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ORIGINAL PAPER

A multiplicity estimate for the Jacobi operator of a nonflat Yang–Mills field over $\mathbb{S}^m$

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Abstract
Here we provide refinements of the stability results of Simons and Xin, concerning the stability of Yang–Mills fields and harmonic maps respectively. The result also implies the earlier Morse index estimates for both cases.

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1 Introduction

The stability is a central issue in the variational problems studied in analysis and geometry. It was proven that there is no nontrivial stable Yang–Mills fields on $\mathbb{S}^m$ for $m \geq 5$ [1, 2, 13] and there is no nonconstant stable harmonic maps from $\mathbb{S}^m$ for $m \geq 3$ [16]. The stability of minimal surfaces was studied also extensively [12].

Later the above results were strengthened by effective estimates on the lower bound of the Morse index. Before we state the results let's first recall some notations and definitions.

Let $u : (M^m, g) \to (N^n, h)$ be a smooth map between two Riemannian manifolds. Define

$$\mathcal{E}(u) = \frac{1}{2}\int_M |du|^2 \, d\mu_g, \text{ where } |du|^2 = \sum_{i=1}^m |du(e_i)|^2.$$

Here $\{e_i\}$ is an orthonormal basis of $T_x M$ for any $x \in M$. The Einstein convention is applied below. If $u_s = u(s, \cdot) : (-\epsilon, \epsilon) \times M \to N$ is a family of maps (a variation), we can consider the first and second variations of $\mathcal{E}(u_s)$. The critical point is called a *harmonic map*. The second variation formula [3, 16] gives that, at a harmonic map $u = u_0$,

$$\frac{d^2}{ds^2}\mathcal{E}(u_s)\Big|_{s=0} = \int_M \langle \nabla^*\nabla V - R^N_{V,du(e_i)} du(e_i), V\rangle \, d\mu_g \tag{1.1}$$

Here V is the variational vector field along $u : M \to N$, namely $V = du_s(\frac{\partial}{\partial s})\big|_{s=0}$, which can be viewed as a section of bundle $E = u^{-1}(TN)$ over M, and $\nabla^*\nabla V = -\nabla^2_{e_i,e_i} V$ is

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-1 times the trace of the Hessian operator of the bundle E. The convention of the curvature is that $\langle R_{X,Y}Y, X\rangle > 0$ for the standard sphere $\mathbb{S}^n$. The second order self-adjoint elliptic operator $\mathcal{J}_u = \nabla^*\nabla - R^N_{(\cdot),du(e_i)}du(e_i)$ is called the Jacobi operator. The harmonic map is called stable if all the eigenvalues of $\mathcal{J}_u$ is nonnegative. The constant map clearly is the minimizer of $\mathcal{E}(u)$. For geometric applications the minimizer within a homotopy class or with other geometric/topological constraints are also considered. The stable maps are the local minimizers. The total number of negative eigenvalues (multiplicity counted) of $\mathcal{J}_u$ is called the Morse index of u (denoted as $\iota(u)$). The dimension of null space $\mathcal{N} = \{V \mid \mathcal{J}_u(V) = 0\}$ is called the nullity of u. The result of Xin [16] asserts that *any stable harmonic map from* $\mathbb{S}^m$ *into any compact Riemannian manifold must be a constant map.* The following result of [5], due to El Soufi, improves Xin's theorem.

Theorem 1.1 *For $m \geq 3$, and any Riemannian manifold (N, h), let $u : \mathbb{S}^m \to (N, h)$ be a nonconstant harmonic map. Then the Morse index of u, $\iota(u) \geq m + 1$.*

Xin's theorem was preceded/motivated by a corresponding important result of Simons for Yang–Mills fields and Yang–Mills connections [1, 13], which is defined to be the critical points of the Yang–Mills functional $\mathcal{YM}(D)$ for connections D. Recall that for a connection D on a principal G-bundle and associated G-vector bundle E over a Riemannian manifold (M, g):

$$\mathcal{YM}(D) = \frac{1}{2}\int_M \|R^D\|^2\, d\mu_g.$$

Here R^D is the curvature of the connection D on E. The norm is taken with respect to the Riemannian metric of M and the $Ad(G)$-invariant metric on the Lie algebra $\mathfrak{g}$ of G (usually a subalgebra of $\mathfrak{so}(n)$ where $n = \dim(E)$). The first and second variations of $\mathcal{YM}(D_s)$ can be defined and calculated similarly for a family of connections D_s. In particular, a critical point is called a Yang–Mills connection with its curvature R^D being called a Yang–Mills field. The second variational formula (cf. Theorem 2.21 of [2]) at a Yang–Mills connection is given by

$$\frac{d^2}{ds^2}\mathcal{YM}(D_s)\Big|_{s=0} = \int_M \langle (d_D)^* d_D B + \mathcal{R}^D(B), B\rangle\, d\mu_g \tag{1.2}$$

where $B = \frac{d}{ds}D_s\big|_{s=0} \in \Omega^1(\mathfrak{g}_E)$ is the variational 1-form of the connections. The operator d_D is the exterior derivative on $\Omega^*(\mathfrak{g})$ and $(d_D)^*)$ is its conjugate. One can consult [2] (also Proposition 2.2 below) for details of the notations. The minimizers include the self-dual (anti-self dual) ones in dimensional four and the flat connections. The associated second order self adjoint operator $\mathcal{J}_D = (d_D)^* d_D + \mathcal{R}^D$ is the corresponding Jacobi operator. One defines the Morse index and nullity of D similarly. Due to the actions of the gauge group, in the definition of the nullity (index and the multiplicity) one needs to restrict to the infinitesimal variations which is orthogonal to the tangent space of the orbits of the gauge groups (cf. (6.10) of [2]). The celebrated result of Simons [1] asserts that *any nontrivial Yang–Mills field on* $\mathbb{S}^m$ *with* $m \geq 5$ *must be unstable*. The following result of [9], due to Nayatani and Urakawa, extends Simons's theorem.

Theorem 1.2 *For any nonflat Yang–Mills connection D on any vector bundle E over the m-sphere $\mathbb{S}^m$, $m \geq 5$ with the standard metric, the Morse index $\iota(D) \geq m + 1$.*

The goal of this note is to prove the following refinement of Theorems 1.1 and 1.2.

Theorem 1.3 *(i) For $m \geq 3$ and any Riemannian manifold (N, h), and any nonconstant harmonic map $u : \mathbb{S}^m \to (N, h)$, let $\mathcal{J}_u$ be the Jacobi operator. Let $E^u_\lambda := \{X \,|\mathcal{J}_u(X) = \lambda X\}$ be the space of the eigenvector fields with eigenvalue λ. Then $\dim(E^u_{-(m-2)}) \geq m + 1$. In particular, the smallest eigenvalue has the estimate $\lambda_1(\mathcal{J}_u) \leq -(m - 2)$.*

(ii) For any nonflat Yang–Mills connection D on any vector bundle E over the m-sphere $\mathbb{S}^m$, $m \geq 5$ with the standard metric, let $\mathcal{J}_D$ be the Jacobi operator. Let $E^D_\lambda := \{B \,|\mathcal{J}_D(B) = \lambda B\}$ be the space of the eigenforms with eigenvalue λ. Then $\dim(E^D_{-(m-4)}) \geq m+1$. In particular, the smallest eigenvalue has the estimate $\lambda_1(\mathcal{J}_D) \leq -(m - 4)$.

Since the multiplicity of negative eigenvalues $-(m - 2)$, or $-(m - 4)$, in each case above, is given by the dimension of the eigenspace, which contributes to the Morse index, the theorem above does imply the earlier results of Simons [13], Xin [16], El Soufi [5], Nayatani and Urakawa [9]. The proof of the part (i) is perhaps known to experts even though we did not find the explicit statement of the result anywhere in the literature. If that was the case, the contribution here would be a very simple proof of the statement. A result for minimal submanifolds in $\mathbb{S}^n$ is also obtained. In particular, Theorem 4.1 generalizes the famous Simons eigenvalue estimate for the Jacobi operator for the minimal hypersurface in $\mathbb{S}^n$, which holds the key to the regularity theory of the minimal hypersurfaces, to high codimensional case. A result (Proposition 4.1), which is dual to Ruh-Vilms' result [11] for manifolds with parallel mean curvature, can facilitate the computations in the proof to give a unified treatment. We remark that besides the stability and Morse index estimate, the nullity estimates for non totally geodesic closed minimal spheres/varieties in higher dimensional spheres were obtained in [12] much earlier. The stability issue for Yang–Mills fields was also studied in dimension four (which is the critical dimension for the Yang–Mills functional) in [15]. It is interesting to study for what manifold N (M) and the property of the map u (of Yang–Mills connection D) the equality cases in the above theorem for either harmonic maps and for the Yang–Mills fields hold. It is also interesting to study the problem with additional topological constraints such as within a nontrivial homotopy class for the harmonic maps. The Morse index estimate for harmonic 2-sphere plays an important role in the application of harmonic maps to geometry [8], where a lower estimate was proved for harmonic 2-sphere into a manifold with positive isotropic curvature. The proof of [8] used a different approach which relied on some curvature conditions of N and a decomposition theorem of holomorphic vector bundles over $\mathbb{S}^2$. The way of using the complex structure of the normal bundle and the construction of holomorphic sections have their precedences in [4] (cf. also Theorem 3.1.5 of [12] and [14]). There exist more recent lower Morse index estimates [6, 7] for two-sphere and projective planes inside a high dimensional spheres, related to the extremal metrics for higher eigenvalues, in terms of the so-called spectral index. Another interesting question is to have a lower estimate on the multiplicity and Morse index for the maps or minimal submanifolds which are not holomorphic or anti-holomorphic between Kähler manifolds when one of them is an irreducible Hermitian symmetric space.

2 Preliminaries

Let E be a Riemannian vector bundle over a Riemannian manifold (M, g) and let D be a connection compatible with the metric. We shall denote the Riemannian curvature of (M, g) by R^M. The curvature of D shall be denoted as R^D. Recall that $R^D_{X,Y} = D_X D_Y - D_Y D_X -$

$D_{[X,Y]}$ is valued in the endomorphism bundle of E. In fact the image is in $\mathfrak{so}(n)$ with n being the dimension of E. Recall that (cf. (2.3) of [10]) for any $A \in \mathfrak{so}(n)$,

$$\langle A, z \wedge w\rangle = \langle A(w), z\rangle. \tag{2.1}$$

In the case of the Riemannian curvature we have $R_{X,Y} = R(X \wedge Y)$ and with our convention

$$\langle R(X \wedge Y), Z \wedge W\rangle = \langle R_{X,Y} W, Z\rangle = R(X, Y, Z, W).$$

Let $\Omega^p(E)$ be the p-forms valued in E. The following Bochner formula is well known (cf. Proposition 1.3.4 of [17]).

Proposition 2.1 *Let $\omega \in \Omega^p(E)$. Then*

$$\Delta_{d_D}\omega \doteqdot \left(d_D d_D^* + d_D^* d_D\right)\omega = -\Delta\omega + S$$

where $\Delta = \sum_{j=1}^n \nabla^2_{e_j,e_j}$, S is defined for any $X_1, \ldots, X_p \in \mathcal{X}(M)$ that

$$S(X_1, \ldots, X_p) = -\sum_{j=1}^{m}\sum_{k=1}^{p}(R_{e_j,X_k} \cdot \omega)(X_1, \ldots, \hat{(e_j)}_k, \ldots, X_p).$$

Here $\{e_j\}$ is an orthonormal basis of T_pM. The curvature term $R_{e_\alpha,X_k} \cdot \omega$ acts on $\wedge^p T^*M \otimes E$ as a derivation. Precisely, $R_{X,Y} \cdot \omega$ involves both the curvature of E and the curvature of (M, g) in the following way

$$(R_{X,Y} \cdot \omega)(X_1, \ldots, X_k, \ldots, X_p) = R^D_{X,Y}\left(\omega(X_1, \ldots, X_k, \ldots, X_p)\right)$$
$$-\sum_{j=1}^{p}\omega(X_1, \ldots, R_{X,Y}X_k, \ldots, X_p).$$

The result follows from the corresponding formulae for d_D and d_D^*.

Lemma 2.1 *For $\omega \in \Omega^k(E)$ and vector fields $X_0, \ldots, X_k$,*

$$d_D\omega(X_0, \ldots, X_k) = \sum_{i=0}^{k}(-1)^i D_{X_i}(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k)) \tag{2.2}$$
$$+ \sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).$$

Lemma 2.2 *Let ∇ be a torsion free connection on TM. For $\omega \in \Omega^k(E)$,*

$$d_D\omega(X_0, X_1, \ldots, X_k) = \sum_{i=0}^{k}(-1)^i(D_{X_i}\omega)(X_0, \ldots, \hat{X}_i, \ldots, X_k). \tag{2.3}$$

Here $(D_Y\omega)(X_1, \ldots, X_k) = D_Y(\omega(X_1, \ldots, X_k)) - \sum_{i=1}^k \omega(X_1, \ldots, \nabla_Y X_i, \ldots X_k)$. If $\{e_s\}$ is any frame with $(g_{st}) = (\langle e_s, e_t\rangle)$,

$$(d_D^*\omega)(X_1, \ldots, X_{k-1}) = -\sum_{s,t} g^{st}(D_{e_s}\omega)(e_t, X_1, \ldots, X_{k-1}).$$

We also apply the formula to the case $E = \mathfrak{so}(E')$ with E' being a Riemaniann vector bundle. The curvature of E' acts on $\mathfrak{so}(E')$ via the adjoint action in this case.

Let $u : (M, g) \to (N, h)$ be a harmonic map. Let $E = u^{-1}(TN)$ be the pull back bundle equipped with the Riemannian metric h. The curvature of E will be the pull back of

R^N, namely u^*R^N. Now $du : TM \to TN$ can be viewed a 1-form valued in E. Namely $du \in \Omega^1(E)$. In fact for any smooth map $d_D du = 0$. The harmonic equation amounts to $d_D^* du = 0$. In this case du is d_D-harmonic with D being the associated connection on E. Hence one can apply Proposition 2.1 and expresses the curvature term more explicitly to arrive the well-known formula for a harmonic map u:

$$\Delta du = -\sum_\alpha R^N_{du(\cdot),du(e_\alpha)} du(e_\alpha) + du(\mathrm{Ric}^M(\cdot)). \tag{2.4}$$

For the case of Yang–Mills fields concerning a Riemannian vector bundle E over (M, g), we apply the formula to the p-forms valued in $\mathfrak{so}(E)$. In particular for the 1-form B and 2-form φ the following results (cf. Theorem 3.2 and Theorem 3.10 of [2]) follow from Proposition 2.1.

Proposition 2.2 *Let $B \in \Omega^1(\mathfrak{so}(E))$. Then*

$$\Delta_{d_D} B = -\Delta B + B(Ric^M(\cdot)) + \mathcal{R}^D(B) \tag{2.5}$$

with $\mathcal{R}^D(B)(X) = \sum_{j=1}^m [R^D_{e_j,X}, B(e_j)]$. Let $\varphi \in \Omega^2(\mathfrak{so}(E))$. Then

$$\Delta_{d_D}\varphi = -\Delta\varphi + \varphi \cdot (2\,\mathrm{Ric}^M \wedge \mathrm{id} - 2R^M) + \mathcal{R}^D(\varphi). \tag{2.6}$$

Here $\mathcal{R}^D(\varphi)(X, Y) = \sum_{j=1}^m \Big([R^D_{e_j,X}, \varphi_{e_j,Y}] - [R^D_{e_j,Y}, \varphi_{e_j,X}]\Big)$.

Here we use the convention as in [10] about $A \wedge B$, namely

$$A \wedge B(x \wedge y) = \frac{1}{2}\left(A(x) \wedge B(y) + B(x) \wedge A(y)\right).$$

We also view the Riemannian curvature as an operator $R^M : \mathfrak{so}(T_pM) \to \mathfrak{so}(T_pM)$ defined as above, namely $\langle R^M(x \wedge y), z \wedge w\rangle = R^M(x, y, z, w)$. Moreover, for a 1-form B valued in $\mathfrak{so}(E)$, the derived action $(R_{X,Y}(B_Z))(s) = R^D_{X,Y}(B_Z(s)) - B_Z(R^D_{X,Y}(s))$ for any $s \in \Gamma(E)$. The action on the 2-form φ valued in $\mathfrak{son}(E)$ is given by a similar formula. These contribute to the operator $\mathcal{R}^D$ in the above result.

3 Proof of Theorem 1.3

We first derive some useful, also well-known, formulae for the linear functions of $\mathbb{R}^{m+1}$ restricted to the unit sphere $\mathbb{S}^m$. We use D to denote the standard derivative/connection on $\mathbb{R}^{m+1}$ and ∇, the derivative/Levi-Civita connection of $\mathbb{S}^m$. For linear function $\ell(x)$, it is well-known that $\langle D\ell, x\rangle = \ell$. Hence $\nabla\ell = D\ell - \ell \cdot x$. Since $D^2\ell = 0$, it is easy to compute that for X, Y tangent to $\mathbb{S}^m$ we have

$$\begin{aligned}(\nabla^2\ell)(X, Y) &= XY\ell - \langle\nabla\ell, \nabla_X Y\rangle = XY\ell - \langle D\ell, \nabla_X Y\rangle \\ &= XY\ell - \langle D\ell, D_X Y\rangle + \langle D\ell, B(X, Y)\rangle = \langle \ell \cdot x, B(X, Y)\rangle = -\ell\langle X, Y\rangle.\end{aligned}$$

Here we have followed the convention of [12] defining the second fundamental form

$$B(X, Y) := D_X Y - \nabla_X Y$$

and used that the second fundamental form of the sphere satisfies that $B(X, Y) = -\langle X, Y\rangle \cdot x$. This then implies that

$$\nabla_X \nabla\ell = -\ell X; \qquad \Delta\nabla\ell = -\nabla\ell. \tag{3.1}$$

The first one is obvious. The second one can be done via the first and the commutator formula (using $\mathrm{Ric}^{\mathbb{S}^m} = (m-1)\,\mathrm{id}$). Or we choose a normal frame $\{e_\alpha\}$ with $\nabla_{e_\beta} e_\gamma = 0$ at the point concerned, and compute

$$\begin{aligned}\Delta\nabla\ell &= \nabla_{e_\alpha}\nabla_{e_\alpha}\nabla\ell = -\nabla_{e_\alpha}(\ell e_\alpha)\\ &= -\langle e_\alpha, \nabla\ell\rangle e_\alpha = -\nabla\ell.\end{aligned} \tag{3.2}$$

Now we prove the part of Theorem 1.3 concerning the harmonic maps. It follows from the two propositions below.

Proposition 3.1 *Assume that $u : \mathbb{S}^m \to (N, h)$ is a harmonic map. The associated section $X_\ell = du(\nabla\ell)$ of E satisfies*

$$\mathcal{J}_u(X_\ell) = -(m-2)X_\ell.$$

Namely if $X_\ell \neq 0$, it is an eigenvector of the Jacobi operator with eigenvalue $-(m-2)$.

Proof Direct calculations with the help of (2.4) yield that

$$\begin{aligned}\Delta(du(\nabla\ell)) &= (\Delta du)(\nabla\ell) + 2(\nabla_{e_j} du)(\nabla_{e_j}\nabla\ell) + du(\Delta\nabla\ell)\\ &= -R^N_{du(\nabla\ell),du(e_j)} du(e_j) + du(\mathrm{Ric}^M(\nabla\ell)) + 2(\nabla_{e_j} du)(\nabla_{e_j}\nabla\ell) + du(\Delta\nabla\ell)\\ &= -R^N_{du(\nabla\ell),du(e_j),} du(e_j) + (m-1)du(\nabla\ell) - 2\ell(\nabla_{e_j} du)(e_j) - du(\nabla\ell)\\ &= -R^N_{du(\nabla\ell),du(e_j)} du(e_j) + (m-2)du(\nabla\ell).\end{aligned}$$

In the above, from line 2 to line 3 we used (3.1), (3.2), and that $\mathrm{Ric}^{\mathbb{S}^m} = (m-1)\,\mathrm{id}$. From line 3 to line 4 we used the harmonic map equation $(d_D)^* du = -(\nabla_{e_j} du)(e_j) = 0$. □

Proposition 3.2 *If for a smooth map u and for some linear function ℓ, $X_\ell = du(\nabla\ell) = 0$, u must be a constant map.*

Proof This was proved in [5] for harmonic maps. We provide an argument below for the above general result for any C^1-maps. It is easy to see that ℓ attains a unique maximum point and a unique minimum point on $\mathbb{S}^m$. Consider the flow Ψ_s generated by $\nabla\ell$. It has two fixed points. One of them is a source $p_{-\infty}$ and the other is a sink p_∞ due to the explicit Hessian of ℓ provided in (3.1). Consider the image curve $u(\Psi_s(p))$. Since it stays in a compact region, $\lim_{s\to\infty} u(\Psi_s(p)) = u(\lim_{s\to\infty}\Psi_s(p)) = u(p_\infty)$, if at $s = 0$, $\Psi_s(p) = p \neq p_{-\infty}$. On the other hand

$$\frac{d}{ds}\left(u(\Psi_s(p))\right) = du\left(\frac{d}{ds}\Psi_s(p)\right) = du\left(\nabla\ell|_{\Psi_s(p)}\right) = 0.$$

This implies that $u(p) = u(p_\infty)$ for any $p \in \mathbb{S}^m$ for all $p \neq p_{-\infty}$, which proves the claim. □

Since the space $\mathcal{H}^1$ of all linear functions of $\mathbb{R}^{m+1}$ is of dimension $m+1$, the gradient of their restrictions on $\mathbb{S}^m$ is of the same dimension due to the homogeneity. The above proposition asserts that $\{X_\ell = du(\nabla\ell)\,|\,\ell \in \mathcal{H}^1\}$ is also a $(m+1)$-dimensional linear space if u is not a constant map. This proves the harmonic map part of Theorem 1.3.

Now we prove the part of Theorem 1.3 concerning Yang–Mills fields. We need the following lemma (cf. Lemma 7.3 of [2]), which can also be obtained by simple calculations.

Lemma 3.1 (Bourguignon–Lawson) *Let $B = \iota_{\nabla f}\varphi$ for some $\varphi \in \Omega^2(\mathfrak{so}(E))$ with $(d_D)^*\varphi = 0$, where f is a smooth function on M. Then $(d_D)^* B = 0$.*

The proof of the theorem follows a parallel strategy. The above result asserts that $B_\ell = \iota_{\nabla \ell} R^D$ belongs to the space of infinitesimal deformations of D (cf. Page 195 of [2]).

Proposition 3.3 *Assume that R^D is Yang–Mills of a Yang–Mills connection D of E. Let $B_\ell = \iota_{\nabla\ell} R^D = R^D_{\nabla\ell,(\cdot)}$. Then*

$$\mathcal{J}_D(B_\ell) = -(m-4)B_\ell.$$

Namely if $B_\ell \neq 0$, it is an eigenform of the Jacobi operator with eigenvalue $-(m-4)$.

Proof By calculations similar to that in the proof of Proposition 3.1 we have that

$$\begin{aligned}\Delta(\iota_{\nabla\ell}R^D) &= \Delta(R^D_{\nabla\ell,(\cdot)}) = \Delta(R^D)_{\nabla\ell,(\cdot)} + 2\left(D_{e_j}R^D\right)_{\nabla_{e_j}\nabla\ell,(\cdot)} + R^D_{\Delta\nabla\ell,(\cdot)}\\ &= \mathcal{R}^D(R^D)_{\nabla\ell,(\cdot)} + (2m-4)R^D_{\nabla\ell,(\cdot)} - R^D_{\nabla\ell,(\cdot)}\\ &= \mathcal{R}^D(R^D)_{\nabla\ell,(\cdot)} + (2m-5)R^D_{\nabla\ell,(\cdot)}.\end{aligned}$$

Recall that the operator $\mathcal{R}^D$ acting on the 1-forms and 2-forms valued in a Lie algebra is defined in Proposition 2.2. From the line 1 to 2 we have used the Yang–Mills equation $(d_D)^*R^D = -\left(D_{e_j}R^D\right)_{e_j,(\cdot)} = 0$ and $\nabla_{e_j}\nabla\ell = -\ell e_j$ to annihilate $2\left(D_{e_j}R^D\right)_{\nabla_{e_j}\nabla\ell,(\cdot)}$. Here we also used (3.1), (3.2), Proposition 2.2, precisely (2.6), and that on $\mathbb{S}^m$, $2\mathrm{Ric}^M \wedge \mathrm{id} - 2R^M = (2m-4)\,\mathrm{I}$, with I being the identity map $\mathrm{I} : \mathfrak{so}(m) \to \mathfrak{so}(m)$. Summarizing the above we have that

$$\Delta B_\ell = \mathcal{R}^D(R^D)_{\nabla\ell,(\cdot)} + (2m-5)B_\ell. \tag{3.3}$$

Now apply Lemma 3.1 and Proposition 2.2 again, precisely (2.5), and have that

$$\begin{aligned}\mathcal{J}_D(B_\ell) &= \Delta_{d_D}(B_\ell) + \mathcal{R}^D(B_\ell)\\ &= (m-1)B_\ell + 2\mathcal{R}^D(B_\ell) - \mathcal{R}^D(R^D)_{\nabla\ell,(\cdot)} - (2m-5)B_\ell.\end{aligned}$$

In the second line above we have also used (3.3). The claimed result follows after we establish that

$$2\mathcal{R}^D(B_\ell) - \mathcal{R}^D(R^D)_{\nabla\ell,(\cdot)} = 0. \tag{3.4}$$

Indeed for any X,

$$\begin{aligned}2\mathcal{R}^D(B_\ell)(X) - \mathcal{R}^D(R^D)_{\nabla\ell,(X)} &= 2\sum_{j=1}^m [R^D_{e_j,X}, B_\ell(e_j)]\\ &\quad - \sum_{j=1}^m \left([R^D_{e_j,\nabla\ell}, R^D_{e_j,X}] - [R^D_{e_j,X}, R^D_{e_j,\nabla\ell}]\right)\\ &= \sum_{j=1}^m [R^D_{e_j,X}, R^D_{\nabla\ell,e_j}] + [R^D_{\nabla\ell,e_j}, R^D_{e_j,X}] = 0.\end{aligned}$$

This completes the proof of the proposition. □

Proposition 3.4 *If for a linear function ℓ, $B_\ell = \iota_{\nabla\ell}R^D \in \Omega^1(\mathfrak{so}(E))$ vanishes on $\mathbb{S}^m$, then R^D is flat.*

Proof This was proved in Proposition 4.3 of [9]. Here we provide an alternate simple argument. Let $\Psi_s(p)$ be the flow generated by $\nabla\ell$. Let $\widetilde{\Psi}_s$ be its lift on the related principal bundle. Let $D_s = \widetilde{\Psi}_{-s} D$ be a family of connections. It is known (cf. (2.34) of [2], and Lemma 3.7 of [9]) that

$$B_\ell = \iota_{\nabla\ell} R^D = \frac{d}{ds}\bigg|_{s=0} D_s.$$

Here the definition of the lifting $\widetilde{\Psi}_s$ depends on D via a horizontal lifting of $\nabla\ell$ requiring that Ψ_s is identity map when $s = 0$. Hence if for some ℓ $B_\ell = 0$, it implies that D_s is constant. This shows that D is the connection of the pull back from the fiber over one point, namely p_∞. This implies that R^D is flat. If one expresses D as the G-invariant field of linear maps/projections $\pi : TP \to \mathcal{V}$ (as on page 199 of [2]), with $\mathcal{V}$ being the canonical vertical subspace, then $B_\ell = 0$ implies that $\widetilde{\Psi}_s^*\pi$ is independent of s. Hence the imagine of π is same as $\pi|_{p_\infty}$. One can also see this by defining the D and D_s via the connexion (namely mappings between the fibers over the two ends of smooth pathes, which smoothly depends on the paths satisfying some additional axioms) as on page 445 of [10]. The argument does not assume that R^D is a Yang–Mills field. □

The part of Theorem 1.3 concerning the Yang–Mills fields now follows exactly as the previous case for harmonic maps.

4 Minimal submanifolds in $\mathbb{S}^n$

The argument of the previous section also implies a similar result for minimal submanifolds in $\mathbb{S}^n$. Let $N(M)$ denote the normal bundle of M. Here the key is the Codazzi equation, namely for X, Y, Z tangent to M

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = 0, \text{ where } B(X, Y) = \bar{\nabla}_X Y - \nabla_X Y \tag{4.1}$$

where $\bar{\nabla}$ being the Levi-Civita connection of $\mathbb{S}^n$ and $\nabla_X Y$ being the induced connection of M via the isometric immersion. Recall that $(\nabla_X B)(Y, Z) = \nabla^\perp(B(X, Y)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$ and $\nabla^\perp_X V = (\bar{\nabla}_X V)^\perp$ for any section V of $N(M)$. Here $(\cdot)^\perp$ stands for the projection to the normal bundle. To put into the setting of our previous discussion we define $\beta : T_pM \to T_pM \otimes N_p(M)$ as

$$\langle \beta(X), Y \otimes V\rangle := \langle B(X, Y), V\rangle := \langle A^V(X), Y\rangle.$$

The last equation defines $A^V : T_pM \to T_pM$, a the symmetric tensor of T_pM (given any $p \in M$) for any $V \in N_p(M)$. The connection D on $TM \otimes N(M)$ is defined via ∇ and $\nabla^\perp$. The 1-form β defined as above is a 1-form valued in $TM \otimes N(M)$. Direct calculation shows that the Codazzi equation is equivalent to $d_D\beta = 0$.

The trace of the second fundamental form $B : T_pM \times T_pM \to N_p(M)$, namely $H := \sum_{j=1}^m B(e_j, e_j) = 0$, for an orthonormal basis $\{e_i\}$ of T_pM, is called the mean curvature. M is a minimal submanifold if and only if $H = 0$. Below we show that $H = 0$ implies that β is a d_∇-harmonic 1-form. In fact for an orthonormal basis $\{e_i\}$ of T_pM with the property

$\nabla_{e_i} e_j = 0$ at a given p,

$$\begin{aligned}\langle -(d_D)^* \beta, X \otimes V \rangle &= \sum_{i=1}^{m} \langle (\nabla_{e_i} B)(e_i, X), V \rangle \\ &= \sum_{i=1}^{m} \nabla_{e_i} \langle \beta(e_i), X \otimes V \rangle - \langle \beta(\nabla_{e_i} e_i), X \otimes V \rangle \\ &= \sum_{i=1}^{m} \nabla_{e_i} \langle B(e_i, X), V \rangle - \langle B(\nabla_{e_i} e_i, X), V \rangle \\ &= \sum_{i=1}^{m} \langle (\nabla_X B)(e_i, e_i), V \rangle = 0. \end{aligned} \tag{4.2}$$

In the above we also used the Codazzi equation. Summarizing the discussion we have that the vanishing of the mean curvature implies that β satisfies $(d_D)^* \beta = 0$. Note that d_D involves the induced connection $\nabla^\perp$ (which defined as $(\bar{\nabla}_X V)^\perp$) on $N(M)$. In fact the argument above also proves the following proposition.

Proposition 4.1 *Let M be an isometric immersed submanifold in $\mathbb{S}^n$ (or in any space forms with constant sectional curvature). Then M has parallel mean curvature H if and only if 1-form β defined above is d_D-harmonic.*

This gives a characterization of submanifolds with parallel mean curvature, similar to that of Ruh-Vilms [11], which is formulated in terms of the harmonicity of the Gauss map into the corresponding Grassmanian manifolds. The proposition above seems easier to use. The second variation of the area $\mathcal{A}$ for a minimal submanifold M (inside another Riemannian manifold N) has the following form ([12], Theorem 3.2.2):

$$\frac{d^2}{ds^2} \mathcal{A}(M_s)\Big|_{s=0} = \int_M \langle \nabla^* \nabla V - \bar{R}(V) - \tilde{A}(V), V \rangle =: \int_M \langle \mathcal{J}_M(V), V \rangle, \tag{4.3}$$

where V is the variational vector field, which in this case belongs to $\Gamma(N(M))$. Here $\nabla^* \nabla = -\Delta$ of $N(M)$ (namely with respect to $\nabla^\perp$). For two sections of the normal bundle V, W, the following defines the operators $\bar{R}$ and $\tilde{A}$, which are symmetric transformations of $N_p(M)$:

$$\bar{R}(V) = \sum_{j=1}^{m} (R^N_{V, e_j} e_j)^\perp, \qquad \langle \tilde{A}(V), W \rangle = \sum_{j=1}^{m} \langle (\bar{\nabla}_{e_j} V)^T, (\bar{\nabla}_{e_j} W)^T \rangle.$$

Here $(\cdot)^T$ stands the projection to $T_p M$ at any given point $p \in M$. In the case that $N = \mathbb{S}^n$, we have that $\bar{R}(V) = mV$. Motivated by Proposition 4.1 and the results of last section we have the following result for minimal submanifolds.

Theorem 4.1 *Let M^m be a minimal submanifold of $\mathbb{S}^n$. Let $N(M)$ denote the normal bundle of M inside $\mathbb{S}^n$. For the associated Jacobi operator $\mathcal{J}_M$, let $E^M_\lambda = \{V \in \Gamma(N) \mid \mathcal{J}_M(V) = \lambda V\}$ be the eigenspace. Then $\dim(E^M_{-m}) \geq n - m$. The equality holds if and only if M is isometric to $\mathbb{S}^m$. In particular $\lambda_1(\mathcal{J}_M) \leq -m$.*

Proof Let V_ℓ be the projection of $\bar{\nabla}\ell$ to the normal bundle $N(M)$. It is a section of $N(M)$. Denote the tangential projection of $\bar{\nabla}\ell$ to TM by T_ℓ. We calculate $\mathcal{J}_M(V_\ell)$. Pick an orthogonal frame $\{e_j\}_{j=1}^m$ with $\nabla_{e_i} e_j = 0$ at a fixed point $p \in M$. Then for a section W of $N(M)$,

$$\langle \mathcal{J}_M(V_\ell), W\rangle = -\left\langle \sum_{j=1}^m \bar\nabla_{e_j}\left(\bar\nabla_{e_j}V_\ell\right)^\perp, W\right\rangle - m\langle V_\ell, W\rangle$$
$$-\sum_{j=1}^m \langle(\bar\nabla_{e_j}V_\ell)^T, (\bar\nabla_{e_j}W)^T\rangle;$$

$$\begin{aligned}
-\left\langle \sum_{j=1}^m \bar\nabla_{e_j}\left(\bar\nabla_{e_j}V_\ell\right)^\perp, W\right\rangle &= -\left\langle \sum_{j=1}^m \bar\nabla_{e_j}\left(\bar\nabla_{e_j}(\bar\nabla\ell - T_\ell)\right)^\perp, W\right\rangle\\
&= \left\langle \sum_{j=1}^m \bar\nabla_{e_j}\left(B(e_j, T_\ell)\right), W\right\rangle\\
&= \left\langle \sum_{j=1}^m B(e_j, \nabla_{e_j}T_\ell), W\right\rangle.
\end{aligned}$$

From line 2 to 3 we have used (3.1) and from line 3 to 4 we have used that the Codazzi equation and $\nabla_X^\perp(\sum_{j=1}^m B(e_j, e_j)) = 0$. Finally

$$\begin{aligned}
\left\langle \sum_{j=1}^m B(e_j, \nabla_{e_j}T_\ell), W\right\rangle &= \sum_{j=1}^m \langle A^W(e_j), \nabla_{e_j}T_\ell\rangle = \sum_{j=1}^m \langle A^W(e_j), \bar\nabla_{e_j}T_\ell\rangle\\
&= \sum_{j=1}^m \langle A^W(e_j), \bar\nabla_{e_j}(\bar\nabla\ell - V_\ell)\rangle\\
&= -\langle W, \ell\sum_{j=1}^m B(e_j, e_j)\rangle + \sum_{j=1}^m \langle(\bar\nabla_{e_j}W)^T, (\bar\nabla_{e_j}V_\ell)^T\rangle\\
&= \sum_{j=1}^m \langle(\bar\nabla_{e_j}W)^T, (\bar\nabla_{e_j}V_\ell)^T\rangle.
\end{aligned}$$

Here we have used that $A^W(e_j) = -(\bar\nabla_{e_j}W)^T$ and $\sum_{j=1}^m B(e_j, e_j) = 0$. Putting the above together we have that

$$\langle \mathcal{J}_M(V_\ell), W\rangle = -m\langle V_\ell, W\rangle$$

for any local section W of $N(M)$. This proves that $V_\ell \in E_{-m}^M$. Now let $\mathcal{S} = \{V_\ell \mid \ell \in \mathcal{H}^1\}$. Since for any $p \in M \subset \mathbb{S}^n$, $\dim(\{\nabla\ell(p) \mid \ell \in \mathcal{H}^1\}) = n$. Hence $\dim(\mathcal{S}_p) = n - m$ with $\mathcal{S}_p = \{V_\ell(p) | V_\ell \in \mathcal{S}\}$. This proves the lower multiplicity estimate since $\dim(\mathcal{S}) \ge \dim(\mathcal{S}_p)$.

It is easy to see that the standard embedding of $\mathbb{S}^m \to \mathbb{S}^n$ attains the equality of the theorem. Let $\mathcal{G} := \{\nabla\ell \mid \ell \in \mathcal{H}^1\}$. As before $\dim(\mathcal{G}) = n + 1$. Let $r_p : \mathcal{G} \to T_p\mathbb{S}^n$ be the restriction map. Let $\mathcal{N}$ and $\mathcal{T}$ be the projections from $\mathcal{G}$ to $N(M)$ and $T(M)$. Then $\mathcal{S} = \mathcal{N}(\mathcal{G})$. If the equality holds we have that $\dim(\mathcal{S}) = n-m$. This implies that $\dim(\ker(\mathcal{N})) = n+1-(n-m)$. Clearly $\ker\mathcal{N} \subset \ker(\mathcal{N}_p \cdot r_p)$. Since $\dim(\ker(\mathcal{N}_p \cdot r_p)) = n + 1 - (n - m)$, we have that $\ker\mathcal{N} = \ker(\mathcal{N}_p \cdot r_p)$ for any $p \in M$. Namely if $(\bar\nabla\ell)^\perp(p) = 0$, $(\bar\nabla\ell)^\perp \equiv 0$ on M. This implies that

$$0 = \nabla_X V_\ell (= \nabla_X^\perp V_\ell) = (\bar\nabla_X(\bar\nabla\ell - T_\ell))^\perp = ((-\ell X) - B(X, T_\ell))^\perp = -B(X, (\bar\nabla\ell)^T).$$

Since such $(\bar\nabla\ell)^T$ spans T_pM as ℓ varies, the above equation implies that M is totally geodesic. □

We remark that the Morse index estimate of minimal submanifolds in $\mathbb{S}^n$ was proved in [12], and the above proof for the equality case is similar to the corresponding rigidity for the Morse index of [12]. Since that the equality holds for the multiplicity lower bound estimate does not necessarily imply that the Morse index (which can be strictly larger than the multiplicity) lower estimate holds the equality, the result above does not follow from Simons's result. When M is a hypersurface, which is not a totally geodesic sphere, an eigenvalue estimate for $\mathcal{J}_M$ was obtained in [12] (cf. Lemma 6.1.7 there). It played a crucial role in Simons proof of Bernstein conjecture by applying it to the analysis on the cone over M. The above result provides a high codimensional analogue. It is expected that the eigenvalue estimate of Theorem 4.1 plays some role in the study of high codimensional minimal surfaces. The Morse index of fully immersed (non-stable) $\mathbb{S}^2$ in $\mathbb{S}^{2n}$ was calculated in [4] to be $2(n(n+2)-3)$.

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