] illustrated examples of Kähler metrics with $\mathrm{Ric}_1 > 0$ on all Hirzebruch surfaces. But on most of them one could not possibly find metrics with $\mathrm{Ric}_2 > 0$. More examples can be found in [1, 37]. But it is easy to see that $S_k > 0$ does follow from $\mathrm{Ric}_k > 0$, and $S_k > 0$ becomes weaker as k increases with S_1 being the same as the holomorphic sectional curvature and S_n being the scalar curvature. Hence if $\mathrm{Ric}_2 > 0$, N is also projective by the result of [40]. Naturally one would ask *whether or not a compact Kähler manifold with* $\mathrm{Ric}_k > 0$ *for some* $k \in \{3, \dots, n-1\}$ *is projective* since the projectivity has been known for the cases of $k = 1, k = 2$ and $k = n$. The following result provides an affirmative answer.

Theorem 4.2. *Let (N^n, h) be a compact Kähler manifold with $\mathrm{Ric}_k > 0$ for some $1 \le k \le n$. Then $h^{p,0} = 0$ for $1 \le p \le n$. In particular, N must be projective.*

Proof. By Theorem 2.2 and that $\mathrm{Ric}_k > 0$ implies $S_k > 0$ we have that $h^{p,0} = 0$ for $p \ge k$. Hence we only need to focus on the case $p < k$. The first part of proof of Theorem 2.2 asserts that if there exists a holomorphic $(p, 0)$-form $\phi \ne 0$, then (3.1) holds. Namely, there exists $x_0 \in N$, and a unitary normal coordinate centered at x_0 such that at x_0:

$$\sum_{j=1}^{p} R_{v\bar{v}j\bar{j}} \le 0 \tag{4.2}$$

for any $v \in T'_{x_0}N$.

Now we pick a k-subspace $\Sigma \subset T'_{x_0}N$ such that it contains the p-dimensional subspace spanned by $\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^p}\}$. Then, by the assumption $\mathrm{Ric}_k > 0$ for all $j \in \{1, \dots, p\}$,

$$\fint_{v \in \mathbb{S}^{2k-1} \subset \Sigma} R_{v\bar{v}j\bar{j}}\, d\theta(v) = \frac{1}{k} \mathrm{Ric}_k\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^{\bar{j}}}\right) > 0.$$

Thus we have that

$$\fint_{v \in \mathbb{S}^{2k-1} \subset \Sigma} \sum_{j=1}^{p} R_{v\bar{v}j\bar{j}}\, d\theta(v) > 0.$$

This is a contradiction to (4.2). The contradiction proves that $h^{p,0} = 0$ for $p < k$. The projectivity follows from $h^{2,0} = 0$ and a theorem of Kodaira (cf. [30, Theorem 8.3 of Chapter 3]). □

For $k = 1, 2, n$, the result is previously known except when $k = 2$, $p \ne 2$. The above proof provides a unified argument for all the previous known cases. The argument above proves a bit more. To state the result, we introduce the following:

Definition 4.3. We call the curvature operator R BC-p positive at x_0 (BC stands for the bisectional curvature) if for any unitary orthogonal p-vectors $\{E_1, \dots, E_p\}$, there exists a $v \in T'_{x_0}N$ such that

$$\sum_{i=1}^{p} R_{v\bar{v}E_i\overline{E}_i} > 0. \tag{4.3}$$

We say that (N, h) is BC-p positive if the above holds all $x_0 \in N$. This can be easily adapted to Hermitian bundle (V, h) over a Hermitian manifold N since the expression in (4.3) makes sense for $v \in V_{x_0}$ and unitary p-vectors $\{E_1, \dots, E_p\}$ of $T'_{x_0}N$.

It is easy to see that BC-1 positivity is the same as RC-positivity for the tangent bundle defined in [50]. In general BC-p positivity amounts to at any $x \in N$,

$$\min_{\Sigma \in G_{p,n}(T'_x N)} \max_{|X|=1} \left(\fint_{Z \in \mathbb{S}^{2p-1} \subset \Sigma} R(X, \overline{X}, Z, \overline{Z})\, d\mu(Z) \right) > 0.$$

Here $G_{p,n}(T'_x N)$ denotes the Grassmannian of rank p subspaces of $T'_x N$. If we endow a compact complex manifold N^n with a Hermitian metric, let R be its curvature, which can be viewed as section of $\bigwedge^{1,1}(\mathrm{End}(T_{x_0}N))$. Then BC-$p$ positivity can be defined for any Hermitian vector bundles. Here we specialize it to $V = T'N$.

Corollary 4.4. *If the curvature of a Hermitian manifold (N^n, h) satisfies the BC-p positivity for some $1 \le p \le n$, then $h^{p,0} = 0$. Hence any Kähler manifold with BC-2 positive curvature must be projective. Moreover, the 2-positivity of* Ric_k *(for some $k \ge 2$) implies the BC-2 positivity, thus the projectivity of N. Similarly,* $\mathrm{Ric}^{\perp}_{k+1} > 0$ *implies BC-p positivity for all $p \ge k$.*

In Theorem 6.3 the last statement is strengthen into *BC-p positivity for all $p \ge 1$*. Here we define

$$\mathrm{Ric}^{\perp}_k (x, \Sigma)(v, \bar{v})=\mathrm{Ric}_k(x, \Sigma)(v, \bar{v})-H(v)/|v|^2$$

for a k-dimensional subspace $\Sigma \subset T'_x N$ and $v \in \Sigma$. The positivity of $\mathrm{Ric}^{\perp}_k$ is defined as Ric_k. Clearly, $\mathrm{Ric}^{\perp}_1 \equiv 0$. Note that $\mathrm{Ric}^{\perp}_2 \ge 0$ is the same as the orthogonal bisectional curvature is nonnegative; $\mathrm{Ric}^{\perp}_n$ is the same as $\mathrm{Ric}^{\perp}$.

Proof. By the proof of Theorem 4.2 and by definition, we only need to show the last statement. The 2-positivity of Ric_k means that for any k-dimensional $\Sigma \subset T'_{x_0} N$ and any two unitary orthogonal $E_1, E_2 \in \Sigma$,

$$\mathrm{Ric}_k(x_0, \Sigma)(E_1, \overline{E}_1) + \mathrm{Ric}_k(x_0, \Sigma)(E_2, \overline{E}_2) > 0.$$

This clearly implies BC-2 positivity, since for any given unitary orthogonal $\{E_1, E_2\}$ there always a k-dimensional Σ containing them, and if $R(v, \overline{v}, E_1, \overline{E}_1) + R(v, \overline{v}, E_2, \overline{E}_2) \le 0$ for all $v \in \Sigma$, it is easy to see that

$$\mathrm{Ric}_k(x_0, \Sigma)(E_1, \overline{E}_1) + \mathrm{Ric}_k(x_0, \Sigma)(E_2, \overline{E}_2) \le 0.$$

If $\mathrm{Ric}^{\perp}_{k+1} > 0$, then it implies that, for a unitary frame $\{E_i\}$ of a $(k+1)$-dimensional Σ with $E_1 = \frac{X}{|X|}$,

$$\sum_{j=2}^{k+1} R(X, \overline{X}, E_j, \overline{E}_j) > 0,$$

which implies BC-k positive. On the other hand, a simple calculation shows that

$$\fint_{\mathbb{S}^{2k+1} \subset \Sigma \subset T'_x N} \mathrm{Ric}^{\perp}_{k+1}(Z, \overline{Z})\, d\theta(Z) = \frac{k}{(k+1)(k+2)} S_{k+1}(x).$$

The case of $\ell > k$ follows by the proposition below if $k + 1 < n$. □

Proposition 4.1. *For a Kähler manifold (N, h), $S_k(x_0) > 0$ implies BC-p positivity for any $p \ge k$, and the ℓ-positivity of* $\mathrm{Ric}_k(x_0)$ *(with $\ell \le k$) implies BC-p positivity for any $\ell \le p \le n$.*

Proof. Note that $S_p > 0$ implies BC-p positivity. The first claim follows from the fact that $S_k > 0$ implies $S_p > 0$ for any $p \ge k$. For the second statement, if $p \ge k$, the result follows from the first. If $\ell \le p < k$, for unitary p-vectors $\{E_1, \dots, E_p\}$ we choose a k-subspace Σ containing them. If $\sum_{i=1}^{p} R(v, \overline{v}, E_i, \overline{E}_i) \le 0$ for all $v \in \Sigma$, it implies that

$$\sum_{i=1}^{p} \mathrm{Ric}_k(x_0, \Sigma)(E_i, \overline{E}_i) \le 0.$$

This violates the ℓ-positivity of $\mathrm{Ric}_k(x_0)$. □

One can also extend the definition of Ric_k to a Hermitian vector bundle over Hermitian manifolds. Let

$$R = R_{\alpha\overline{\beta}i}{}^{j}\, dz^{\alpha} \wedge d\overline{z}^{\beta} \otimes e_i^* \otimes e_j$$

be the curvature of a Hermitian vector bundle (V, h) over a Hermitian manifold.

Definition 4.5. Let $\Sigma \subset T'_{x_0}N$ and $\sigma \subset V_{x_0}$ be two k-dimensional subspaces. Define for $X \in T'_{x_0}N$, $v = v^i e_i \in V_{x_0}$ with $\{e_k\}_{k=1}^{L}$ being a unitary frame of V_{x_0}, $L = \dim(V_{x_0})$, the first and second Ric_k as follows:

$$\mathrm{Ric}^1_k(x_0, \sigma)(X, \overline{X}) = \sum_{i=1}^{k} R_{X\overline{X}s}{}^{r} a_i^s \overline{a_i^t} h_{r\overline{t}},$$

$$\mathrm{Ric}^2_k(x_0, \Sigma)(v, \overline{v}) = \sum_{\alpha=1}^{k} R_{E_\alpha \overline{E}_\alpha i}{}^{j} v^i \overline{v}^l h_{j\overline{l}},$$

with $\{E_\alpha\}_{\alpha=1}^{k}$ being a unitary frame of Σ, and $\{\tilde{e}_i\}_{i=1}^{k}$ being a unitary frame of σ. Here

$$\tilde{e}_i = \sum_{k=1}^{L} a_i^k e_k.$$

Note that Ric^1 is a $(1,1)$-form of N, and that it coincides with the first Chern–Ricci of a Hermitian manifold if $k = n$ and $V = T'N$. Further, observe that for $V = T'N$, $\mathrm{Ric}^1_k|_\sigma$ is $\mathrm{Ric}_k(x_0, \sigma)$ when N is Kähler, and generalizes Ric_k to the case of N being just Hermitian.

Corollary 4.4 generalizes the main theorem of [40]. Towards the rational connectedness we prove the following result.

Theorem 4.6. *Let (N^n, h) be a compact projective manifold with* $\mathrm{Ric}_k > 0$ *for some* $k \in \{1, \dots, n\}$. *Then* (4.1) *holds, and N must be rationally connected.*

Proof. Before the general case, we start with a proof for the special case $k = 1$ by proving the above criterion in Theorem 4.1 directly via the $\partial\overline{\partial}$-Bochner formula. Let s be a holomorphic section in $H^0(N, ((T'N)^*)^{\otimes p} \otimes L^{\otimes \ell})$. Locally it can be expressed as

$$s = \sum_{I_p} a_{I_p,\ell}\, dz^{i_1} \otimes \cdots \otimes dz^{i_p} \otimes e^{\ell}$$

with $I_p = (i_1, \dots, i_p) \in \mathbb{N}^p$, and e being a local holomorphic section of L and $e^\ell = e \otimes \cdots \otimes e$ being the ℓth power of e. Equip L with a Hermitian metric a and let C_a be the corresponding curvature form. The point-wise norm $|s|^2$ is with respect to the induced metric of $((T'N)^*)^{\otimes p}$ and $L^{\otimes \ell}$). The $\partial\overline{\partial}$-Bochner formula implies that for any $v \in T'_xN$,

$$\begin{aligned}\partial_v \overline{\partial}_{\overline{v}} |s|^2 = |\nabla_v s|^2 &+ \sum_{I_p} \sum_{t=1}^{n} \sum_{\alpha=1}^{p} \langle a_{I_p,\ell} R_{v\overline{v} i_\alpha \overline{t}}\, dz^{i_1} \otimes \cdots \otimes dz^{i_{\alpha-1}} \\ &\qquad \otimes dz^{t} \otimes \cdots \otimes dz^{i_p} \otimes e^{\ell}, \overline{s} \rangle \\ &- \sum_{I_p} \langle a_{I_p,\ell}\, \ell C_a(v, \overline{v})\, dz^{i_1} \otimes \cdots \otimes dz^{i_p} \otimes e^{\ell}, \overline{s} \rangle. \end{aligned} \tag{4.4}$$

Apply the above equation at the point x_0, where $|s|^2$ attains its maximum, with respect to a normal coordinate centered at x_0. Pick a unit vector v such that $H(v)$ attains its minimum on $\mathbb{S}^{2n-1} \subset T'_{x_0}N$. By the assumption $H > 0$, there exists a $\delta > 0$ such that $H(v) \geq \delta$ for any unit vector and any $x \in N$. Diagonalize $R_{v\overline{v}(\cdot)\overline{(\cdot)}}$ by a suitable chosen unitary frame $\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}\}$. By applying the first and second derivative tests, it shows that if at $v \in \mathbb{S}^{2n-1}$, $H(v)$ attains its minimum, then $R_{v\overline{v}w\overline{w}} \geq \frac{\delta}{2}$, and $R_{v\overline{v}v\overline{w}} = 0$, for any w with $|w| = 1$, and $\langle w, \overline{v}\rangle = 0$. This implies that

$$R_{v\overline{v}i_\alpha\overline{i}_\alpha} = |\mu_1|^2 R_{v\overline{v}v\overline{v}} + |\beta_1|^2 R_{v\overline{v}w\overline{w}} \geq \frac{\delta}{2}$$

where we write $\frac{\partial}{\partial z^{i_\alpha}} = \mu_1 v + \beta_1 w$ with $|\mu_1|^2 + |\beta_1|^2 = 1$, $w \in \{v\}^\perp$ and $|w| = 1$. (This perhaps goes back to the work of Berger. See also for example [50] or [37, Corollary 2.1].) If A is the upper bound of $C_a(v, \overline{v})$, we have that

$$0 \geq \partial_v \overline{\partial}_{\overline{v}} |s|^2 \geq \left(\frac{p\delta}{2} - \ell A\right)|s|^2.$$

This is a contradiction for $p \geq \frac{3A\ell}{\delta}$ if $s \neq 0$. Hence we can conclude that for any $p \geq C(L)\ell$ with $C(L) = \frac{3A}{\delta}$, $H^0(N, ((T'N)^*)^{\otimes p} \otimes L^{\otimes \ell}) = \{0\}$.

For the general case, namely $\mathrm{Ric}_k > 0$ for some $k \in \{1, \dots, n\}$, we combine the argument above with the second variation result of [40]. At the point x_0 where the maximum of $|s|^2$ is attained, we pick Σ such that $S_k(x_0, \Sigma)$ attains its minimum $\delta_1 > 0$. For simplicity of the notations, we denote the average of a function $f(X)$ over the unit sphere $\mathbb{S}^{2k-1}$ in Σ by $⨍ f(X)$. The second variation consideration in [40] gives the following useful estimates.

Proposition 4.2 ([40, Proposition 3.1]). *Let $\{E_1, \dots, E_m\}$ be a unitary frame at x_0 such that $\{E_i\}_{1 \leq i \leq k}$ spans Σ. Then for any $E \in \Sigma$, $E' \perp \Sigma$, and any $k+1 \leq p \leq m$, we have*

$$⨍ R(E, \overline{E}', Z, \overline{Z})\, d\theta(Z) = ⨍ R(E', \overline{E}, Z, \overline{Z})\, d\theta(Z) = 0, \tag{4.5}$$

$$⨍ R(E_p, \overline{E}_p, Z, \overline{Z})\, d\theta(Z) \geq \frac{S_k(x_0, \Sigma)}{k(k+1)}. \tag{4.6}$$

Proof. For the convenience of the reader we include the proof. The proof uses the first and second variation out of the fact that $S_k(x_0, \Sigma)$ is minimum. Let $a \in \mathfrak{u}(m)$ be an element of the Lie algebra of $\mathsf{U}(m)$. Consider the function

$$f(t) = ⨍ H(e^{ta}X)\, d\theta(X).$$

By the choice of Σ, $f(t)$ attains its minimum at $t = 0$. This implies that $f'(0) = 0$ and $f''(0) \geq 0$. Hence

$$⨍ \big(R(a(X), \overline{X}, X, \overline{X}) + R(X, \overline{a}(\overline{X}), X, \overline{X})\big)\, d\theta(X) = 0 \tag{4.7}$$

and

$$\begin{aligned} &⨍ \big(R(a^2(X), \overline{X}, X, \overline{X}) + R(X, \overline{a}^2(\overline{X}), X, \overline{X}) + 4R(a(X), \overline{a}(\overline{X}), X, \overline{X})\big)\, d\theta(X) \\ &\quad + ⨍ \big(R(a(X), \overline{X}, a(X), \overline{X}) + R(X, \overline{a}(\overline{X}), X, \overline{a}(\overline{X}))\big)\, d\theta(X) \geq 0. \end{aligned} \tag{4.8}$$

We exploit these by looking into some special cases of a. Let $W \perp \Sigma$ and $Z \in \Sigma$ be two fixed vectors. Let $a = \sqrt{-1}(Z \otimes \overline{W} + W \otimes \overline{Z})$. Then

$$a(X) = \sqrt{-1}\langle X, \overline{Z}\rangle W, \quad a^2(X) = -\langle X, \overline{Z}\rangle Z.$$

Applying (4.8) to the above a and also the one with W being replaced by $\sqrt{-1}W$, and add the resulting two estimates together, we have that

$$4 \fint |\langle X, \overline{Z}\rangle|^2 R(W, \overline{W}, X, \overline{X})\, d\theta(X) \geq \fint \langle X, \overline{Z}\rangle R(Z, \overline{X}, X, \overline{X}) + \langle Z, \overline{X}\rangle R(X, \overline{Z}, X, \overline{X}).$$

Applying the above to $Z \in \mathbb{S}^{2k-1} \subset \Sigma$ and taking the average of the result, we then have

$$\frac{4}{k} \fint R(W, \overline{W}, X, \overline{X})\, d\theta(X) \geq \frac{2}{k} \fint R(X, \overline{X}, X, \overline{X}).$$

This proves (4.6). By combining (4.7) (with a as above) and the one with W being replaced by $\sqrt{-1}W$, we obtain two equalities

$$\fint \langle X, \overline{Z}\rangle R(W, \overline{X}, X, \overline{X}) = \fint \langle Z, \overline{X}\rangle R(X, \overline{W}, X, \overline{X}) = 0.$$

Now write $X = x_1E_1 + x_2E_2 + \cdots + x_kE_k$. Let $Z = E_i$, $W = E_\ell$ (for $i = 1, 2$, $\ell \geq k+1$). Direct calculation (with $Z = E_1$) shows that

$$\fint R_{\ell\bar{1}1\bar{1}}|x_1|^4 + 2\sum_{j=2}^{k} R_{\ell\bar{1}j\bar{j}}|x_1|^2|x_j|^2 = 0.$$

Applying the integral identities in the proof of the Berger's lemma (cf. [40, Lemma 1.1]), the above equation (together with the case $Z = E_i$ with $2 \leq i \leq k$) implies that

$$\sum_{j=1}^{k} R_{\ell\bar{i}j\bar{j}} = 0, \quad 1 \leq i \leq k,\ k+1 \leq \ell \leq n. \tag{4.9}$$

This and its conjugate imply (4.5). □

As [40], we may choose the frame so that $\fint R_{v\bar{v}(\cdot)\overline{(\cdot)}}$ is diagonal. Integrating (4.4) over the unit sphere $\mathbb{S}^{2k-1} \subset \Sigma$, we have that

$$0 \geq \fint \partial_v \bar{\partial}_{\bar{v}}|s|^2\, d\theta(v) \geq \sum_{I_p} |a_{I_p,\ell}|^2 \fint \left(\sum_{\alpha=1}^{p} R_{v\bar{v}i_\alpha\bar{i}_\alpha} - \ell C_a(v, \bar{v})\right) d\theta(v).$$

Here we have chosen a unitary frame $\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}\}$ so that $\fint R_{v\bar{v}(\cdot)\overline{(\cdot)}}\, d\theta(v)$ is diagonal.

As in the paper [40], decompose $\frac{\partial}{\partial z^i}$ into the sum of vectors $\mu_i E_i \in \Sigma$ and $\beta_i E_i' \in \Sigma^\perp$ with $|E_i| = |E_i'| = 1$ and $|\mu_i|^2 + |\beta_i|^2 = 1$. If we denote the lower bound of Ric_k by $\delta_2 > 0$, by (4.5) and (4.6)

$$\begin{aligned}
\fint R_{v\bar{v}1\bar{1}}\, d\theta(v) &= |\mu_1|^2 \fint R_{v\bar{v}E_1\overline{E}_1}\, d\theta(v) + |\beta_1|^2 \fint R_{v\bar{v}E_1'\overline{E}_1'}\, d\theta(v) \\
&= \frac{|\mu_1|^2}{k} \mathrm{Ric}_k(E_1, \overline{E}_1) + |\beta_1|^2 \fint R_{v\bar{v}E_1'\overline{E}_1'}\, d\theta(v) \\
&\geq \frac{|\mu_1|^2}{k}\delta_2 + \frac{|\beta_1|^2}{k(k+1)}\delta_1 \\
&\geq \frac{\min(\delta_1, \delta_2)}{k(k+1)}.
\end{aligned}$$

The above estimate holds for any $\frac{\partial}{\partial z^{i\alpha}}$ as well. Hence combining two estimates above, we have

$$0 \geq \fint \partial_v \bar{\partial}_{\bar{v}} |s|^2 \, d\theta(v) \geq \left(p \frac{\min(\delta_1, \delta_2)}{k(k+1)} - \ell A \right) |s|^2.$$

The same argument as the special case $k = 1$ leads to a contradiction if $p \geq C(L)\ell$ for suitable chosen $C(L)$, provided that $s \neq 0$. This proves the vanishing theorem claimed in Theorem 4.1 for manifolds with $\mathrm{Ric}_k > 0$. □

The simply-connectedness part of Theorem 1.1 follows from Theorem 4.2, Theorem 6.1, and the argument of [20] (recalled in the introduction) via Hirzebruch's Riemann–Roch theorem. It can also be inferred from the rational connectedness and [12, Corollary 4.29]. It is expected that the construction via the projectivization in [36, 49] would give more examples of Kähler manifolds with $\mathrm{Ric}_k > 0$.

Regarding rational connectedness we should point out that there exists a recent work [51], in which it was proved that if $T'N$ is *uniformly* RC-*positive* in the sense that for any $x \in N$, there exists an X such that $R(X, \overline{X}, V, \overline{V}) > 0$ for any $V \in T_x N$, then N is projective and rationally connected. As pointed out above, BC-2 positivity (which follows from the uniform RC-positivity) already implies the projectivity. The uniform RC-positivity is equivalent to

$$\delta \doteqdot \min_{x \in N} \Big(\max_{|X|=1,\, X \in T'_x N} \Big(\min_{|V|=1, V \in T'_x N} R(X, \overline{X}, V, \overline{V}) \Big) \Big) > 0.$$

Hence one can derive Theorem 4.1 from (4.4) directly by letting $v = X$ with X being the vector which attains the maximum in the above definition, and $p \geq \frac{2A}{\delta}$. This provides a direct proof of [51, Theorem 1.3].

Since the boundedness of smooth Fano varieties (namely there are finitely many deformation types) was also proved in [24], it is natural to ask *whether or not the family of Kähler manifolds with* $\mathrm{Ric}_k > 0$ *(for some k, particularly for n large and* $n - k \neq 0$ *small) is bounded.* Before one proves that every Kähler manifold with $\mathrm{Ric}^{\perp} > 0$ is Fano, it remains an interesting future project to *investigate the rational connectedness of compact Kähler manifolds with* $\mathrm{Ric}^{\perp} > 0$. For manifold with $\mathrm{QB} > 0$, as a simple consequence of the results in the next section and the result of [7, 24] we have the following corollary.

Corollary 4.7. *Any compact Kähler manifold* (N, h) *with quasi-positive QB, or more generally with quasi-positive* $\mathrm{Ric}^{\perp}$*, and* $\rho(N) = 1$ *must be rationally connected.*

The same conclusion holds if $\mathrm{Ric}^{\perp} \geq 0$, (N^n, h) is locally irreducible and $\rho(N) = 1$.

5. Compact Kähler manifolds with $h^{1,1} = 1$ and CQB

Recall the following result from [37], which is a consequence of a formula of Berger.

Lemma 5.1. *Let* (N^n, h) *be a Kähler manifold of complex dimension n. At any point* $p \in N$,

$$\frac{n-1}{n(n+1)} S(p) = \frac{1}{\mathrm{Vol}(\mathbb{S}^{2n-1})} \int_{|Z|=1, Z \in T'_p N} \mathrm{Ric}^{\perp}(Z, \overline{Z}) \, d\theta(Z)$$

where $S(p) = \sum_{i=1}^{n} \mathrm{Ric}(E_i, \overline{E}_i)$ *(with respect to any unitary frame* $\{E_i\}$*) denotes the scalar curvature at* p.

Note that the first Chern form $c_1(N) = \frac{\sqrt{-1}}{2\pi} r_{i\bar{j}} dz^i \wedge d\bar{z}^j$, with $r_{i\bar{j}} = \mathrm{Ric}(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j})$. Let $\omega_h = \frac{\sqrt{-1}}{2\pi} h_{i\bar{j}}$ be the Kähler form (the normalization is to make the Kähler and Riemannian settings coincide). A direct computation via a unitary frame gives

$$c_1(N)(y) \wedge \omega_h^{n-1}(y) = \frac{1}{n} S(y)\, \omega_h^n(y).$$

We also let $V(N) = \int_N \omega_h^n$. The normalization above makes sure that the volume of an algebraic subvariety has its volume being an integer.

Recall that for any line bundle L its degree $d(L)$ is defined as

$$d(L) = \int_N c_1(L) \wedge \omega_h^{n-1}.$$

When $h^{1,1}(N) = 1$, it implies that $[c_1(N)] = \ell[\omega_h]$ for some constant ℓ. Hence we have that $d(K_N^{-1}) = \ell V(N)$.

Under assumption (i) of Theorem 2.3, we know that $S(y) > 0$ somewhere and $S(y) \geq 0$ for all $x \in N$ by Lemma 5.1, which then implies that $d(K_N^{-1}) > 0$, hence $\ell > 0$. This shows that $[c_1(N)] > 0$. Now Yau's solution to the Calabi's conjecture [44, 52] implies that N admits a Kähler metric such that its Ricci curvature is $\ell\omega_h > 0$.

The proof for statement (ii) is similar. The existence of negative Kähler–Einstein metric follows from the Aubin–Yau theorem [44, 52].

To prove Corollary 2.5, we observe that if $\ell = 0$ in the above argument, it implies that $S(y) \equiv 0$. Hence by Lemma 5.1 we have that $\mathrm{Ric}^{\perp} \equiv 0$. By [36, Theorem 6.1] it implies that N is flat for $n \geq 3$, or $n = 2$ and N is either flat or locally a product. This contradicts to the assumption of local irreducibility.

Note that the same argument can be applied to conclude the same result Ric_k and $\mathrm{Ric}_k^{\perp}$.

Proposition 5.1. *Let* (N, h) *be a compact Kähler manifold of complex dimension* n. *Assume further that* $h^{1,1}(N) = 1$. *Then:*

(i) *If* Ric_k *(or* $\mathrm{Ric}_k^{\perp}$*) is quasi-positive for some* $1 \leq k \leq n$*, then* N *must be Fano.*

(ii) *If* Ric_k *(or* $\mathrm{Ric}_k^{\perp}$*) is quasi-negative, then* N *must be projective with ample canonical line bundle* K_N.

In particular, in case (i) N *admits a Kähler metric with positive Ricci, and in case* (ii) N *admits a Kähler–Einstein metric with negative Einstein constant.*

Before introducing two new curvatures, we first observe that in (2.2) if we replace A by its traceless part $\mathring{A} = A - \lambda\,\mathrm{id}$ with $\lambda = \frac{\mathrm{trace}(A)}{n}$, it remains the same. Namely, $\mathrm{QB}(A) = \mathrm{QB}(\mathring{A})$. Hence QB is defined on the quotient space $S^2(\mathbb{C}^n)/\{\mathbb{C}\,\mathrm{id}\}$, with $S^2(\mathbb{C}^n)$ being the space of Hermitian symmetric transformations of $\mathbb{C}^n$. Now $\mathrm{QB} > 0$ means that $\mathrm{QB}(A) > 0$ for all $A \neq 0$ as an equivalence class. This suggests a refined positivity $\mathrm{QB}_k > 0$ for any $1 \leq k \leq n$, defined as $\mathrm{QB}(A) > 0$ for any $A \notin \{\mathbb{C}\,\mathrm{id}\}$ of rank not greater than k. Clearly, for $k < n$, a nonzero Hermitian symmetric matrix with rank no greater than k cannot be in $\{\mathbb{C}\,\mathrm{id}\}$. It is easy to see $\mathrm{QB}_1 > 0$ is equivalent to $\mathrm{Ric}^{\perp} > 0$ and $\mathrm{QB}_n > 0$ is equivalent to $\mathrm{QB} > 0$. Naturally, a possible approach towards the classification of $\mathrm{Ric}^{\perp} > 0$ is through the family of Kähler manifolds with $\mathrm{QB} > 0$ and $\mathrm{QB}_k > 0$.

Now we discuss the first of two new curvatures. Recall that *cross quadratic bisectional curvature* CQB is defined as a Hermitian quadratic form on linear maps $A : T''N \to T'N$:

$$\mathrm{CQB}_R(A) = \sum_{\alpha,\beta=1}^{n} R(A(\overline{E}_\alpha), \overline{A(\overline{E}_\alpha)}, E_\beta, \overline{E}_\beta) - R(E_\alpha, \overline{E}_\beta, A(\overline{E}_\alpha), \overline{A(\overline{E}_\beta)})$$

for any unitary frame $\{E_\alpha\}$ of $T'M$. This is similar to (2.2). But here we allow A to be any linear maps. We say R has $\mathrm{CQB} > 0$ if $\mathrm{CQB}(A) > 0$ for any $A \neq 0$. For any $X \neq 0$, if we choose $\{E_\alpha\}$ with $E_1 = \frac{X}{|X|}$, and let A be the linear map satisfying $A(\overline{E}_1) = E_1$ and $A(\overline{E}_\alpha) = 0$ for any $\alpha \geq 2$, it is easy to see that

$$\mathrm{CQB}_R(A) = \frac{\mathrm{Ric}^{\perp}(X, \overline{X})}{|X|^2}.$$

Hence $\mathrm{CQB} > 0$ implies that $\mathrm{Ric}^{\perp} > 0$.[3)] Theorem 5.4 below shows that $\mathrm{CQB} > 0$ holds for all classical Kähler C-spaces with $b_2 = 1$, unlike QB, which fails to be positive on about 20 % of Kähler C-spaces with $b_2 = 1$. The expression CQB is motivated by the work of Calabi and Vesentini [6] where the authors studied the deformation rigidity of compact quotients of Hermitian symmetric spaces of noncompact type. We can introduce the concept $\mathrm{CQB}_k > 0$ (or $\mathrm{CQB}_k < 0$), defined as $\mathrm{CQB}(A) > 0$ for any A with rank not greater than k. Express A as $\sum_{s=1}^{k} X_s \otimes Y_s$; then $\mathrm{CQB}_k > 0$ is equivalent to

$$\sum_{s,t=1}^{k} \mathrm{Ric}(X_s, \overline{X}_t)\langle Y_s, \overline{Y}_t\rangle - R(X_s, \overline{X}_t, Y_s, \overline{Y}_t) > 0, \quad \sum_{s=1}^{k} X_s \otimes Y_s \neq 0.$$

Proposition 5.2. *The following statements hold.*

(i) *The condition* $\mathrm{CQB}_1 > 0$ *implies that* $\mathrm{Ric}^{\perp} > 0$*; in particular, N satisfies* $h^{p,0} = 0$, $\pi_1(N) = \{0\}$, *and N is projective.*

(ii) *If N is compact with $n \geq 2$, and* $\mathrm{CQB}_2 > 0$, *then Ricci curvature is* 2*-positive.*

Proof. Part (i) is proved in the paragraph above together with Theorem 2.2. For part (ii), for any unitary frame $\{E_\alpha\}$, let A be the map defined as $A(\overline{E}_1) = E_2$ and $A(\overline{E}_2) = -E_1$, and $A(E_\alpha) = 0$ for all $\alpha > 2$. Then the direct checking shows that $\mathrm{CQB} > 0$ is equivalent to

$$\mathrm{Ric}(E_1, \overline{E}_1) + \mathrm{Ric}(E_2, \overline{E}_2) > 0.$$

Since this holds for any unitary frame, we have the 2-positivity of the Ricci curvature. □

The work of Calabi and Vesentini [6] proves the following result.

Theorem 5.1. *Let (N, h) be a compact Kähler manifold with quasi-negative CQB (i.e.* $\mathrm{CQB} \leq 0$ *and* < 0 *at least at one point). Then*

$$H^1(N, T'N) = \{0\}.$$

In particular, N is deformation rigid in the sense that it does not admit nontrivial infinitesimal holomorphic deformation.

[3)] In [38], it was shown that $\mathrm{CQB} > 0$ implies $\mathrm{Ric} > 0$.

Proof. Let $\phi = \sum_{i,\alpha=1}^{n} \phi^i_{\overline{\alpha}} dz^{\overline{\alpha}} \otimes E_i$ be a $(0,1)$-form taking value in $T'N$ with $\{E_i\}$ being a local holomorphic basis of $T'N$. The Arizuki–Nakano formula gives

$$(\Delta_{\overline{\partial}}\phi - \Delta_{\partial}\phi)^i_{\overline{\beta}} = R^{\;i\overline{\tau}}_{j\;\;\overline{\beta}}\phi^j_{\overline{\tau}} - \mathrm{Ric}^i_j\, \phi^j_{\overline{\tau}}.$$

Under a normal coordinate we have that

$$\langle \Delta_{\overline{\partial}}\phi - \Delta_{\partial}\phi, \overline{\phi}\rangle = -\big(\mathrm{Ric}_{j\overline{i}}\, \phi^j_{\overline{\beta}}\overline{\phi^i_{\overline{\beta}}} - R_{j\overline{i}\tau\overline{\beta}}\phi^j_{\overline{\tau}}\overline{\phi^i_{\overline{\beta}}}\big).$$

Hence if $\Delta_{\overline{\partial}}\phi = 0$, we then have

$$0 = \int_N |\partial\phi|^2 + \int_N |\partial^*\phi|^2 - \int_N \big(\mathrm{Ric}_{i\overline{j}}\, \phi^i_{\overline{\beta}}\overline{\phi^j_{\overline{\beta}}} - R_{j\overline{i}\tau\overline{\beta}}\phi^j_{\overline{\tau}}\overline{\phi^i_{\overline{\beta}}}\big).$$

Letting $A(\overline{E}_\beta) = \phi^i_{\overline{\beta}} E_i$, the assumption amounts to that the expression in the third integral above is nonpositive and negative over a open subset U, where CQB < 0 if $(\phi^i_{\overline{\tau}}) \neq 0$ on U. This forces $(\phi^i_{\overline{\beta}}) \equiv 0$ on U, hence $\phi = 0$ by the unique continuation since ϕ is harmonic. □

It has been proved in [37] that if $\mathrm{Ric}^\perp < 0$, then $H^0(N, T'N) = \{0\}$. By [6, Table 1] and the proof of Theorem 5.4 below, all locally Hermitian symmetric spaces of noncompact type satisfy CQB < 0. Moreover the above theorem generalizes the result of Calabi and Vesentini since there are examples of non-Hermitian symmetric manifolds with CQB < 0. Flipping the sign we have the following corollary.

Corollary 5.2. *Let (N, h) be a compact Kähler manifold with quasi-positive CQB. Then*

$$\mathcal{H}^{0,1}_{\overline{\partial}}(N, T'N) = \mathcal{H}^{1,0}_{\overline{\partial}}(N, \Omega) = H^0(N, \Omega^1(\Omega)) = \{0\},$$

where $\Omega = (T'N)^*$. If only $\mathrm{Ric}^\perp > 0$ *is assumed, then $H^0(N, \Omega) = \{0\}$.*

In fact, we can strengthen the argument to prove the following result.

Theorem 5.3. *Assume that (N, h) is a compact Kähler manifold with* CQB > 0*. Then for any ample line bundle L there exists $C(L)$ such that*

$$H^0(N, ((T'N)^*)^{\otimes p} \otimes L^{\otimes \ell}) = \{0\}$$

for any $p \geq C(L)\ell$, with ℓ being any positive integer. In particular, N is rationally connected.

Proof. First we observe that a holomorphic section of $((T'N)^*)^{\otimes(p+1)} \otimes L^{\otimes \ell}$ can be viewed as a holomorphic $(1,0)$-form valued in $((T'N)^*)^{\otimes p} \otimes L^{\otimes \ell}$. Write it as

$$\varphi = \varphi^{I_p}_\alpha dz^\alpha \otimes dz^{i_1} \otimes dz^{i_2} \otimes \cdots \otimes dz^{i_p} \otimes e^\ell.$$

Applying the Arizuki–Nakano formula to the $\overline{\partial}$-harmonic φ as above, using the formula for the curvature of the tensor products, and under a normal coordinate, we have that

$$0 \leq \langle \Box_\partial \varphi, \overline{\varphi}\rangle \leq \int_M \big(\Omega^I_J \varphi^I_\alpha \overline{\varphi^J_\alpha} - \Omega^I_{J\,\gamma\overline{\alpha}}\varphi^I_\alpha\overline{\varphi^J_\gamma}\big) + A\ell|\varphi|^2 \leq \int_M \big(-p\delta|\varphi|^2 + A\ell|\varphi|^2\big),$$

where $\Omega^I_{J\gamma\overline{\alpha}} dz^\gamma \wedge \overline{z}^\alpha$ is the curvature of $((T'N)^*)^{\otimes p}$ and Ω^I_J is the corresponding mean curvature, $\delta > 0$ is the lower bound of CQB, A is an upper bound of the scalar curvature of L (equipped with a Hermitian metric of positive curvature). This implies that $\varphi = 0$ if $\frac{p}{\ell}$ is sufficiently large, hence the result. □

Recently it was proved that CQB > 0 implies that M is Fano, which gives an alternate proof of the above result.

Question 1. The results above naturally lead to the following questions:

(a) Does $H^1(N, T'N) = \{0\}$ hold under the weaker assumption that $\mathrm{Ric}^{\perp} < 0$?

(b) Is a harmonic map f of sufficiently high rank from a Kähler manifold (M, g) into a compact manifold with negative CQB must be holomorphic or conjugate holomorphic?

(c) Is there any nonsymmetric (locally) example of manifolds with CQB < 0?

(d) Do all Kähler C-spaces (the canonical Kähler metric) with $b_2 = 1$ satisfy CQB > 0 (below we provide a partial answer to this)?

The ultimate goal is to prove a classification theorem for compact Kähler manifolds with CQB > 0.

Concerning (c), in a recent preprint [38] a nonsymmetric example has been constructed. Moreover examples show that b_2 can be arbitrarily large under the condition CQB > 0. Concerning (d), we have the following affirmative answer for all classical Kähler C-spaces.

Theorem 5.4. *Let N^n be a compact Hermitian symmetric space ($n \geq 2$), or classical Kähler C-space with $n \geq 2$ and $b_2 = 1$. Then the Kähler–Einstein metric (unique up to constant multiple) has* CQB > 0.

Proof. If we write $A(\overline{E}_\beta) = A^i_\beta E_i$, it is easy to see if we change to a different unitary frame $\widetilde{\overline{E}}_\alpha = B^\beta_\alpha \overline{E}_\beta$, the effect on A is BAB^{tr} with B being a unitary transformation. Now

$$\mathrm{CQB}(A) = \mathrm{Ric}_{i\bar{j}} A^i_\beta \overline{A^j_\beta} - R_{j\bar{i}\tau\bar{\beta}} A^j_\tau \overline{A^i_\beta}.$$

Given that there exists a normal form under congruence for symmetric and skew symmetric matrices, it is helpful to write A into sum of the symmetric and skew-symmetric parts. For the special case $\mathrm{Ric} = \lambda h$, namely the metric is Kähler–Einstein with $\lambda > 0$, if we decompose A into the symmetric part A_1 and the skew-symmetric part A_2, noting that $R_{j\bar{i}\tau\bar{\beta}}$ is symmetric in j, τ and i, β we have

$$\mathrm{CQB}(A) = \lambda|A_1|^2 + \lambda|A_2|^2 - R_{j\bar{i}\tau\bar{\beta}} (A_1)^j_\tau \overline{(A_1)^i_\beta} \geq \lambda|A_1|^2 - R_{j\bar{i}\tau\bar{\beta}} (A_1)^j_\tau \overline{(A_1)^i_\beta}.$$

Now note that $R_{j\bar{i}\tau\bar{\beta}} (A_1)^j_\tau \overline{(A_1)^i_\beta}$ is the Hermitian symmetric action Q on the symmetric tensor (matrix) A considered in [19] and [6]. Precisely Q is defined by

$$Q(X \cdot Y, \overline{Z \cdot W}) = R_{X\overline{Z}Y\overline{W}}$$

for $X \cdot Y = \frac{1}{2}(X \otimes Y + Y \otimes X)$, and then is extended to all symmetric tensors. Let ν denote the biggest eigenvalue of Q. As in [36], to verify the result we just need to compare λ and ν. This can be done for all Hermitian symmetric spaces by [6, Table 2]. Note that λ here is $\frac{R}{2n}$ in Calabi and Vesentini's paper [6]. For the classical homogeneous examples which are not Hermitian symmetric we can use the comparison done in [36] with the data supplied by [19] and [11]. If we use the notation of [19] and [11], only the three types below need to be checked:

$$(B_r, \alpha_i)_{r\geq 3, 1<i<r}, \quad (C_r, \alpha_i)_{r\geq 3, 1<i<r}, \quad (D_r, \alpha_i)_{r\geq 4, 1<i<r-1}.$$

The verification in [36, Section 2] applies verbatim. □

The above result strengthens the one in [36] since $\mathrm{CQB} > 0$ is stronger than $\mathrm{Ric}^{\perp} > 0$. Note that the result also holds for the exceptional (non-Hermitian symmetric) Kähler C-space (F_4, α_4) since for such a space $\lambda = \frac{11}{2}$ and the biggest eigenvalue of Q is 1. A future natural project is to classify all the compact Kähler manifolds with $\mathrm{CQB} > 0$. The example in [36], shows that there certainly are compact Kähler manifolds with $\mathrm{Ric}^{\perp} > 0$, but not homogeneous.

The second related curvature is a dual version of CQB, which is motivated by the study of the compact dual of the noncompact Hermitian symmetric spaces in [6]. We denote it by ${}^d\mathrm{CQB}$. It is defined as a quadratic Hermitian form on the space of linear maps $A : T'N \to T''N$:

$$ {}^d\mathrm{CQB}_R(A) \doteqdot R(\overline{A(E_i)}, A(E_i), E_k, \overline{E}_k) + R(E_i, \overline{E}_k, \overline{A(E_i)}, A(E_k)). $$

Similarly, we can introduce the concept ${}^d\mathrm{CQB}_k > 0$. The analogy of $\mathrm{Ric}^{\perp}$ is

$$ \mathrm{Ric}^{+}(X, \overline{X}) \doteqdot \mathrm{Ric}(X, \overline{X}) + \frac{H(X)}{|X|^2}. $$

We say ${}^d\mathrm{CQB}_k > 0$ if ${}^d\mathrm{CQB}(A) > 0$ for any $A \neq 0$ with rank not greater than k. Letting A be the map which satisfies $A(E_1) = \overline{E}_1$ and $A(E_i) = 0$ for all $i \geq 2$, it is easy to see that ${}^d\mathrm{CQB}_1 > 0$ implies that $\mathrm{Ric}^{+} > 0$. We discuss geometric implications of these two curvature notions in details next.

6. Manifolds with ${}^d\mathrm{CQB} > 0$ and the properties of Ric^{+}

We have seen that $\mathrm{Ric}_k^{\perp}$ interpolates between the orthogonal bisection sectional curvature and $\mathrm{Ric}^{\perp}$. In a similar manner we can define Ric_k^{+}, which interpolates between the holomorphic sectional curvature and Ric^{+} as Ric_k does. First we show that the diameter estimate in [37] for manifolds with $\mathrm{Ric}^{\perp}$ can be extended to Ric^{+} (Ric_k, Ric_k^{+}). The argument via the second variational formulae in the proof of the Bonnet–Meyer theorem proves the compactness of the Kähler manifolds if the Ric^{+} is uniformly bounded from below by a positive constant.

Theorem 6.1. *Let (N^n, h) be a Kähler manifold with*

$$ \mathrm{Ric}^{+}(X, \overline{X}) \geq (n+3)\lambda |X|^2 $$

with $\lambda > 0$. Then N is compact with diameter bounded from the above by $\sqrt{\frac{2n}{(n+3)\lambda}} \cdot \pi$. Moreover, for any geodesic $\gamma(\eta) : [0, \ell] \to N$ with length $\ell > \sqrt{\frac{2n}{(n+3)\lambda}} \cdot \pi$, the index $i(\gamma) \geq 1$.

Similarly, for a Kähler manifold (N^n, h) with

$$ \mathrm{Ric}_k \geq (k+1)\lambda > 0, $$

its diameter is bounded from above by $\sqrt{\frac{2k-1}{(k+1)\lambda}} \cdot \pi$; for a Kähler manifold (N^n, h) with

$$ \mathrm{Ric}_k^{\perp} \geq (k-1)\lambda > 0 $$

its diameter is bounded from above by $\sqrt{\frac{2}{\lambda}} \cdot \pi$; for a Kähler manifold (N^n, h) with

$$ \mathrm{Ric}_k^{+} \geq (k+3)\lambda > 0 $$

its diameter is bounded from above by $\sqrt{\frac{2k}{(k+3)\lambda}} \cdot \pi$.

Note that the result (for Ric^+) is slightly better than $\sqrt{\frac{2n-1}{(n+1)\lambda}}\pi$, the one predicted by the Bonnet–Meyer estimate assuming

$$\mathrm{Ric}(X, \overline{X}) \geq (n+1)\lambda|X|^2$$

for $n \geq 2$. But it is roughly about $\sqrt{2}$ times the one predicted by the Tsukamoto's theorem in terms of the lower bound of the holomorphic sectional curvature. Let $N = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, namely the product of n copies of $\mathbb{P}^1$, its diameter is $\sqrt{\frac{n}{2}}\pi$. An easy computation shows that it has $\mathrm{Ric} = 2$ and $H \geq \frac{2}{n}$. This shows that the upper bound provided by Tsukamoto's theorem holds equality on both $\mathbb{P}^n$ and $N = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. The product of n-copies of $\mathbb{P}^1$ also illustrates a compact Kähler manifold (after proper scaling) with $\mathrm{Ric} = n+1$, but its diameter is roughly about $\sqrt{2}$ times of that of $\mathbb{P}^n$. The product example and $\mathbb{P}^n$ indicate that the above estimate on the diameter is far from being sharp.

We prove Theorem 2.7 via a vanishing theorem with weaker assumptions. For that we introduce the scalar curvatures $S_k^+(x, \Sigma)$ which is defined as

$$S_k^+(x, \Sigma) = k \fint_{Z \in \Sigma, |Z|=1} \mathrm{Ric}^+(Z, \overline{Z})\, d\theta(Z)$$

for any k-dimensional subspace $\Sigma \subset T_x N$. Similarly, we say $S_k^+ > 0$ if $S_k^+(x, \Sigma) > 0$ for any x and Σ.

Theorem 6.2. *Assume that $S_k^+ > 0$. Then N is BC-p, positive and one has $h^{p,0} = 0$ for $k \leq p \leq n$.*

Proof. The first part of the proof follows similarly as in that of Theorem 2.2. Assuming the existence of a nonzero holomorphic $(p, 0)$-form ϕ leads to the conclusion that at the point x_0 where the maximum of the comass $\|\phi\|_0$ is attained we have that

$$0 \geq \sum_{j=1}^{p} R_{v\bar{v}j\bar{j}} \tag{6.1}$$

for any $v \in T'_{x_0}N$, for a particularly chosen frame $\{\frac{\partial}{\partial z_\ell}\}_{\ell=1,\ldots,n}$ with

$$\Sigma = \mathrm{span}\left\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_p}\right\}.$$

This implies that $S_p(x_0, \Sigma) \leq 0$, by applying the above to $v = \{\frac{\partial}{\partial z_i}\}_{1 \leq i \leq p}$.

Now a similar calculation as that of Section 2 shows that

$$\begin{aligned}
\frac{1}{p} S_p^+(x_0, \Sigma) \qquad\qquad & \qquad (6.2)\\
&= \fint_{Z \in \Sigma, |Z|=1} \mathrm{Ric}^+(Z, \overline{Z})\, d\theta(Z)\\
&= \fint_{Z \in \Sigma, |Z|=1} (\mathrm{Ric}(Z, \overline{Z}) + H(Z))\, d\theta(Z)\\
&= \fint \frac{1}{\mathrm{Vol}(\mathbb{S}^{2n-1})} \left(\int_{\mathbb{S}^{2n-1}} (nR(Z, \overline{Z}, W, \overline{W}) + H(Z))\, d\theta(W) \right) d\theta(Z)\\
&= \frac{1}{\mathrm{Vol}(\mathbb{S}^{2n-1})} \int_{\mathbb{S}^{2n-1}} \left(\fint (nR(Z, \overline{Z}, W, \overline{W}) + H(Z))\, d\theta(Z) \right) d\theta(W)\\
&= \frac{1}{p}(\mathrm{Ric}_{1\bar{1}} + \mathrm{Ric}_{2\bar{2}} + \cdots + \mathrm{Ric}_{p\bar{p}}) + \frac{2}{p(p+1)} S_p(x_0, \Sigma).
\end{aligned}$$

Using estimate (3.1) similarly as in Section 2 (cf. (3.3)), we have that

$$\mathrm{Ric}_{1\bar{1}} + \mathrm{Ric}_{2\bar{2}} + \cdots + \mathrm{Ric}_{p\bar{p}} \leq S_p(x_0, \Sigma).$$

Thus together with (6.2) it implies that

$$0 < S_k^+(x_0) \leq S_p^+(x_0, \Sigma) \leq \frac{p+3}{p+1} S_p(x_0, \Sigma).$$

This is a contradiction. □

Theorem 2.7 follows from the above theorem since $\mathrm{Ric}^+ > 0$ implies that $S_p^+ > 0$ for all $1 \leq p \leq n$. In the theorem below we extend the projectivity part of Theorem 1.1 to $\mathrm{Ric}_k^\perp$ and Ric_k^+.

Theorem 6.3. *Let (N, h) be a compact Kähler manifold. If $\mathrm{Ric}_k^\perp > 0$, or $\mathrm{Ric}_k^+ > 0$ for some $1 \leq k \leq n$, then N is BC-p positive for $p \geq 2$. In particular, $h^{p,0} = 0$ for $1 \leq p \leq n$, and N is projective and simply-connected.*

Proof. We only provide the proof for $\mathrm{Ric}_k^\perp$ since the proof for Ric_k^+ is similar. As before it suffices to prove that N is BC-p positive for all $1 \leq p \leq n$ since $h^{p,0} = 0$ follows from this. By Theorem 2.2 and Corollary 4.4, the BC-p positivity (for all $1 \leq p \leq n$) is known for $k = 2, n$. Thus we only need to prove it for $3 \leq k \leq n-1$. By Corollary 4.4 again we have BC-p positivity for $p \geq k-1$. Hence we only need to prove it for $p \leq k-2$.

The proof is essentially the same as the proof of Theorem 2.2. We prove by the contradiction argument. Assume that there exists unitary p-vectors $\{E_1, \dots, E_p\}$ such that (6.1) hold for any $v \in T'_x N$. Let $\Sigma = \mathrm{span}\{E_1, \dots, E_p\}$. Since $k-2 \geq p$, we extend them into unitary k-vectors $\{E_1, \dots, E_k\}$. Let Σ' be the k-dimensional subspace spanned by them. Clearly, $\Sigma \subset \Sigma'$. We denote by Ric' the Ricci curvature restricted to Σ'. We also define similarly $(\mathrm{Ric}^\perp)'$, S'_p and $(S^\perp)'_p$ correspondingly. In particular, $(\mathrm{Ric}^\perp)'$ is $\mathrm{Ric}_k^\perp(x, \Sigma')$, $(S^\perp)'_p$ is the average of $(\mathrm{Ric}^\perp)'$ on the unit sphere of a p-dimensional subspace of Σ', and

$$S'_p(x, \Sigma) = \sum_{i,j=1}^{p} R_{i\bar{i}j\bar{j}}.$$

Now the proof of Theorem 2.2 implies that for any $x \in N$,

$$(S^\perp)'_p(x) \leq \frac{p-1}{p+1} S'_p(x, \Sigma) = \frac{p-1}{p+1} \sum_{i,j=1}^{p} R_{i\bar{i}j\bar{j}} \leq 0.$$

On the other hand, the assumption implies that $(\mathrm{Ric}^\perp)' > 0$, hence $(S^\perp)'_p > 0$. This induces a contradiction. Hence we have N is BC-p positive for any $1 \leq p \leq n$ if $\mathrm{Ric}_k^\perp > 0$. □

Applying argument similar to that of the last section, we also have the following result.

Proposition 6.1. *Let (N^n, h) be a compact Kähler manifold of complex dimension n with quasi-positive Ric^+ (or Ric_k^+). Assume further that $h^{1,1}(N) = 1$ (or $\rho(N) = 1$). Then $c_1(N) > 0$, namely N is Fano.*

The proof of this result and the following lemma, which plays the analogue role of Lemma 5.1, is the same as that of the last section.

Lemma 6.1. *Let (N^n, h) be a Kähler manifold of complex dimension n. At any point $p \in N$,*

$$\frac{n+3}{n(n+1)} S(p) = \frac{1}{\mathrm{Vol}(\mathbb{S}^{2n-1})} \int_{|Z|=1,\, Z\in T'_pN} \mathrm{Ric}^+(Z, \overline{Z})\, d\theta(Z),$$

where $S(p) = \sum_{i=1}^n \mathrm{Ric}(E_i, \overline{E}_i)$ (with respect to any unitary frame $\{E_i\}$) denotes the scalar curvature at p.

Following the argument in [36, Appendix], we also have that *a* Ric^+*-Einstein Kähler metric must be of constant curvature*. In particular, the one with zero scalar curvature must be flat. Hence we have the same result as Corollary 2.5 if we replace $\mathrm{Ric}^\perp$ by Ric^+.

Corollary 6.4. *Let (N, h) be a compact Kähler manifold of complex dimension n with $\mathrm{Ric}^+ \geq 0$ (or $\mathrm{Ric}^+_k \geq 0$). Assume further that $h^{1,1}(N) = 1$ and N is locally irreducible. Then $c_1(N) > 0$, namely N is Fano. Similar result holds under the assumption $\mathrm{Ric}^+ \leq 0$.*

The result similar to Corollary 4.7 holds for $\mathrm{Ric}^+ > 0$ and $\rho(N) = 1$, in view of Theorem 2.7, Proposition 6.1 and Corollary 6.4.

Corollary 6.5. *Any compact Kähler manifold (N, h) with quasi-positive Ric^+ (or quasi-positive Ric^+_k) and $\rho(N) = 1$, must be rationally connected.*

The same holds if $\mathrm{Ric}^+ > 0$ is replaced with $\mathrm{Ric}^+ \geq 0$ and (N^n, h) is locally irreducible. For compact Kähler manifolds with $\mathrm{Ric}^+ < 0$, we have the result below.

Proposition 6.2. *Let (N, h) be a compact Kähler manifold with $\mathrm{Ric}^+ < 0$. Then N does not admits any nonzero holomorphic vector field.*

The proof is the same as that of [37]. A dual version of Theorem 5.1 is the following result.

Theorem 6.6. *The following statements hold.*

(i) *For (N, h) a compact Kähler manifold with quasi-positive ${}^d\mathrm{CQB}$,*

$$H^1(N, T'N) = \{0\}.$$

In particular, N is deformation rigid in the sense that it does not admit nontrivial infinitesimal holomorphic deformation.

(ii) *If a compact Kähler manifold (N, h) has ${}^d\mathrm{CQB}_2 > 0$, its Ricci curvature is 2-positive.*

(iii) *If (N, h) is compact with ${}^d\mathrm{CQB}_1 > 0$, then N is projective and simply-connected.*

Proof. For (i) one may use the conjugate operator $\# : A^{0,1}(T'N) \to A^{1,0}((T'N)^*)$ which is defined for $\phi = \phi^i_{\overline{\alpha}} dz^{\overline{\alpha}} \otimes E_i$, with $\{E_i\}$ being a unitary frame of $T'N$, as

$$\#\phi = \overline{\phi^i_{\overline{\alpha}}} dz^\alpha \otimes \overline{E}_i.$$

Since $\#(\partial\phi) = \overline{\partial}(\#(\phi))$, it implies that $\partial^*(\#(\phi)) = \#(\overline{\partial}^*\phi)$. Together $\#$ induces an isomorphism between $\mathcal{H}^{p,q}_{\overline{\partial}}(N, T'N)$ and $\mathcal{H}^{q,p}_{\partial}(N, (T'N)^*)$. To prove the result, it is sufficient to show that any $\psi \in \mathcal{H}^{1,0}_{\partial}(N, (T'N)^*)$, $\psi = 0$. Now we apply the Kodaira–Bochner formula for Δ_∂ operator, and get for $\psi = \psi_\alpha^{\bar{i}} dz^\alpha \otimes \overline{E}_i$ that

$$(\Delta_\partial \psi)_\gamma^{\bar{i}} = -h^{\alpha\bar{\beta}} \nabla_{\bar{\beta}} \nabla_\alpha \psi_\gamma^{\bar{i}} + R^{\bar{i}}{}_{\bar{j}\gamma}{}^{\sigma} \psi_\sigma^{\bar{j}} + (\mathrm{Ric})_\gamma^\sigma \psi_\sigma^{\bar{i}}. \tag{6.3}$$

Taking product with $\overline{\psi}$, as before under the unitary frame, if $\Delta_\partial \psi = 0$ we have that

$$0 = \int_N |\nabla \psi|^2 + \int_N \big[(\mathrm{Ric})_{\alpha\bar{\sigma}} \psi_\sigma^{\bar{i}} \overline{\psi_\alpha^{\bar{i}}} + R_{i\bar{j}\alpha\bar{\sigma}} \psi_\sigma^{\bar{j}} \overline{\psi_\alpha^{\bar{i}}}\big].$$

The claimed result follows in the similar way as in the proof of Theorem 5.1.

For part (ii), for any unitary frame $\{E_i\}$, let A be the rank 2 skew-symmetric transformation: $A(E_1) = \overline{E}_2$, $A(E_2) = -\overline{E}_1$, and $A(E_k) = 0$ for all $k \geq 3$. Then as in the CQB > 0 case, the second part in the expression of ^{d}CQB vanishes and the first part yields

$$\mathrm{Ric}(E_1, \overline{E}_1) + \mathrm{Ric}(E_2, \overline{E}_2).$$

Part (iii) follows from that ${}^d\mathrm{CQB}_1 > 0$ is the same as $\mathrm{Ric}^+ > 0$ and Theorem 2.7. □

By a similar argument (comparing the Einstein constant with the smallest eigenvalue of the symmetric curvature Q obtained in tables of [19]) as in the proof of Theorem 5.4 we also have the following corollary concerning Kähler C-spaces.

Theorem 6.7. *Let N^n be a classical Kähler C-space with $n \geq 2$ and $b_2 = 1$, or a compact exceptional Hermitian symmetric space with $n \geq 2$. Then the (unique up to constant multiple) Kähler–Einstein metric satisfies ^{d}CQB > 0. In particular, for a classical Kähler C-space N with $b_2 = 1$, $H^q(N, T'N) = \{0\}$ with $1 \leq q \leq n$, and N is deformation rigid in the sense that it does not admit nontrivial infinitesimal holomorphic deformation.*

Proof. To check ^{d}CQB > 0, writing $A(E_i) = A_i^t \overline{E}_t$, we then apply an argument similar to the case of CQB. First decompose A into $A_1 + A_2$, the symmetric and the skew symmetric parts. As in the proof of Theorem 5.4,

$${}^d\mathrm{CQB}(A) \geq \lambda |A_1|^2 + R_{i\bar{k}s\bar{t}} \overline{(A_1)_i^s} (A_1)_k^t.$$

Here λ is the Einstein constant of the canonical metric. The problem is now reduced to check that $\lambda + \nu_1 > 0$ with ν_1 being the smallest eigenvalue of Q. Recall the Hermitian symmetric linear operator Q is defined as

$$Q(X \cdot Y, \overline{Z \cdot W}) = R_{X\overline{Z}Y\overline{W}}$$

for $X \cdot Y = \frac{1}{2}(X \otimes Y + Y \otimes X)$ and extended linearly on the space of symmetric tensors. This quadratic curvature was considered previously in [6, 19]. We apply their results below. The Hermitian symmetric case again follows from [6, Table 2]. For the nonsymmetric classical Kähler C-spaces, we check the condition $\lambda + \nu_1 > 0$ as follows. Note that in [11] and [19] the same normalization for the canonical metric was used. For $(B_r, \alpha_i)_{r\geq 3, 1<i<r}$, $\lambda = 2r - i$. According to [19, Table 4], $\nu_1 = -2(r-i) + 1$ or -2. Since $2r \geq 2i + 2$, clearly $2r - i > 2$. Also $2r - i - 2r + 2i + 1 = i + 1 > 0$. This implies the result for both cases of ν_1, namely $\nu_1 = -2(r-i) + 1$ and $\nu_1 = -2$.

For $(C_r, \alpha_i)_{r\geq 3, 1<i<r}$, $\lambda = 2r - i + 1$. According to [19, Table 7], $\nu_1 = -2(r - i + 1)$. Hence $\lambda + \nu_1 = i - 1 > 0$ for $i \geq 2$. This verifies the result.

For $(D_r, \alpha_i)_{r\geq 4, 1<i<r-1}$, $\lambda = 2r-i-1$. According to [19, Table 10], $\nu_1 = -2(r-i)+2$ or -2. Since $2r-i-3 \geq i-1 > 0$ and $2r-i-1-2r+2i+2 = i+1 > 0$, this also verifies the result.

This proved the $H^1(N, T'N) = \{0\}$. For $q > 1$, the argument of [6] implies that one only needs to check that $\lambda + \frac{q+1}{2q}\nu_1 > 0$. This is a consequence of the $q = 1$ case above. $\square$

For the exceptional space (F_4, α_4) since $\lambda = \frac{11}{2}$ and $\nu_1 = -5$, the above result also holds. Hence it should not be surprising that the result in the corollary holds for the rest (22 of them total) exceptional Kähler C-spaces. The deformation rigidity result above holds infinitesimally. It would be interesting to see for a deformation with each fiber except the central is assumed to be biholomorphic to a fixed manifold with ${}^d\mathrm{CQB} > 0$, whether or not the central fiber must be the same manifold (as the main theorem of [42]). If we express $A = \sum_{\ell=1}^{k} \overline{X}_\ell \otimes \overline{Y}_\ell$, we have that ${}^d\mathrm{CQB}_k > 0$ if and only if

$$\sum_{\ell,j=1}^{k} \mathrm{Ric}(X_\ell, \overline{X}_j)\langle Y_\ell, \overline{Y}_j\rangle + R(X_\ell, \overline{X}_j, Y_\ell, \overline{Y}_j) > 0, \quad \sum_{\ell=1}^{k} \overline{X}_\ell \otimes \overline{Y}_\ell \neq 0.$$

Non-locally Hermitian symmetric examples of compact Kähler manifold with ${}^d\mathrm{CQB} < 0$ have been constructed in [38].

Questions similar to those in Question 1 can be asked for ${}^d\mathrm{CQB}$. We also add the following question:

Question 2. For which $k \in \{1, \ldots, n\}$, Ric_k, $\mathrm{Ric}_k^\perp$, Ric_k^+, $\mathrm{CQB}_k \geq 0$, and ${}^d\mathrm{CQB}_k \geq 0$ are preserved under the Kähler–Ricci flow? Is a compact manifold with $\mathrm{Ric}_k^\perp < 0$ projective?

Concerning this, some Ricci-flow invariant cones have been constructed recently in [38]. It is also known that $\mathrm{Ric}_2^\perp$ is preserved by the Kähler–Ricci flow.

A. Appendix: Estimates on the harmonic (1, 1)-forms of low rank

We prove a vanishing theorem for harmonic $(1,1)$-forms of low rank related to the condition $\mathrm{QB}_k > 0$ introduced earlier. This is particularly relevant given that in [38] examples of arbitrary large b_2 was constructed with $\mathrm{CQB} > 0$ (in particular with $\mathrm{Ric}^\perp > 0$). First recall that

$$\mathrm{QB}_R(A) = \sum_{\alpha,\beta=1}^{n} R(A(E_\alpha), \overline{A(E_\alpha)}, E_\beta, \overline{E}_\beta) - R(E_\alpha, \overline{E}_\beta, A(E_\beta), \overline{A(E_\alpha)})$$

vanishes for $A = \lambda\,\mathrm{id}$. Hence when define $\mathrm{QB}_k(A) > 0$ we require the above expression positive for A in $S^2(\mathbb{C}^n) \setminus \{\lambda\,\mathrm{id}\}$, and that A has rank not greater than k. The space of harmonic $(1,1)$-forms $\mathcal{H}^{1,1}_{\bar\partial}$ can be decomposed further. First we observe that an $(1,1)$-form

$$\Omega = \sqrt{-1}A_{i\bar{j}}dz^i \wedge dz^{\bar{j}}$$

can be decomposed as

$$\Omega = \Omega_1 - \sqrt{-1}\Omega_2 = \frac{\sqrt{-1}}{2}B_{i\bar{j}}dz^i \wedge dz^{\bar{j}} - \sqrt{-1}\left(\frac{\sqrt{-1}}{2}C_{i\bar{j}}dz^i \wedge dz^{\bar{j}}\right)$$

with

$$B_{i\bar{j}} = A_{i\bar{j}} + \overline{A_{j\bar{i}}}, \quad C_{i\bar{j}} = \sqrt{-1}(A_{i\bar{j}} - \overline{A_{j\bar{i}}}).$$

If Ω is harmonic, then $\partial\Omega = \bar{\partial}\Omega = 0$. It can be verified that Ω_1 and Ω_2 are both harmonic (cf. [30, Theorem 5.4 in Chapter 3]). This shows that Ω can be decomposed into the sum of a Hermitian symmetric one with $-\sqrt{-1}$ times another Hermitian symmetric one. Namely,

$$\mathcal{H}^{1,1}_{\bar{\partial}} = \mathcal{H}^{1,1}_{\bar{\partial},s} - \sqrt{-1}\mathcal{H}^{1,1}_{\bar{\partial},s},$$

where $\mathcal{H}^{1,1}_{\bar{\partial},s}$ is the spaces of harmonic Ω with $(A_{i\bar{j}})$ being Hermitian symmetric. Within $\mathcal{H}^{1,1}_{\bar{\partial},s}$ we shall consider $\mathcal{H}^{1,1}_{\bar{\partial},s} \setminus \{\mathbb{C}\omega\}$. To prove $b_2 = 1$ under the assumption $\mathrm{QB} > 0$, it suffices to show that

$$\mathcal{H}^{1,1}_{\bar{\partial},s} \setminus \{\mathbb{C}\omega\} = \{0\}.$$

We can stratify the space into ones with rank bounded from above. Let $\mathcal{H}^{1,1}_{s,k}$ denote the subspace of $\mathcal{H}^{1,1}_{\bar{\partial},s}$ which consists of

$$\Omega = \frac{\sqrt{-1}}{2} A_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$$

with $(A_{i\bar{j}})$ being Hermitian symmetric and of rank no greater than k everywhere on N. The following result can be shown.

Theorem A.1. *Assume that (N^n, g) is a compact Kähler manifold with quasi-positive* QB_k *with $k < n$. Then $\mathcal{H}^{1,1}_{s,k}(N) = \{0\}$. In particular,* $\mathrm{Ric}^{\perp} > 0$ *implies that $\mathcal{H}^{1,1}_{s,1}(N) = \{0\}$.*

Proof. Assume that Ω is a nonzero element in $\mathcal{H}^{1,1}_{s,k}(N)$. Applying the Δ operator to $\|\Omega\|^2$, by the Kodaira–Bochner formula we have that

$$\frac{1}{2}(\nabla_\gamma\nabla_{\bar{\gamma}} + \nabla_{\bar{\gamma}}\nabla_\gamma)\|\Omega\|^2(x) = \|\nabla_\gamma\Omega\|^2(x) + \|\nabla_{\bar{\gamma}}\Omega\|^2(x) + 2\mathrm{QB}(\Omega)(x).$$

Integrating on N, we have that

$$0 = \int_N \left[\|\nabla_\gamma\Omega\|^2(x) + \|\nabla_{\bar{\gamma}}\Omega\|^2(x)\right] d\mu(x) + 2\int_N \mathrm{QB}(\Omega)(x)\, d\mu(x) > 0.$$

The last strictly inequality is due to the fact that by the unique continuation we know at a neighborhood U where $\mathrm{QB}_k > 0$, and Ω is identically zero. The contradiction implies that $\Omega \equiv 0$. □

For any holomorphic line bundle L over N with a Hermitian metric a, its first Chern form

$$c_1(L,a) = -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log a$$

is a Hermitian symmetric $(1,1)$-form. If η is the harmonic representative of $c_1(L,a)$, then η is Hermitian symmetric by the uniqueness of the Hodge decomposition and Kähler identities (cf. [30, Chapter 3]). The following is a simple observation towards possible topological meanings of the rank of η (the minimum k such that $\eta \in \mathcal{H}^{1,1}_{\bar{\partial},k}$, denoted as $\mathrm{rk}(L)$).

Proposition A.1. *Recall that the numerical dimension of L is defined as*

$$\mathrm{nd}(L) = \max\{k = 0, \dots, n : c_1(L)^k \neq 0\}.$$

Then $\mathrm{rk}(L) \geq \mathrm{nd}(L)$.

The proof of the above theorem also shows that if $\mathrm{QB}_k \geq 0$, then any element in $\mathcal{H}^{1,1}_{s,k}(N)$ must be parallel. Thus we have the dimension estimate

$$\dim(\mathcal{H}^{1,1}_{s,k}(N)) \leq k^2.$$

In fact, the existence of a non-vanishing $(1,1)$-form of rank at most k has a strong implication due to the De Rham decomposition.

Corollary A.2. *Assume that* $\mathrm{QB}_k \geq 0$ *and* $\mathcal{H}^{1,1}_{s,k}(N) \neq \{0\}$. *Then* N *must be locally reducible. In particular, if* N *is locally irreducible and* $\mathrm{Ric}^{\perp} \geq 0$, *then* $\mathcal{H}^{1,1}_{s,1}(N) = \{0\}$.

Proof. By the above, we know that the nonzero $\Omega \in \mathcal{H}^{1,1}_{s,k}(N)$ must be parallel. Its null space is invariant under the parallel transport. This provides a nontrivial parallel distribution, hence the local splitting. □

The product example $\mathbb{P}^2 \times \mathbb{P}^2$, which satisfies $\mathrm{Ric}^{\perp} > 0$ and supports nontrivial rank 2 harmonic $(1,1)$-forms, shows that the above result is sharp for $\mathrm{Ric}^{\perp} > 0$. Irreducible examples of dimension greater than 4 were constructed via the projectivized bundles in [36].

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