Non-existence of Some Quasi-conformal Harmonic Diffeomorphisms

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1 Introduction

The property of harmonic maps between complete Riemannian manifolds has been studied extensively by many authors(Cf. [Ch], [Sh], [T], etc). In the present paper we show some non-existence results for quasi-conformal harmonic diffeomorphism between complete Riemannian manifolds. In dimension two, harmonic maps are closely related to the deformation theory of Riemann surfaces. One of the questions that arises naturally is: *whether Riemann surfaces which are related by harmonic diffeomorphism are necessarily quasi-conformally related?* See R. Schoen's article [S] for a general discussion on this subject, where other questions were also discussed. The result we show in this paper provides some partial answers to the high dimension generalization of this type of questions. In particular we prove the following result of which can be thought as a Liouville type

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theorem for harmonic diffeomorphisms.

Theorem 1.1 *Let M^n be a complete manifold with $Ricci_M \geq 0$, and let N^n be a simply-connected manifold with nonpositive sectional curvature, where n is the dimension of both manifolds. If there is a point $p \in M$ such that $\lim_{r\to\infty} V_p(r) = o(r^n)$, then there is no quasi-conformal harmonic diffeomorphism from M into N with polynomial growth energy density.*

It is not surprising that the growth rate of energy density plays a role here. For example in [W], T. Wan proved that a harmonic diffeomorphism between hyperbolic spaces of dimension two is quasi-conformal if and only if it has bounded energy density. The *only if* part of Wan's theorem was generalized to high dimension in [L-T-W]. There they proved that if the Ricci curvature of the domain manifold is bounded from below and the first eigenvalue of the target manifold is positive, then any quasi-conformal harmonic diffeomorphism into the target manifold has bounded energy density. These results and some other related results in [H-T-T-W] all indicate that the growth condition on the energy density is a natural assumption and is closely related to the study of quasi-conformal diffeomorphisms. On the other hand, we can show by examples that Theorem 1.1 will not be true if any of the assumptions is removed. We also should point out that, besides the curvature assumption, we only use the fact that the target manifold N satisfies Sobolev-inequality in the proof of Theorem 1.1. So Theorem 1.1 still holds for more general target manifolds, for example, when N is a minimal submanifolds of $\mathbf{R}^K$, since in this case, we know from [M-S] Sobolev inequality holds on N.

On the other hand, it is well-known that there is no non-constant holomorphic map from a complete Kähler manifold M into a complex Hermitian manifold N if M has nonnegative Ricci curvature and N has holomorphic bisectional curvature bounded from above by a negative constant. This follows easily from the generalized Schwarz lemma(Cf. [Y]). Using a modified argument of Theorem 1.1 we can generalize this result to quasi-conformal diffeomorphisms as follows:

Theorem 1.2 *Let M^n be a complete Riemannian manifold with $\lim_{r\to\infty} V_p(r) = o(r^n)$, and let N^n be a complete Riemannian manifold with $\lambda_1(N) > 0$, where n is the dimension of both manifolds, $p \in M$ is any fixed point, $V_p(r)$ is the volume of the ball of radius r centered at p and $\lambda_1(N)$ is the lower bound of the spectrum of the Laplacian-Beltrami operator. Then there is no quasi-conformal diffeomorphism form M into N.*

We know that if N is simply-connected and has sectional curvature bounded from above by some negative constant then one has $\lambda_1(N) > 0$. That is why we call Theorem 1.2 a generalization of the above mentioned result on holomorphic maps(which is derived from the generalized Schwarz lemma). The interesting thing is that Theorem 1.2 is invariant under the quasi-isometries while Theorem 1.1 is not.

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2 The Proof of the Theorems

Proof of Theorem 1.1. We prove by contradiction.

Assume that there is a harmonic diffeomorphism u from M into N. Let $a^2(x) = \inf_{\{v\in TM_x \,|\, \|v\|=1\}} |(du)^* \circ (du)(v)|^2$. By choosing a suitable orthonormal frame $\{e_i\}$ around x in TM we have that

$$(du)^* \circ (du) = \begin{pmatrix} \lambda_1^2 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & \lambda_n^2 \end{pmatrix}.$$

We can assume that $\lambda_1^2 \geq \lambda_2^2 \geq \dots \lambda_n^2 = a^2(x)$. By definition we have $e(u) = \Sigma_{i=1}^n \lambda_i^2$ and $J(u) = \lambda_1 \dots \lambda_n$.

Let $\phi(x)$ be a smooth function with compact support defined on M. Then $\phi \circ u^{-1}(y)$ is a smooth function with compact support defined on N. By the assumption on N we know that the Sobolev inequality holds (Cf. [H-S]), i. e. we have a constant S such that

$$\int_N |\nabla \varphi| \; dv_N \geq S \left(\int_N |\varphi|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}, \quad \text{for all } \varphi \in C_0^\infty(N).$$

Applying to $\phi \circ u^{-1}$ we have

$$\int_N |(\nabla \phi \circ u^{-1})(y)| \; dv_N \geq S \left(\int_N |(\phi \circ u^{-1})(y)|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}. \tag{2.1}$$

On the other hand, direct calculation shows that

$$\begin{aligned} \int_N |\nabla_N (\phi \circ u^{-1})(y)| \; dv_N &\leq \int_M |(\nabla_M \phi)(x)| a^{-1}(x) J(x) dv_M \\ &\leq \int_M |\nabla_M \phi| \; |\lambda_1(x) \ldots \lambda_{n-1}(x)| dv_M. \end{aligned}$$

Using the arithmetic-geometric inequality $(\lambda_1 \ldots \lambda_{n-1})^{\frac{2}{n-1}} \leq \frac{\sum_{i=1}^{n-1} \lambda_i^2(x)}{n-1} \leq \frac{e(u)}{n-1}$ we have

$$\int_M |\nabla_N (\phi \circ u^{-1})(y)| dv_N \leq C(n) \int_M |(\nabla_M \phi)(x)| \; e^{\frac{n-1}{2}} \; dv_M. \tag{2.2}$$

Combining (2.1) and (2.2) we have

$$\left(\int_N |(\phi \circ u^{-1})(y)|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq C(n, S) \int_M |\nabla_M \phi| \; e^{\frac{n-1}{2}} \; dv_M. \tag{2.3}$$

Now we estimate $\left(\int_N |(\phi \circ u^{-1})(y)|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}$ using the quasi-conformity of u as follows.

Recall the definition of the quasi-conformal constant to be $\alpha = \sup_{x \in M} \frac{\lambda_1}{\lambda_n}$. Then we have $\frac{\lambda_1}{\lambda_i} \leq \alpha$, and

$$J(x) = \lambda_1 \ldots \lambda_n \geq (a(x))^n;$$

$$e(x) = \sum_{i=1}^n \lambda_i^2 \leq a(x)^2((n-1)\alpha^2 + 1).$$

Combining them we have

$$(2.4)\qquad \left(\int_N |(\phi\circ u^{-1})(y)|^{\frac{n}{n-1}}\ dv_N\right)^{\frac{n-1}{n}} \geq C(n,\alpha)\left(\int_M |\phi(x)|^{\frac{n}{n-1}}\ e^{\frac{n}{2}}\ dv_M\right)^{\frac{n-1}{n}}.$$

(2.4) together with (2.3) implies

$$(2.5)\qquad \left(\int_M |\phi|^{\frac{n}{n-1}}\ e^{\frac{n}{2}}\ dv_M\right)^{\frac{n-1}{n}} \leq C(n,\alpha,S)\ \int_M |\nabla_M\phi|\ e^{\frac{n-1}{2}}\ dv_M.$$

The rest of the proof is just deriving a contradiction out of (2.5).
Let $S(r)=\sup_{B_p(r)} e(u)$. It is well known that $e(u)$ is a subharmonic function under our curvature assumptions, from the Bochner formula for harmonic maps. By the mean-value inequality for subharmonic functions on manifolds with nonnegative *Ricci* curvature we have

$$S(\frac{r}{2}) \leq C(n)\ \frac{1}{V_p(r)}\int_{B_p(r)} e(u)(x)\ dv_M.$$

By choosing ϕ to be

$$\phi(x)=\begin{cases} 1 & \text{for } x\in B_p(r),\\ 0 & \text{for } x\in M\setminus B_p(2r),\end{cases}$$

$$|\nabla\phi| \leq \frac{C}{r}, \text{with } C=2,$$

the inequality (2.5) yields

$$\left(\frac{1}{V_p(r)}\int_{B_p(r)} e(u)^{\frac{n}{2}}\ dv_M\right)^{\frac{n-1}{n}} \leq C(n,\alpha,S)\frac{(V_p(r))^{\frac{1}{n}}}{r}\frac{V_p(2r)}{V_p(r)}\frac{1}{V_p(2r)}\int_{B_p(2r)} e(u)^{\frac{n-1}{2}}\ dv_M.$$

Combining with Li-Schoen's mean-value inequality and the volume doubling property of *Ricci* nonnegative manifolds we have

$$(2.6)\qquad \left(S(\frac{r}{2})\right)^{\frac{n-1}{2}} \leq C\frac{(V_p(r))^{\frac{1}{n}}}{r}\left(S(2r)\right)^{\frac{n-1}{2}}.$$

Now we use the polynomial growth condition on the energy density of u. The polynomial growth assumption simply means that there exists a constant K such that $S(r) \leq K(1+r^d)$ for some $d \geq 0$. But it is not hard to show that the polynomial growth condition implies that there exists a constant $A > 0$ and $r_j \to \infty$ such that

$$\frac{S(2r_j)}{S(\frac{r_j}{2})} \leq A.$$

Applying (2.6) to r_j we have

$$\left(S(\frac{r_j}{2})\right)^{\frac{n-1}{2}} \left(1 - C\frac{(V_p(r_j))^{\frac{1}{n}}}{r_j}(A)^{\frac{n-1}{2}}\right) \leq 0.$$

Letting $r_j \to \infty$ and using the fact $V_p(r) = o(r^n)$ we have

$$\lim_{r_j \to \infty} S(\frac{r_j}{2}) \leq 0.$$

Since $e(u)(x)$ is a subharmonic function and it achieves its maximum at infinity we have $e(u)(x) \equiv 0$, which is a contradiction since u is a quasi-conformal diffeomorphism. q.e.d.

Corollary 2.1 *There is no quasi-conformal harmonic diffeomorphism from* $\mathbf{S}^k \times \mathbf{R}^{n-k}$ *into* $\mathbf{R}^n$ *with polynomial growth energy density.*

In order to prove Theorem 1.2 we need the following lemma which is well-known to the experts. But for the completeness we include a simple proof here.

Lemma 2.2 *If there exists a positive constant* A_p *such that*

$$\int_N |\varphi|^p \, dv_N \leq A_p \int_N |\nabla\varphi|^p \, dv_N, \quad \text{for any } \varphi \in C_0^\infty(N). \tag{2.7}$$

Then there exists a positive constant A_q *such that*

$$\int_N |g|^q \, dv_N \leq A_q \int_N |\nabla g|^q \, dv_N \quad \text{for any } g \in C_0^\infty(N), \tag{2.8}$$

provided $q \geq p$. *In other words,* L_p*-Poincaré implies* L_q*-Poincaré if* $q \geq p$.

Proof. Let $\psi = |g|^{\frac{q}{p}}$. Then

$$|\nabla \psi| = \frac{q}{p}|g|^{\frac{q-p}{p}}|\nabla g|.$$

Applying (2.7) to ψ we have

$$\begin{aligned}
\int_N |g|^q \, dv_N &= \int_N |\psi|^p \, dv_N \\
&\leq A_p \int_N |\nabla \psi|^p dv_N \\
&= A_p \left(\frac{q}{p}\right)^p \int_N |\nabla g|^p |g|^{q-p} dv_N \\
&\leq A_p \left(\frac{q}{p}\right)^p \left(\int_N |\nabla g|^q\right)^{\frac{p}{q}} \left(\int_N |g|^q\right)^{\frac{q-p}{q}}.
\end{aligned}$$

Then we have (2.8) with $A_q = (A_p)^{\frac{q}{p}} \left(\frac{q}{p}\right)^q$.

Now we can begin to prove Theorem 1.2. By the assumption on N we have the L_2-Poincaré inequality

$$\lambda_1 \int_N |\varphi|^2 \leq \int_N |\nabla \varphi|^2, \quad \text{for any } \varphi \in C_0^\infty(N).$$

Applying Lemma 2.2 we have that

$$\int_N |\varphi|^n dv_N \leq C(\lambda_1, n) \int_N |\nabla \varphi|^n dv_n.$$

As in the proof of Theorem 1.1 we apply above inequality to $\phi \circ u^{-1}$. Then we have

$$\int_N |(\phi \circ u^{-1})(y)|^n \, dv_N \leq C(\lambda_1, n) \int_N |\nabla_N (\phi \circ u^{-1})(y)|^n \, dv_N. \tag{2.9}$$

Similar calculation as in the proof of (2.2) shows

$$\int_N |\nabla_N (\phi \circ u^{-1})(y)|^n \, dv_N \leq C(\alpha) \int_M |(\nabla_M \phi)(x)|^n dv_M.$$

On the other hand same calculation as in the proof of (2.4) shows that

$$\int_M |\phi(x)|^n e(u)^{\frac{n}{2}} dv_M \leq \int_N |(\phi \circ u^{-1})(y)|^n dv_N.$$

Combining the preceding two inequalities and choosing ϕ as in the proof of Theorem 1.1 we will have

$$\frac{V_p(r)}{r^n} \geq C(\lambda_1, \alpha, n) \int_{B_p(r)} e(u)^{\frac{n}{2}}. \tag{2.10}$$

Now our assumption on the volume growth yields $e(u) \equiv 0$, which completes the proof.

Combining the proof of Theorem 1.1 and Theorem 1.2 we can have the following corollary.

Corollary 2.3 *Let M^n be a complete Riemannian manifold with nonnegative Ricci curvature, and let N^n be a complete Riemannian manifold with nonpositive sectional curvature and $\lambda_1(N) > 0$, where n is the dimension of the both manifolds. Then there exists no quasi-conformal harmonic diffeomorphism from M into N.*

Proof. From (2.10) and the mean value inequality of Li-Schoen we can write

$$\begin{aligned} \sup_{B_p(\frac{r}{2})} e(u)^{\frac{n}{2}} &\leq \frac{C(M)}{V_p(r)} \int_{B_p(r)} e(u)^{\frac{n}{2}} \\ &\leq C(\lambda_1, \alpha, n, M) \frac{1}{r^n}. \end{aligned}$$

Letting $r \to \infty$, we have $e(u) \equiv 0$. We complete the proof.

Finally we present two examples. They will show that in Theorem 1.1 both the volume growth assumption and the polynomial growth assumption of the energy density are indeed necessary.

Example 1. This example shows that if we do not assume $V_p(r) = o(r^n)$ we can't have our theorem. Let $M = N = R^n$ and u be the identity map between R^n's then u has bounded energy density and satisfies all the other assumptions of Theorem 1.1.

Example 2. This example shows that if we do not have the growth condition on the energy density our theorem also fails. Here we let $M = S^1 \times R$ and $N = R^2$, u mapping from M into N (in fact onto $R^2 \setminus \{0\}$) is given by $\eta = \exp(r)$ and $\omega = \theta$, where $dr^2 + d\theta^2$ is the metric on M and $d\eta^2 + \eta^2 d\omega^2$ is the metric on N. We can see easily $e(u)$ has exponential growth energy density and satisfies all the other assumptions in Theorem 1.1.

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