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General Schwarz Lemmata and their applications

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Dedicated to Professor Luen-Fai Tam on the occasion of his 70th birthday.

We prove estimates interpolating the Schwarz Lemmata of Royden–Yau and the ones recently established by the author. These more flexible estimates provide additional information on (algebraic) geometric aspects of compact Kähler manifolds with nonnegative holomorphic sectional curvature, nonnegative Ric_ℓ or positive S_ℓ.

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1. Introduction

There are many generalizations of the classical Schwarz Lemma on holomorphic maps between unit balls via the work of Ahlfors, Chen–Cheng–Look, Lu, Mok–Yau, Royden, Yau, etc. (see [10] and [23, 28] and references therein). The one obtained by Royden [23] states:

Theorem 1.1. *Let $f : M^m \to N^n$ be a holomorphic map. Assume that the holomorphic sectional curvature of N, $H(Y) \le -\kappa|Y|^4$, $\forall\, Y \in T'N$ and the Ricci curvature of M, $\mathrm{Ric}^M(X, \overline{X}) \ge -K|X|^2$, $\forall\, X \in T'M$ with $\kappa, K > 0$. Let $d = \dim(f(M))$. Then*

$$\|\partial f\|^2(x) \le \frac{2d}{d+1}\frac{K}{\kappa}. \tag{1.1}$$

In [17], the author proved a new version which only involves the holomorphic sectional curvature of domain and target manifolds. Recall that for the tangent map $\partial f : T'_x M \to T'_{f(x)} N$, we define its maximum norm square to be

$$\|\partial f\|_0^2(x) \doteqdot \sup_{v\neq 0} \frac{|\partial f(v)|^2}{|v|^2}. \tag{1.2}$$

Theorem 1.2. *Let (M, g) be a complete Kähler manifold such that the holomorphic sectional curvature $H^M(X)/|X|^4 \geq -K$ for some $K \geq 0$. Let (N^n, h) be a Kähler manifold such that $H^N(Y) < -\kappa|Y|^4$ for some $\kappa > 0$. Let $f : M \to N$ be a holomorphic map. Then*

$$\|\partial f\|_0^2(x) \leq \frac{K}{\kappa}, \quad \forall\, x \in M, \tag{1.3}$$

provided that the bisectional curvature of M is bounded from below if M is not compact. In particular, if $K = 0$, any holomorphic map $f : M \to N$ must be a constant map.

The assumption on the bisectional curvature lower bound can be replaced with the existence of an exhaustion function $\rho(x)$ which satisfies that

$$\limsup_{\rho\to\infty} \left(\frac{|\partial\rho| + [\sqrt{-1}\partial\bar{\partial}\rho]_+}{\rho} \right) = 0. \tag{1.4}$$

The proof uses a viscosity consideration from PDE theory. It is also reminiscent of Pogorelov's Lemma [22] (cf. [7, Lemma 4.1.1]) for Monge–Ampère equation, since the maximum eigenvalue of $\nabla^2 u$ is the $\|\cdot\|_0$ for the normal map ∇u for any smooth u. A consequence of Theorem 1.2 asserts that *the equivalence of the negative amplitude of the holomorphic sectional curvature implies the equivalence of the metrics*. Namely, if M^m admits two Kähler metrics g_1 and g_2 satisfying that

$$-L_1|X|^4_{g_1} \leq H_{g_1}(X) \leq -U_1|X|^4_{g_1}, \quad -L_2|X|^4_{g_2} \leq H_{g_2}(X) \leq -U_2|X|^4_{g_2}$$

then for any $v \in T'_x M$, we have the estimates:

$$|v|^2_{g_2} \leq \frac{L_1}{U_2}|v|^2_{g_1}; \quad |v|^2_{g_1} \leq \frac{L_2}{U_1}|v|^2_{g_2}.$$

This result can be viewed as a stability statement of the classical result asserting that a complete Kähler manifold with the negative constant holomorphic sectional curvature must be a quotient of the complex hyperbolic space form. Motivated by Rauch's work which induces much work towards the 1/4-pinching theorem and, the above stability of Kähler metrics it is natural to ask *whether or not a Kähler manifold M with its homomorphic sectional curvature being close to -1 is biholomorphic to a quotient of the complex hyperbolic space.* Besides the Liouville type theorem for holomorphic maps into manifolds with negative holomorphic sectional curvature, we shall show in Sec. 5 further implications of this estimate towards the structure of the fundamental groups of manifolds with nonnegative holomorphic sectional curvature.

Before we state another recent result of the author, we first recall some basic notions from Grassmann algebra [5, 26]. Let $\mathbb{C}^m$ be a complex Hermitian space (later, we will identify the holomorphic tangent spaces T'_xM and $T'_{f(x)}N$ with $\mathbb{C}^m$ and $\mathbb{C}^n$). Let $\wedge^\ell\mathbb{C}^m$ be the spaces of ℓ-multi-vectors $\{v_1 \wedge \cdots \wedge v_\ell\}$ with $v_i \in \mathbb{C}^m$. For $\mathbf{a} = v_1 \wedge \cdots \wedge v_\ell, \mathbf{b} = w_1 \wedge \cdots w_\ell$, the inner product can be defined as $\langle \mathbf{a}, \overline{\mathbf{b}}\rangle = \det(\langle v_i, \bar{w}_j\rangle)$. This endows $\wedge^\ell\mathbb{C}^n$ an Hermitian structure, hence a norm $|\cdot|$. There are also other norms, such as the *mass* and the *comass*, which shall be denoted as $|\cdot|_0$ as in [26], and could be useful for some problems. We refer [26, Secs. 13 and 14] for detailed discussions. Assume that $f : (M^m, g) \to (N^n, h)$ is a holomorphic map between two Kähler manifolds. Let $\partial f : T'M \to T'N$ be the tangent map. Let $\Lambda^\ell\partial f : \wedge^\ell T'_xM \to \wedge^\ell T'_{f(x)}N$ be the associated map defined as $\Lambda^\ell\partial f(v_1 \wedge \cdots \wedge v_\ell) = \partial f(v_1) \wedge \cdots \wedge \partial f(v_\ell)$. Define $\|\cdot\|_0$ as

$$\|\Lambda^\ell\partial f\|_0(x) \doteqdot \sup_{\mathbf{a}=v_1\wedge\cdots\wedge v_\ell\neq 0, \mathbf{a}\in\wedge^\ell T'_xM} \frac{|\Lambda^\ell\partial f(\mathbf{a})|}{|\mathbf{a}|}.$$

The notion $\|\cdot\|_0$ is adapted to be consistent with the comass notion in [26]. By the singular value decomposition, we may choose normal coordinates centered at x_0 and $f(x_0)$ such that at x_0, $df(\frac{\partial}{\partial z^\alpha}) = \lambda_\alpha \delta_{i\alpha}\frac{\partial}{\partial w^i}$. If we order $\{\lambda_\alpha\}$ such that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_m|$, $\|\Lambda^\ell\partial f\|_0(x_0) = |\lambda_1\cdots\lambda_\ell|$. It is also easy to see that $\|\partial f\|^2 \doteqdot g^{\alpha\bar{\beta}}h_{i\bar{j}}\frac{\partial f^i}{\partial z^\alpha}\overline{\frac{\partial f^j}{\partial z^\beta}} = \sum_{\alpha=1}^m |\lambda_\alpha|^2$. The following was proved in [17, Corollary 3.4].

Theorem 1.3. *Let $f : M^m \to N^n$ $(m \leq n)$ be a holomorphic map with M being a complete manifold. Assume that Ric^M is bounded from below and the scalar curvature $S^M(x) \geq -K$. Assume further that $\mathrm{Ric}^N_m(x) \leq -\kappa < 0$. Then we have the estimate*

$$\|\Lambda^m\partial f\|_0^2(x) \leq \left(\frac{K}{m\kappa}\right)^m.$$

Here, recall that in [17] $\mathrm{Ric}(x, \Sigma)$ is defined as the Ricci curvature of the curvature tensor restricted to the k-dimensional subspace $\Sigma \subset T'_xM$. Precisely for any $v \in \Sigma$, $\mathrm{Ric}(x, \Sigma)(v, \bar{v}) \doteqdot \sum_{i=1}^k R(E_i, \overline{E}_i, v, \bar{v})$ with $\{E_i\}$ being a unitary basis of Σ. We say that $\mathrm{Ric}_k(x) < 0$ if $\mathrm{Ric}(x, \Sigma) < 0$ for every k-dimensional subspace Σ. Clearly, $\mathrm{Ric}_k(x) < 0$ implies that $S_k(x) < 0$, and it coincides with H when $k = 1$, with the Ricci curvature Ric when $k = \dim(N)$. Here, $S_k(x, \Sigma)$ is defined to be the scalar curvature of the curvature operator restricted to $\Sigma \subset T'_xN$. One can refer to [17, 18, 21] for the definitions and related results on the geometric significance of Ric_ℓ and S_ℓ.

Note that Theorem 1.3 has at least two limits in studying the holomorphic maps. The first it applies only to the case that $\dim(N)$, the dimension of the target manifold is at least as big as the dimension of the domain. The second limit is that it can only be applied to detect whether or not the map is full-dimensional, namely $\dim(f(M)) = \dim(M)$ or not. The first goal of this paper is to prove a family of estimates for holomorphic maps between Kähler manifolds containing the above three results as special cases. The result below removes the above mentioned constraints of Theorem 1.3.

Theorem 1.4. *Let $f : M^m \to N^n$ be a holomorphic map with M being a complete manifold. When M is noncompact assume either the bisectional curvature is bounded from below or* (1.4) *holds for some exhaustion function ρ. Let $\ell \leq \dim(M)$ be a positive integer.*

(i) *Assume that the holomorphic sectional curvature of N, $H^N(Y) \leq -\kappa|Y|^4$ and M has*, $\mathrm{Ric}_\ell^M \geq -K$, *for some $K \geq 0, \kappa > 0$. Then*

$$\sigma_\ell(x) \leq \frac{2\ell'}{\ell'+1}\frac{K}{\kappa},$$

where $\sigma_\ell(x) = \sum_{\alpha=1}^{\ell} |\lambda_\alpha|^2(x)$, *and* $\ell' = \min\{\ell, \dim(f(M))\}$. *In particular, if $K = 0$, the map f must be a constant.*

(ii) *Assume that $S_\ell^M(x) \geq -K$ and that* $\mathrm{Ric}_\ell^N(x) \leq -\kappa$ *for some $K \geq 0, \kappa > 0$. Then*

$$\|\Lambda^\ell \partial f\|_0^2(x) \leq \left(\frac{K}{\ell\kappa}\right)^\ell.$$

In particular, if $K = 0$, the map f has rank smaller than ℓ.

Note that part (i) above recovers Theorem 1.1 for $\ell = \dim(M)$, and recovers Theorem 1.2 for $\ell = 1$. Hence, it provides a family of estimates interpolating between Theorems 1.1 and 1.2. Similarly part (ii) recovers Theorem 1.3 when $\ell = \dim(M)$, and recovers Theorem 1.2 for $\ell = 1$, noting that in the case $\ell = \dim(M)$, the assumption on the lower bound of bisectional curvature can be weakened to a lower bound of the Ricci curvature (from the proof this is obvious). Hence, part (ii) provides a family of estimates interpolating between Theorems 1.2 and 1.3. Part (ii) also implies that any Kähler manifold with $\mathrm{Ric}_\ell \leq -\kappa < 0$ must be ℓ-hyperbolic, a result proved in [17]. Moreover, it can also be applied to M with $\dim(M) > \ell$ or even $\dim(M) > \dim(N)$ concluding more detailed degeneracy information of the map, re-enforcing the relationship between the ℓ dimensional "holomorphic" area of N and the Ric_ℓ^N.

The proof of the result (in Sec. 4) is built upon extensions of $\partial\bar{\partial}$-Bochner formulae of [17], which are proved in Sec. 3 after some preliminaries in Sec. 2. In Sec. 5, we show that the estimates can be used to rule out the existence of certain holomorphic mappings under some curvature conditions (cf. Theorem 5.1). In particular Theorem 1.2 (cf. [17, Corollary 5.4]) implies that *if a compact Kähler manifold* (M, g) *has* $H \geq 0$, *then there is no onto homomorphism from its fundamental group to the fundamental group of any oriented Riemann surface* (*complex curve*) *of genus greater than one.* The more flexible Theorem 1.4 extends this statement to include all Kähler manifolds with $\mathrm{Ric}_\ell \geq 0$ (for some $\ell \in \{1, \ldots, m\}$). Note that a similar statement was proved for Riemannian manifold with positive isotropic curvature in [6]. In [18, 25] it was proved that if the holomorphic sectional curvature $H > 0$ or more generally $\mathrm{Ric}_\ell > 0$, then $\pi_1(M) = \{0\}$. The result here provides some information for the nonnegative case. Note that the examples in [8] indicate that the class of Kähler manifolds with $H > 0$ (most of them are not Fano) seems to be much larger than that with $\mathrm{Ric} > 0$. There has been very little known for manifold M with $H \geq 0$ (or $\mathrm{Ric}_\ell \geq 0$ for $\ell < \dim(M)$) comparing with the situation for compact manifolds with $\mathrm{Ric} \geq 0$. In fact, when M is a compact Kähler manifold with nonnegative bisectional curvature, Mok's classification result [14] implies that the fundamental group $\pi_1(M)$ must be a Bieberbach one. In [19, Corollary 5.1] a paper by Tam and the author, this was extended (as a result of F. Zheng) to the case when M is a noncompact complete Kähler manifold, but under the nonnegativity of sectional curvature. For compact Riemannian manifolds with nonnegative Ricci curvature Cheeger–Gromoll [4] proved that $\pi_1(M)$ must be a finite extension of a Bieberbach group. *Could this be proven for a compact Kähler manifold with* $\mathrm{Ric}_\ell \geq 0$ *with* $\ell < \dim(M)$? Note that such a statement cannot be possibly true for Kähler manifold with $B^\perp \geq 0$ (hence nor with $\mathrm{Ric}^\perp \geq 0$). In a recent preprint [15], the question has been answered positively for $H \geq 0$, assuming additionally that M is a projective variety. Given that there are many nonalgebraic Kähler manifolds with $H \geq 0$, our result for general Kähler manifolds is not contained in [15]. An equally interesting question is when a Kähler manifold with $H \geq 0$ is projective.

In [1], two invariants were defined for a Kähler manifold M. One is the so-called Albanese dimension $a(M) \doteqdot \dim_{\mathbb{C}}(\mathrm{Alb}(M))$ (we use the complex dimension instead), the dimension of the image of the Albanese map $\mathrm{Alb} : M \to \mathbb{C}^{\dim(H^{1,0}(M))}/H_1(M, \mathbb{Z})$. The other invariant is the genus of M, $g(M)$ which is defined as the maximal $\dim(U)$ with U being an isotropic subspace of $H^1(M, \mathbb{C})$. The above consequence of Theorem 5.1 can be rephrased as that for M with $H^M(X) \geq 0$, or more generally $\mathrm{Ric}_\ell \geq 0$, we must have $g(M) \leq 1$. The same conclusion is obtained in Sec. 6 for Kähler manifold M with the Picard number $\rho(M) = 1$ and $S_2 > 0$, or $h^{1,1}(M) = 1$. A corollary of Theorem 5.1 concludes that if $S^M_\ell > 0$, then $a(M) \leq \ell - 1$. (This is also a consequence of the vanishing theorem proved in [21].) These results endow the curvature Ric_ℓ and S_ℓ some algebraic geometric/topological implications.

In Sec. 5, we also illustrate that the C^2-estimate for the complex Monge–Ampère equation is a special case of our computation in Sec. 3. In Sec. 6, we derive some estimates on the minimal "energy" needed for a nonconstant holomorphic map between certain Kähler manifolds extending earlier results in [17].

2. Preliminaries

We collect some needed algebraic results. For holomorphic map $f : (M^m, g) \to (N^n, h)$, let $\partial f(\frac{\partial}{\partial z^\alpha}) = \sum_{i=1}^n f^i_\alpha \frac{\partial}{\partial w^i}$ with respect to local coordinates $(z^1, \dots, z^m)$ and $(w^1, \dots, w^n)$. The Hermitian form $A_{\alpha\bar\beta} dz^\alpha \wedge dz^{\bar\beta}$ with $A_{\alpha\bar\beta} = f^i_\alpha \overline{f^j_\beta} h_{i\bar j}$ is the pull-back of Kähler form ω_h via f. By the singular value decomposition for $x_0 \in M$

and $f(x_0) \in N$, we may choose normal coordinates centered at x_0 and $f(x_0)$ such that $\partial f(\frac{\partial}{\partial z^\alpha}) = \lambda_\alpha \delta^i_\alpha \frac{\partial}{\partial w^i}$. Then $|\lambda_\alpha|$ are the singular values of $\partial f : (T'_{x_0}M, g) \to (T'_{f(x_0)}N, h)$. It is easy to see that $|\lambda_1|^2 \geq \cdots \geq |\lambda_m|^2$ are the eigenvalues of A (with respect to g).

Proposition 2.1. *For any* $1 \leq \ell \leq m$ *the following holds*:

$$\sigma_\ell \doteqdot \sum_{\alpha=1}^{\ell} |\lambda_\alpha|^2 \geq \sum_{1 \leq \alpha, \beta \leq \ell} g^{\alpha\bar{\beta}} A_{\alpha\bar{\beta}} \doteqdot U_\ell.$$

Proof. Arguing invariantly, we choose unitary basis of $T'_{x_0}M$ with respect to g. Then the left-hand side is the partial sum of the eigenvalues of A in descending order, and the right-hand side is the trace of the first $\ell \times \ell$ block of $(A_{\alpha\bar{\beta}})$. Hence, the result is well known (cf. [9, Corollary 4.3.34]). $\square$

For a linear map $L : \mathbb{C}^m \to \mathbb{C}^n$ between two Hermitian linear spaces, $\Lambda^\ell L : \wedge^\ell \mathbb{C}^m \to \wedge^\ell \mathbb{C}^n$ is define as the linear extension of the action on simple vectors: $\Lambda^\ell L(\mathbf{a}) \doteqdot L(v_1) \wedge \cdots \wedge L(v_\ell)$ with $\mathbf{a} = v_1 \wedge \cdots \wedge v_\ell$. The metric on $\wedge^\ell \mathbb{C}^m$ is defined as $\langle \mathbf{a}, \overline{\mathbf{b}} \rangle = \det(\langle v_i, \bar{w}_j \rangle)$. If $\{e_\alpha\}$ is a unitary frame of $\mathbb{C}^m$, the $\{e_\lambda\}$, with $\lambda = (\alpha_1, \ldots, \alpha_\ell)$, $\alpha_1 \leq \cdots \leq \alpha_\ell$, being the multi-index, and $e_\lambda = e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_\ell}$, is a unitary frame for $\wedge^\ell \mathbb{C}^m$. The Binet–Cauchy formula implies that this is consistent with the Hermitian product $\langle \mathbf{a}, \overline{\mathbf{b}} \rangle$ defined in the previous section. The norm $\|\Lambda^\ell L\|_0$ is the operator norm with respect to the Hermitian structures of $\wedge^\ell \mathbb{C}^m$ and $\wedge^\ell \mathbb{C}^m$ defined above, which equals to the Jacobian of a Lipschitz map f, when $\ell = m$ or n, applying to $L = \partial f$ (cf. [5, Sec. 3.1]).

For the local Hermitian matrices $A = (A_{\alpha\bar{\beta}})$ and $G = (g_{\alpha\bar{\beta}})$, we denote A_ℓ and G_ℓ be the upper-left $\ell \times \ell$ blocks of them.

Proposition 2.2. *For any* $1 \leq \ell \leq m$ *the following holds*:

$$\|\Lambda^\ell \partial f\|_0^2 = \Pi_{\alpha=1}^{\ell} |\lambda_\alpha|^2 \geq \frac{\det(A_\ell)}{\det(G_\ell)} \doteqdot W_\ell. \tag{2.1}$$

Proof. For the inequality in (2.1), as in the above proposition, we may choose a unitary frame of $T'_{x_0}M$ such that $G = \mathrm{id}$. Then the claimed result is also a well-known statement about the partial products of the descending eigenvalues. The result can be seen by applying [13, 4.1.6] to $(A + \epsilon G)^{-1}$ and let $\epsilon \to 0$ (see also [9, Problem 4.3.P15]).

For the equality (2.1), first observe that

$$\|\Lambda^\ell \partial f\|_0^2(x) \geq \frac{|\partial f(v_1) \wedge \cdots \wedge \partial f(v_\ell)|^2}{|v_1 \wedge \cdots \wedge v_\ell|^2} = \Pi_{\alpha=1}^{\ell} |\lambda_\alpha|^2$$

if $\{v_\alpha\}$ are the eigenvectors of A with eigenvalues $\{|\lambda_\alpha|^2\}$. On the other hand for general orthonormal vectors $\{v_\alpha\}$, the above paragraph implies $\frac{|\partial f(v_1) \wedge \cdots \wedge \partial f(v_\ell)|^2}{|v_1 \wedge \cdots \wedge v_\ell|^2} \leq \Pi_{\alpha=1}^{\ell} |\lambda_\alpha|^2$. Combining them, we have the equality in (2.1). $\square$

3. $\partial\bar{\partial}$-Bochner Formulae

Here, we generalize the $\partial\bar{\partial}$-Bochner formula derived in [17] on $\|\partial f\|^2$ and $\|\Lambda^m \partial f\|_0^2$ to σ_ℓ and $\|\Lambda^\ell \partial f\|_0^2$. Since both $\sigma_\ell(x)$ and $\|\Lambda^\ell \partial f\|_0^2(x)$ are only continuous in general, we first derive formula on their barriers supplied by Propositions 2.1 and 2.2.

Proposition 3.1. *Under the normal coordinates near x_0 and $f(x_0)$ such that $\partial f(\frac{\partial}{\partial z^\alpha}) = \lambda_\alpha \delta^i_\alpha \frac{\partial}{\partial w^i}$ with $|\lambda_1| \geq \cdots \geq |\lambda_\alpha| \geq \cdots \geq |\lambda_m|$ being the singular values of $\partial f : (T'_{x_0}M, g) \to (T'_{f(x_0)}N, h)$, let $U_\ell(x)$ and $W_\ell(x)$ be the functions defined in the last section in a small neighborhood of x_0. Then at x_0, for $v \in T'_{x_0}M$, and nonzero U_ℓ and W_ℓ,*

$$\left\langle \sqrt{-1}\partial\bar{\partial}\log U_\ell, \frac{1}{\sqrt{-1}} v \wedge \bar{v} \right\rangle = \frac{U_\ell \sum_{1\leq i\leq n, 1\leq \alpha\leq \ell} |f^i_{\alpha v}|^2 - |\sum_{\alpha=1}^{\ell} \overline{\lambda_\alpha} f^\alpha_{\alpha v}|^2}{U_\ell^2} + \sum_{\alpha=1}^{\ell} \frac{|\lambda_\alpha|^2}{U_\ell}(-R^N(\alpha,\bar{\alpha},\partial f(v),\overline{\partial f(v)}) + R^M(\alpha,\bar{\alpha},v,\bar{v})); \tag{3.1}$$

$$\left\langle \sqrt{-1}\partial\bar{\partial}\log W_\ell, \frac{1}{\sqrt{-1}} v \wedge \bar{v} \right\rangle = \sum_{\alpha=1}^{\ell} \sum_{\ell+1\leq i\leq n} \frac{|f^i_{\alpha v}|^2}{|\lambda_\alpha|^2} + \sum_{\alpha=1}^{\ell}(-R^N(\alpha,\bar{\alpha},\partial f(v),\overline{\partial f(v)}) + R^M(\alpha,\bar{\alpha},v,\bar{v})). \tag{3.2}$$

Proof. The calculation is similar to that of [17]. Here, we include the details of the first. Choose holomorphic normal coordinate $(z_1, z_2, \ldots, z_m)$ near a point p on the domain manifold M, correspondingly $(w_1, w_2, \ldots, w_n)$ near $f(p)$ in the target. Let $\omega_g = \sqrt{-1} g_{a\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ and $\omega_h = \sqrt{-1} h_{i\bar{j}} dw^i \wedge d\bar{w}^j$ be the Kähler forms of M and N, respectively. Correspondingly, the Christoffel symbols are given

$${}^M\Gamma^\beta_{\alpha\gamma} = \frac{\partial g_{\alpha\bar{\delta}}}{\partial z^\gamma} g^{\bar{\delta}\beta} = \Gamma^\beta_{\gamma\alpha}; \quad {}^N\Gamma^j_{ik} = \frac{\partial h_{i\bar{l}}}{\partial w^k} h^{\bar{l}k} = \Gamma^j_{ki}.$$

We always uses Einstein's convention when there is an repeated index. The symmetry in the Christoffel symbols is due to Kählerity. If the appearance of the indices can distinguish the manifolds, we omit the superscripts ^{M} and ^{N}. Correspondingly, the curvatures are given by

$${}^M R^\beta_{\alpha\bar{\delta}\gamma} = -\frac{\partial}{\partial \bar{z}^\delta} \Gamma^\beta_{\alpha\gamma}; \quad {}^N R^j_{i\bar{l}k} = -\frac{\partial}{\partial \bar{w}^l} \Gamma^j_{ik}.$$

At the points x_0 and $f(x_0)$, where the normal coordinates are centered we have that

$$R_{\bar{\beta}\alpha\bar{\delta}\gamma} = -\frac{\partial^2 g_{\bar{\beta}\alpha}}{\partial z^\gamma \partial \bar{z}^\delta}; \quad R_{\bar{j}i\bar{l}k} = -\frac{\partial^2 h_{\bar{j}i}}{\partial w^k \partial \bar{w}^l}.$$

Direct calculation shows that at the point x_0 (here repeated indices α, β are summed from 1 to ℓ, while i, j, k, l are summed from 1 to n)

$$(\log U_\ell)_\gamma = \frac{g^{\alpha\bar{\beta}}{}_{,\gamma} A_{\alpha\bar{\beta}} + g^{\alpha\bar{\beta}} f^i_{\alpha\gamma} h_{i\bar{j}} \overline{f^j_\beta} + g^{\alpha\bar{\beta}} f^i_\alpha \overline{f^j_\beta} f^k_\gamma h_{i\bar{j},k}}{U_\ell} = \frac{f^i_{\alpha\gamma}\overline{f^i_\alpha}}{U_\ell};$$

$$(\log U_\ell)_{\bar{\gamma}} = \frac{g^{\alpha\bar{\beta}}{}_{,\bar{\gamma}} A_{\alpha\bar{\beta}} + g^{\alpha\bar{\beta}} f^i_\alpha h_{i\bar{j}} \overline{f^j_{\beta\gamma}} + g^{\alpha\bar{\beta}} f^i_\alpha \overline{f^j_\beta}\,\overline{f^k_\gamma} h_{i\bar{j},\bar{k}}}{U_\ell} = \frac{\overline{f^i_{\alpha\gamma}} f^i_\alpha}{U_\ell};$$

$$(\log U_\ell)_{\gamma\bar{\gamma}} = \frac{R^M_{\alpha\bar{\beta}\gamma\bar{\gamma}} f^i_\alpha \overline{f^i_\beta} + |f^i_{\alpha\gamma}|^2 - R^N_{i\bar{j}k\bar{l}} f^i_\alpha \overline{f^j_\beta} f^k_\gamma \overline{f^l_\gamma}}{U_\ell} - \frac{|\sum_{1\le\alpha\le\ell; 1\le i\le n} f^i_{\alpha\gamma}\overline{f^i_\alpha}|^2}{U_\ell^2}.$$

The claimed equation then follows. □

Corollary 3.2. *Let $f : M \to N$ be a holomorphic map between two Kähler manifolds.*

(i) *If the bisectional curvature of N is nonpositive and the bisectional curvature of M is nonnegative, then $\log \sigma_\ell(x)$ is a plurisubharmonic function.*

(ii) *Assume that $\mathrm{Ric}^N_\ell \le 0$ and $\mathrm{Ric}^M_\ell \ge 0$. If $\|\Lambda^\ell \partial f\|^2_0$ not identically zero, then for every x, there exists a $\Sigma \subset T'_x M$ with $\dim(\Sigma) \ge \ell$ such that $\log \|\Lambda^\ell \partial f\|^2_0(x)$ is plurisubharmonic on Σ.*

4. Proof of Theorem 1.4

Since in general σ_ℓ and $\|\Lambda^\ell \partial f\|_0$ are not smooth, we adopt the viscosity consideration as in [17, Sec. 5] to prove the result. We also need to modify the algebraic argument in the [17, Appendix] for some point-wise estimates needed. Another difference of the argument is that we shall apply the maximum principle to a degenerate operator. First, we need a Royden type lemma.

Lemma 4.1. *If the holomorphic sectional curvature R^N has a upper bound $-\kappa$, with respect to the normal coordinates as in Proposition 2.1 at x_0 (and $f(x_0)$),*

$$\sum_{1\le\alpha,\beta,\gamma,\delta\le\ell} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} R^N_{i\bar{j}k\bar{l}} f^i_\alpha \overline{f^j_\beta} f^k_\gamma \overline{f^l_\delta} \le -\frac{\ell'+1}{2\ell'}\kappa U_\ell^2, \quad \textit{when } \kappa > 0;$$

$$\le -\kappa U_\ell^2 \quad \textit{when } \kappa \le 0.$$

Here $\ell' = \min\{\ell, \dim(f(M))\}$.

Proof. We follow the argument in [17, Appendix], which is due to F. Zheng. The left-hand side can be written as $\sum_{1\le\alpha,\beta\le\ell'} R^N_{\alpha\bar{\alpha}\beta\bar{\beta}} |\lambda_\alpha|^2 |\lambda_\beta|^2$. In the space

$$\Sigma \doteqdot \mathrm{span}\left\{\partial f\left(\frac{\partial}{\partial z^1}\right), \ldots, \partial f\left(\frac{\partial}{\partial z^{\ell'}}\right)\right\},$$

consider the vector $Y = \sum_{1\le i\le \ell'} w^i\lambda_i \frac{\partial}{\partial w^i}$ with $(w^1,\ldots,w^{\ell'}) \in \mathbb{S}^{2\ell'-1} \subset \Sigma$. Then direct calculations show that

$$\begin{aligned}\sum_{1\le\alpha,\beta\le\ell'} R^N_{\alpha\bar\alpha\beta\bar\beta}|\lambda_\alpha|^2|\lambda_\beta|^2 &= \frac{\ell'(\ell'+1)}{2}\cdot\frac{1}{\mathrm{Vol}(\mathbb{S}^{2\ell'-1})}\int_{\mathbb{S}^{2\ell'-1}} R^N(Y,\overline{Y},Y,\overline{Y})\\ &\le -\kappa\frac{\ell'(\ell'+1)}{2}\cdot\frac{1}{\mathrm{Vol}(\mathbb{S}^{2\ell'-1})}\int_{\mathbb{S}^{2\ell'-1}}|Y|^4\\ &= \frac{-\kappa}{2}\left(U_\ell^2 + \sum_{1\le\alpha\le\ell'}|\lambda_\alpha|^4\right).\end{aligned}$$

The result follows from elementary inequalities $\sum_{1\le\alpha\le\ell'}|\lambda_\alpha|^4 \le U_\ell^2 \le \ell'\sum_{1\le\alpha\le\ell'}|\lambda_\alpha|^4$. □

To prove part (i), let $\eta(t) : [0,+\infty) \to [0,1]$ be a function supported in $[0,1]$ with $\eta' = 0$ on $[0,\frac{1}{2}]$, $\eta' \le 0$, $\frac{|\eta'|^2}{\eta} + (-\eta'') \le C_1$. The construction of such η is elementary. Let $\varphi_R(x) = \eta(\frac{r(x)}{R})$. When the meaning is clear, we omit subscript R in φ_R. Clearly, $\sigma_\ell\cdot\varphi$ attains a maximum somewhere at x_0 in $B_p(R)$. With respect to the normal coordinates near x_0 and $f(x_0)$, $(U_\ell\varphi)(x_0) = (\sigma\varphi)(x_0)$, and $(U_\ell\varphi)(x) \le (\sigma_\ell\varphi)(x) \le (\sigma\varphi)(x_0) \le (U_\ell\varphi)(x_0)$ for x in the small normal neighborhood. The maximum principle, then implies that at x_0

$$\nabla(U_\ell\varphi) = 0; \sum_{1\le\alpha\le\ell}\frac{1}{2}(\nabla_\alpha\nabla_{\bar\alpha} + \nabla_{\bar\alpha}\nabla_\alpha)\log(U_\ell\varphi) \le 0.$$

Now applying the $\partial\bar\partial$ formula (3.1), the above lemma and the complex Hessian comparison theorem of Li–Wang [12], together with the argument in [17], imply the result. It is clear from the proof that if $\ell = m = \dim(M)$, only the Laplacian comparison theorem is needed. Hence, one only needs to assume that the Ricci curvature of M is bounded from below.

The proof of part (ii) is similar. For the sake of the completeness, we include the argument under the assumption (1.4). In this case, we let $\varphi = \eta(\frac{\rho}{R})$. Now φ has support in $D(2R) \doteqdot \{\rho \le 2R\}$. Hence, $W_\ell\cdot\varphi$ attains its maximum somewhere, say at $x_0 \in D(2R)$. Now at x_0 we have

$$\begin{aligned} 0 &\ge \sum_{\gamma=1}^{\ell}\frac{\partial^2}{\partial z^\gamma\partial z^{\bar\gamma}}\,(\log(W_\ell\,\varphi)) \ge \sum_{\alpha,\gamma=1}^{\ell} R^M_{\alpha\bar\alpha\gamma\bar\gamma} - R^N_{\alpha\bar\alpha\gamma\bar\gamma}|\lambda_\gamma|^2 + \sum_{\gamma=1}^{\ell}\frac{\partial^2\log\varphi}{\partial z^\gamma\partial z^{\bar\gamma}}\\ &\ge -K + \ell\cdot\kappa\cdot W_\ell^{1/\ell} + \frac{\eta''}{R^2\varphi}|\nabla\rho|^2 + \frac{\ell\eta'}{R\varphi}\left([\sqrt{-1}\partial\bar\partial\rho]_+\right) - \frac{|\eta'|^2}{\varphi^2R^2}\cdot|\nabla\rho|^2\\ &\ge -K + \ell\cdot\kappa\cdot W_\ell^{1/\ell} - \frac{C_1}{\varphi R^2}|\nabla\rho|^2 - \frac{C_1}{\varphi R}\cdot C(m)([\sqrt{-1}\partial\bar\partial\rho]_+).\end{aligned}$$

Multiplying φ on both sides of the above, we have that

$$\sup_{D(R)} \|\Lambda^\ell \partial f\|_0^2(x) \leq \left(\frac{K + +\frac{C_1}{\varphi R^2}|\nabla\rho|^2 + \frac{C_1}{\varphi R}\cdot C(m)([\sqrt{-1}\partial\bar{\partial}\rho]_+)}{\ell\kappa}\right)^\ell.$$

The result follows by observing that $\frac{|\nabla\rho|^2}{R^2} \leq \frac{4|\nabla\rho|^2}{\rho^2} \to 0$ and $\frac{[\sqrt{-1}\partial\bar{\partial}\rho]_+}{R} \leq 2\frac{[\sqrt{-1}\partial\bar{\partial}\rho]_+}{\rho} \to 0$ as $R \to \infty$.

5. Applications

First, we show that the Pogorelov type estimate of [17] can be adapted to derive the C^2-estimate for the Monge–Ampère equation related to the existence of Kähler–Einstein metrics and prescribing the Ricci curvature problem. Recall that the geometric problems reduce to a complex Monge–Ampère equation

$$\frac{\det(g_{\alpha\bar{\beta}} + \varphi_{\alpha\bar{\beta}})}{\det(g_{\alpha\bar{\beta}})} = e^{t\varphi+f}$$

with $t \in [-1, 1]$, f being a fixed function with prescribed complex Hessian. $g'_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} + \varphi_{\alpha\bar{\beta}}$ is another Kähler metric with $[\omega_{g'}] = [\omega_g]$. We apply our previous setting to the map $\mathrm{id} : (M, g) \to (M, g')$. The computation in [2, 27] (see also the exposition in [24]) is on $\mathcal{L}\|\partial f\|^2$. By the computation from Secs. 3 and 4, at the point where $\|\partial\,\mathrm{id}\,\|_0^2$ is attained we have that

$$0 \geq \frac{\partial^2}{\partial z^\gamma \partial\bar{z}^\gamma}\log(1 + \varphi_{1\bar{1}}) \geq R_{1\bar{1}\gamma\bar{\gamma}} - R'_{1\bar{1}\gamma\bar{\gamma}}(1 + \varphi_{\gamma\bar{\gamma}}).$$

Here, R' is the curvature of g' and $|\lambda_\gamma|^2 = 1 + \varphi_{\gamma\bar{\gamma}}$. Since we do not have information on R' in general, but only $\mathrm{Ric}^{g'}(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial \bar{z}^1}) = \mathrm{Ric}^g_{1\bar{1}} - t\varphi_{1\bar{1}} - f_{1\bar{1}}$, we multiply $\frac{1}{1+\varphi_{\gamma\bar{\gamma}}}$ on the both sides of the above inequality and then sum γ from 1 to m arriving at

$$\begin{aligned} 0 &\geq \sum_{\gamma=1}^m \frac{1}{1 + \varphi_{\gamma\bar{\gamma}}} R^g_{1\bar{1}\gamma\bar{\gamma}} - \frac{\mathrm{Ric}^g_{1\bar{1}}}{1 + \varphi_{1\bar{1}}} + t\frac{\varphi_{1\bar{1}}}{1 + \varphi_{1\bar{1}}} + \frac{f_{1\bar{1}}}{1 + \varphi_{1\bar{1}}} \\ &\geq -C(M, g, f)\sum_{\gamma=1}^m \frac{1}{1 + \varphi_{\gamma\bar{\gamma}}} - 1. \end{aligned}$$

Now we apply/repeat the same consideration/calculation to $Q \doteqdot \log\sigma_1 - (C(M, g, f) + 1)\varphi$. Then at the point x_0, where Q attains its maximum, we have that

$$0 \geq -C(M, g, f)\sum_{\gamma=1}^m \frac{1}{1 + \varphi_{\gamma\bar{\gamma}}} - (C(M, g, f) + 2) + (C(M, g, f) + 1)\sum_{\gamma=1}^m \frac{1}{1 + \varphi_{\gamma\bar{\gamma}}},$$

which then implies that

$$\sum_{\gamma=1}^m \frac{1}{1 + \varphi_{\gamma\bar{\gamma}}} \leq C(M, g, f) + 2.$$

This implies that at the maximum point of $\sigma_1 e^{-(C(M,g,f)+1)\varphi}$,

$$\begin{aligned}\sigma_1 e^{-(C(M,g,f)+1)\varphi} &= \sigma_1 \frac{\omega_g^m}{\omega_{g'}^m} e^{t\varphi+f} e^{-(C(M,g,f)+1)\varphi} \\ &\leq \left(\frac{1}{m-1}\sum_{\gamma=2}^m \frac{1}{1+\varphi_{\gamma\bar{\gamma}}}\right)^{m-1} e^{t\varphi+f} e^{-(C(M,g,f)+1)\varphi} \\ &\leq \left(\frac{C(M,g,f)+2}{m-1}\right)^{m-1} e^{t\varphi+f} e^{-(C(M,g,f)+1)\varphi}.\end{aligned}$$

If we write $K = (\frac{C(M,g,f)+2}{m-1})^{m-1}$, $\kappa = C(M,g,f)+2$, the above implies

$$1+\varphi_{\gamma\bar{\gamma}}(x) \leq \sigma_1(x) \leq K e^{\kappa(\varphi(x)-\varphi(x_0))} e^{t\varphi(x_0)+f(x_0)}, \quad \forall\, \gamma \in \{1,\ldots,m\}. \tag{5.1}$$

As mentioned in the introduction, Theorem 1.4 removes the constrains that $\dim(M) \leq \dim(N)$ in the previous results proved in [17]. As in [20], we denote by $B^\perp$ the orthogonal bisectional curvature. We say $B^\perp \leq \kappa$ if for any $X, Y \in T'N$ with $\langle X, \overline{Y}\rangle = 0$, $R(X,\bar{X},Y,\bar{Y}) \leq \kappa |X|^2|Y|^2$. The following is a corollary of the proof Theorem 1.4.

Theorem 5.1. *Let $f : (M,g) \to (N,h)$ be a holomorphic map.*

(ii) *Assume that M is compact. Under the assumptions either $\mathrm{Ric}_\ell^M > 0$, and the holomorphic sectional curvature $H^N \leq 0$, or $\mathrm{Ric}_\ell^M \geq 0$ and $H^N < 0$, f must be constant. The same result also holds if $(B^M)^\perp > 0$ and $(B^N)^\perp \leq 0$ or $(B^M)^\perp \geq 0$ and $(B^N)^\perp < 0$.*

(ii) *If M is compact with $S_\ell^M \geq 0$ and $\mathrm{Ric}_\ell^N < 0$, or $S_\ell^M > 0$, $\mathrm{Ric}_\ell^N \leq 0$ then $\dim(f(M)) < \ell$. The same result holds if $\mathrm{Ric}_\ell^M \geq 0$ and $S_\ell^N < 0$, or $\mathrm{Ric}_\ell^M > 0$ and $S_\ell^N \leq 0$.*

Proof. Since M is compact σ_ℓ attains a maximum somewhere, say at x_0. If f is not constant, $\sigma_\ell(x_0) > 0$. Applying (3.1), using the normal coordinates around x_0 and $f(x_0)$ specified as in the last two sections, we have that

$$\begin{aligned}0 &\geq \sum_{\gamma=1}^{\ell} \frac{\partial^2}{\partial z^\gamma, \partial \bar{z}^\gamma}(\log U_\ell) \\ &\geq \sum_{1\leq \alpha,\gamma\leq \ell} \frac{-R^N_{\alpha\bar{\alpha}\gamma\bar{\gamma}}|\lambda_\alpha|^2|\lambda_\gamma|^2}{U_\ell} + \sum_{\alpha=1}^{\ell} \frac{\mathrm{Ric}^M(x_0,\Sigma)(\alpha,\bar{\alpha})|\lambda_\alpha|^2}{U_\ell}.\end{aligned}$$

Here $\Sigma = \mathrm{span}\{\frac{\partial}{\partial z^1},\ldots,\frac{\partial}{\partial z^\ell}\}$. By Lemma 4.1, if $H^N < 0$, the first term is positive, the second one is nonnegative since $\mathrm{Ric}_\ell^M \geq 0$. Hence a contradiction. From the proof, the same holds if $H^N \leq 0$ and $\mathrm{Ric}_\ell^M > 0$. For the case concerning $B^\perp$ the proof is similar.

For (ii), if $\operatorname{rank}(f) \geq \ell$, $\|\Lambda^\ell \partial f\|_0$ has a nonzero maximum somewhere, say at x_0. Then applying (3.2), using the normal coordinates around x_0 and $f(x_0)$ specified as in the last two sections, we have that

$$0 \geq \sum_{\gamma=1}^{\ell} \frac{\partial^2}{\partial z^\gamma, \partial \bar{z}^\gamma}(\log W_\ell) \geq \sum_{1\leq\gamma\leq\ell}(-\mathrm{Ric}_\ell^N(x_0,\Sigma)|\lambda_\gamma|^2) + \mathrm{Scal}^M(x_0,\Sigma).$$

This leads to a contradiction under the assumptions either $S_\ell^M \geq 0$ and $\mathrm{Ric}_\ell^N < 0$, or $S_\ell^M > 0$, $\mathrm{Ric}_\ell^N \leq 0$. For the second part, we introduce the operator:

$$\mathcal{L}_\ell = \sum_{\gamma=1}^{\ell} \frac{1}{2|\lambda_\gamma|^2}(\nabla_\gamma \nabla_{\bar{\gamma}} + \nabla_{\bar{\gamma}}\nabla_\gamma).$$

Since at x_0 $\mathrm{W}_\ell \neq 0$, the above operator is well defined in a small neighborhood of x_0. As before applying $\mathcal{L}$ at x_0 implies that

$$0 \geq \mathcal{L}_\ell(\log W_\ell) \geq (-\mathrm{Scal}^N(x_0, \partial f(\Sigma)) + \sum_{1\leq\gamma\leq\ell} \frac{\mathrm{Ric}^M(x_0,\Sigma)(\gamma,\bar{\gamma})}{|\lambda_\gamma|^2}.$$

The above also induces a contradiction under either $\mathrm{Ric}_\ell^M \geq 0$ and $S_\ell^N < 0$, or $\mathrm{Ric}_\ell^M > 0$ and $S_\ell^N \leq 0$. □

This can be combined with the following result of Siu–Beauville (cf. [1, Theorem 1.5]) to infer information regarding the fundamental group of the manifolds with $\mathrm{Ric}_\ell \geq 0$.

Theorem 5.2 (Siu–Beauville). *Let M be a compact Kähler manifold. There exists a compact Riemann surface C_g of genus greater than one and a surjective holomorphic map $f : M \to C'$ with $g(C') \geq g(C)$ with connected fibers if and only if there exists a surjective homomorphism $h : \pi_1(M) \to \pi_1(C_g)$.*

Corollary 5.3. (i) *Let (M, g) be a compact Kähler manifold with* $\mathrm{Ric}_\ell \geq 0$ *for some $1 \leq \ell \leq m$. Then there exists no surjective homomorphism $h : \pi_1(M) \to \pi_1(C_g)$. Furthermore, there is no subspace $V \subset H^1(M,\mathbb{C})$ with $\wedge^2 V = 0$ in $H^2(M,\mathbb{C})$ and* $\dim(V) \geq 2$. *Namely $g(M) \leq 1$. Similarly, if* $\mathrm{Ric}_\ell \geq 0$, $\pi_1(M)$ *cannot be of the type of an amalgamated product $\Gamma_1 *_\Delta \Gamma_2$ with the index of Δ in Γ_1 greater than one and index of Δ in Γ_2 greater than two.*

(ii) *Let (M, g) be a compact Kähler manifold with $S_\ell^M > 0$ for some $1 \leq \ell \leq m$. Then $a(M) \leq \ell - 1$.*

(iii) *If $S_n^M \geq 0$, then any harmonic map $f : M \to N$ with N being a locally Hermitian symmetric space, cannot have* $\operatorname{rank}(f) = \dim(N)$.

Proof. The first part of (i) follows from part (i) of Theorem 5.1. Namely, apply it to $N = C_g$ and combine it with the above Siu–Beauville's result. The second part follows by combining [1, Theorem 5.1 with Theorem 1.4] due to Catanese (cf. [3, Theorem 1.10]). For the second part involving the amalgamated product, apply

[1, Theorem 6.27], namely, a result of Gromov–Schoen below instead, to conclude that there exists an equivariant holomorphic map from $\widetilde{M}$ into the Poincaré disk. This induce a contradiction with part (i) of Theorem 5.1 since the maximum principle argument still applies (see also [16]). The statement of (ii) is an easy consequence of part (ii) of Theorem 5.1.

For part (iii), by Siu's result on the holomorphicity of the harmonic maps between Kähler manifolds, namely [1, Theorem 6.13], any such a harmonic map must be holomorphic. Then part (ii) of Theorem 5.1 induces a contradiction noting that the canonical metric on N is Kähler–Einstein with negative Einstein constant. □

Theorem 5.4 (Gromov–Schoen). *Let M be a compact Kähler manifold with fundamental group $\Gamma = \Gamma_1 *_\Delta \Gamma_2$ with the index of Δ in Γ_1 greater than one and index of Δ in Γ_2 greater than two. Then there exists a representation $\rho : \pi_1(M) \to \mathrm{Aut}(\mathbb{D})$, where $\mathbb{D} = \{z \,|\, |z| = 1\}$, with discrete cocompact image, and a holomorphic equivariant map from the universal cover $\widetilde{M} \to \mathbb{D}$, which also descends to a surjective map $M \to \rho(\Gamma)/\mathbb{D}$.*

In fact, the vanishing theorem of [21] implies that for Kähler manifolds with $S_\ell > 0$, there does not exist a k-wedge subspace in $H^{1,0}$ (in the sense of [3]) for any $k \geq \ell$. Moreover, such manifolds have to be Albanese primitive for $k \geq \ell$.

For noncompact manifolds, Theorems 1.3 and 1.4 can also be applied, together with [1, Theorems 4.14 and 4.28], to infer some restriction on Kähler manifolds with nonnegative holomorphic sectional curvature or with $\mathrm{Ric}_\ell \geq 0$.

Corollary 5.5. *Assume that M is a complete Kähler manifold with bounded geometry with $\mathrm{Ric}_\ell^M \geq 0$. Then*

(i) $H^1(M, \mathbb{C}) = \{0\}$ *implies that* $\mathcal{H}^1_{L^2}(M) = \{0\}$;
(ii) *And* $\dim(\mathcal{H}^1_{L^2,\mathrm{ex}}(M)) \leq 1$.

Here, $\mathcal{H}_{L^2}(M)$ is the space of the harmonic L^2-forms and $\mathcal{H}^1_{L^2,\mathrm{ex}}(M)$ is the space of the L^2 harmonic exact forms. The statements are trivial when M is compact.

6. Mappings from Positively Curved Manifolds

In [20], the orthogonal $\mathrm{Ric}^\perp$ was studied. Recall that $\mathrm{Ric}^\perp(X, \overline{X}) = \mathrm{Ric}(X, \overline{X}) - H(X)/|X|^2$. We say $\mathrm{Ric}^\perp \geq K$ if $\mathrm{Ric}^\perp(X, \overline{X}) \geq K|X|^2$. It is easy to see that $B^\perp \geq \kappa$ implies that $\mathrm{Ric}^\perp \geq (m-1)\kappa$. Similar upper estimate also holds if $B^\perp$ is bounded from above. It was also shown in [20] via explicit examples that $B^\perp$ is independent of the holomorphic sectional curvature H, as well as the Ricci curvature. Similarly, $\mathrm{Ric}^\perp$ is independent of Ric, as well as H. It was proved in [20] that for manifold whose $\mathrm{Ric}^\perp$ has a positive lower bound, the manifold is compact with an effective diameter upper bound. (See [25] for the corresponding result for holomorphic sectional curvature.) It is not hard to see that for Kähler manifolds with $\mathrm{Ric}_\ell \geq K > 0$, they must be compact with an upper diameter estimate.

Applying $\partial\bar{\partial}$-Bochner formulae, we have the following estimates in the spirit of [17].

Theorem 6.1. (i) *Assume that* $\mathrm{Ric}_\ell^M(X, \overline{X}) \ge K|X|^2$, *and* $H^N(Y) \le \kappa|Y|^4$, *with* $K, \kappa > 0$. *Then for any nonconstant* $f : M \to N$

$$\max_{x \in M} \sigma_\ell(x) \ge \frac{K}{\kappa}.$$

(ii) *Assume that* $(B^M)^\perp \ge K$, *and* $(B^N)^\perp \le \kappa$, *with* $K, \kappa > 0$. *Then for any nonconstant* $f : M \to N$, $\dim(f(M)) = m$. *Moreover for any* $\ell < \dim(M)$

$$\max_{x \in M} \sigma_\ell(x) \ge \ell\frac{K}{\kappa}.$$

(iii) *Assume that* $\mathrm{Ric}_\ell^M \ge K$, *and that* $\mathrm{Ric}_\ell^N \le \kappa$, *with* $K, \kappa > 0$. *Then for any holomorphic map* $f : M \to N$ *with* $\dim(f(M)) \ge \ell$

$$\max_x \|\Lambda^\ell \partial f\|_0^2(x) \ge \left(\frac{K}{\kappa}\right)^\ell.$$

(iv) *Assume that* $(\mathrm{Ric}^M)^\perp \ge K$, *and that* $(B^N)^\perp \le \kappa$, *with* $K, \kappa > 0$. *Then for any holomorphic map* $f : M \to N$ *with* $\dim(f(M)) \ge m - 1$, $\dim(f(M)) = m$. *Moreover*

$$\max_x \|\Lambda^m \partial f\|_0^2(x) \ge \left(\frac{K}{(m-1)\kappa}\right)^m.$$

In the case $\dim(M) = \dim(N)$, *only* $(\mathrm{Ric}^N)^\perp \le (m-1)\kappa$ *is needed. In general* $(B^N)^\perp \le \kappa$ *can be weakened to* $(\mathrm{Ric}_m^N)^\perp \le (m-1)\kappa$. *Here,* $(\mathrm{Ric}_\ell^N)^\perp$ *is the orthogonal Ricci curvature of the curvature tensor* R^N *restricted to* m*-dimensional subspaces.*

Proof. First, observe that under any assumption of the above theorem M is compact. From Lemma 4.1 and (3.1), part (i) follows. For part (ii), at the point x_0, where $\sigma_\ell(x)$ attains its maximum, applying (3.1) to $v = \frac{\partial}{\partial z^m}$, we have that

$$0 \ge -\kappa|\lambda_m|^2 + K$$

which implies that $|\lambda_m|^2 \ge \frac{K}{\kappa}$. Then claimed estimate follows from $\sigma_\ell \ge \ell|\lambda_m|^2$.

For part (iii), we apply (3.2) at the point x_0, where $\|\Lambda^\ell \partial f\|_0^2(x)$ attains its maximum. In particular, we apply it to $v = \frac{\partial}{\partial z^\ell}$ and let $\Sigma = \mathrm{span}\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^\ell}\}$. Hence at x_0

$$0 \ge -\mathrm{Ric}^N(x_0, f(\Sigma))|\lambda_\ell|^2 + \mathrm{Ric}^M(x_0, \Sigma).$$

Hence, we derive that $|\lambda_\ell|^2 \ge \frac{K}{\kappa}$. The claimed result then follows.

The part (iv) can be proved similarly. □

The part (ii) of the theorem is not as strong as it appears, since $B^{\perp} > 0$ implies that $h^{1,1}(M) = 1$. On the other hand, we have the following observation.

Proposition 6.2. *Let M be a Kähler manifold with $h^{1,1}(M) = 1$. Then any holomorphic map $f : M \to N$, with $\dim(f(M)) < \dim(M)$ must be a constant map. Hence $g(M) \leq 1$, if $\dim(M) \geq 2$. In particular, if the Picard number $\rho(M) = 1$ and $S_2^M > 0$, any holomorphic map $f : M \to N$, with $\dim(f(M)) < \dim(M)$ must be a constant map.*

Proof. In fact $f^*\omega_h$, with ω_h being the Kähler form of N, is a d-closed positive $(1,1)$-form. By the assumption $[f^*\omega_h]$ proportional to $[\omega_g]$. Hence, it must be either zero or a positive multiple of $[\omega_g]$. Since the second case implies that $\dim(f(M)) = m$, only the first case can occur, which implies that f is a constant map.

Note that this implies that for any Kähler manifold M with $\dim(M) \geq 2$ and $h^{1,1}(M) = 1$, the genus $g(M) \leq 1$, in view of the result of Catanese (cf. [3, Theorem 1.10]) since otherwise there exists a nonconstant holomorphic map $f : M \to C_g$ with C_g being a Riemann surface of genus $g(M)$. Since the first Chern class map $c_1 : H^1(M, \mathcal{O}^*) \to \mathcal{H}^{1,1}(M) \cap H^2(M, \mathbb{Z})$ is onto, and $S_2^M > 0$ implies that $H^2(M, \mathbb{C}) = \mathcal{H}^{1,1}(M)$, the assumption then implies $h^{1,1}(M) = 1$. The last result then follows from the first. □

Taking $\kappa \to 0$, the part (ii) of Theorem 6.1 also implies that any holomorphic map from a compact manifold with $B^{\perp} > 0$ into one with $B^{\perp} \leq 0$ must be a constant map (cf. Theorem 5.1). Given that $B^{\perp}$ is independent of H and Ric, this does not follow from Yau–Royden's estimate Theorem 1.1, nor from Theorem 1.2. The part (iv) provides an additional information on compact Kähler manifolds with $\mathrm{Ric}^{\perp} > 0$.

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