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Gap theorem on Kähler manifolds with nonnegative orthogonal bisectional curvature

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Abstract. In this paper we prove a gap theorem for Kähler manifolds with nonnegative orthogonal bisectional curvature and nonnegative Ricci curvature, which generalizes an earlier result of the first author [*L. Ni*, An optimal gap theorem, Invent. Math. **189** (2012), no. 3, 737–761]. We also prove a Liouville theorem for plurisubharmonic functions on such a manifold, which generalizes a previous result of L.-F. Tam and the first author [*L. Ni* and *L.-F. Tam*, Plurisubharmonic functions and the structure of complete Kähler manifolds with nonnegative curvature, J. Differential Geom. **64** (2003), no. 3, 457–524] and complements a recent result of Liu [*G. Liu*, Three-circle theorem and dimension estimate for holomorphic functions on Kähler manifolds, Duke Math. J. **165** (2016), no. 15, 2899–2919].

1. Introduction

In [13, 14], the following result was proved.

Theorem 1.1. *Let (M^m, g) be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Then M is flat if for some $o \in M$,*

$$\frac{1}{V_o(r)} \int_{B_o(r)} S(y)\, d\mu(y) = o(r^{-2}), \tag{1.1}$$

where $V_o(r)$ is the volume of $B_o(r)$ and $S(y)$ is the scalar curvature.

A result of this type was originated by Mok, Siu and Yau in [10], where it was proved that M is isometric to $\mathbb{C}^m$ under much stronger assumptions that (M^m, g) (with $m \geq 1$) is of maximum volume growth (meaning that $V_o(r) \geq \delta r^{2m}$ for some $\delta > 0$) and $\mathcal{S}(x)$ decays pointwisely as $r(x)^{-2-\epsilon}$ for some $\epsilon > 0$. Let $n = 2m$ be the real dimension. A Riemannian version of this result in [10] was proved by Greene and Wu [7] shortly afterwards (see also [6]

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for related results). In [11, Theorem 5.1], with a parabolic method introduced on solving the so-called Poincaré–Lelong equation, the result of [10] was improved to the cases covering manifolds of more general volume growth. Since then there are several further works aiming to prove the optimal result. See for example [3, 16]. In particular, the Ricci flow method was applied in one of these papers. In [17], using a Liouville theorem concerning the plurisubharmonic functions on a complete Kähler manifold, and the solution of Poincaré–Lelong equation obtained therein, Theorem 1.1 was proved with an additional exponential growth assumption on the integral of *the square of the scalar curvature* over geodesic balls, which was removed in [13] using a different method.

The approach of [13] toward Theorem 1.1 is via the asymptotic behavior of the optimal solution obtained by evolving a $(1,1)$-form with the initial data being the Ricci form through the heat flow of the Hodge–Laplacian operator. The key component of the proof is the monotonicity obtained in [12] (see also [15]), which makes the use of the nonnegativity of the bisectional curvature crucially. On the other hand, in [19], the authors proved that the method of deforming a $(1,1)$-form via the Hodge–Laplacian heat equation and studying the asymptotic behavior of the solution can be applied to solve the Poincaré–Lelong equation and obtain an optimal solution for it. Namely, the following result was proved.

Theorem 1.2. *Let (M^m, g) be a complete noncompact Kähler manifold with nonnegative Ricci curvature and nonnegative quadratic orthogonal bisectional curvature. Suppose that ρ is a smooth closed real $(1,1)$-form on M and let $f = \|\rho\|$ be the norm of ρ. Suppose that*

$$\int_0^\infty k_f(r)\,dr < \infty, \tag{1.2}$$

where

$$k_f(r) = \frac{1}{V_o(r)} \int_{B_o(r)} |f|\,d\mu,$$

for some fixed point $o \in M$. Then there is a smooth function u so that $\rho = \sqrt{-1}\partial\bar{\partial}u$. Moreover, for any $0 < \epsilon < 1$, u satisfies

$$\begin{aligned} &\alpha_1 r \int_{2r}^\infty k_{\|\rho\|}(s)\,ds + \beta_1 \int_0^{2r} s k_{\|\rho\|}(s)\,ds \\ &\quad \geq u(x) \geq \beta_3 \int_0^{2r} s k_{\|\rho\|}(s)\,ds - \alpha_2 r \int_{2r}^\infty k_{\|\rho\|}(s)\,ds - \beta_2 \int_0^{\epsilon r} s k_{\|\rho\|}(s)\,ds \end{aligned} \tag{1.3}$$

for some positive constants $\alpha_1(n), \alpha_2(n,\epsilon)$ and $\beta_i(n)$, $1 \leq i \leq 3$, where $r = r(x)$.

Recall that a Kähler manifold (M^m, g) is said to have nonnegative quadratic orthogonal bisectional curvature (NQOB for short) if, at any point $x \in M$ and any unitary frame $\{e_i\}$, $\sum_{i,j=1}^m R_{i\bar{i}j\bar{j}}(a_i - a_j)^2 \geq 0$ for all real numbers a_i. A Kähler manifold (M^m, g) is said to have nonnegative orthogonal bisectional curvature (NOB for short) if for any orthogonal $(1,0)$ vector fields X, Y, one has $R(X, \bar{X}, Y, \bar{Y}) \geq 0$. The example constructed in [8] shows that the curvature condition (NOB) is stronger than (NQOB). On the other hand, examples constructed in this paper show that the (NOB) is weaker than the nonnegativity of the bisectional curvature.

A natural question is whether or not the gap theorem remains true under the assumption of Theorem 1.2, or less ambitiously under the nonnegativity of the orthogonal bisectional curvature, and the nonnegativity of the Ricci curvature.

Related to the gap theorem, a Liouville-type theorem was proved in [17] for plurisubharmonic functions.

Theorem 1.3. *Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Let u be a continuous plurisubharmonic function on M. Suppose that*

$$\limsup_{x\to\infty} \frac{u(x)}{\log r(x)} = 0. \tag{1.4}$$

Then u must be constant.

Very recently, using a partial maximum principle the same Liouville result was proved [9] for complete Kähler manifolds with nonnegative holomorphic sectional curvature. On the other hand, there exists an algebraic curvature [22] which has positive holomorphic sectional curvature, positive orthogonal bisectional curvature (hence positive Ricci curvature), but with negative bisectional curvature for some pair of vectors. In the last part of this paper we also illustrate an example metric, which is unitary symmetric Kähler, with (NOB), but whose holomorphic sectional curvature is negative somewhere. This indicates that the (NOB) condition is in some sense independent of the nonnegativity of the holomorphic sectional curvature. The (NOB) condition is also a Kähler analogue of the nonnegativity of the isotropic curvature (cf. [23]). Generalizing the Liouville theorem to manifolds with (NOB) becomes an interesting itself. Note that the solution constructed in Theorem 1.2 above for ρ being the Ricci form satisfies the estimate (1.4) if the scalar curvature satisfies assumption (1.1). In fact, assumption (1.1) implies that

$$r\int_{2r}^{\infty} k_{\|\rho\|}(s)\,ds = o(1), \qquad \int_0^{2r} s k_{\|\rho\|}(s)\,ds = o(\log r).$$

Hence to obtain the gap theorem under the weaker assumptions of nonnegative orthogonal bisectional curvature and nonnegative Ricci curvature, one only needs to prove the above Liouville theorem for the manifolds with these weaker assumptions.

The main purpose of this paper is to prove Theorem 1.1 and Theorem 1.3 under the weaker assumptions of nonnegative orthogonal bisectional curvature and nonnegative Ricci curvature. In fact, the following more general result is proved.

Theorem 1.4. *Let (M, g) be a complete Kähler manifold with nonnegative orthogonal bisectional curvature and nonnegative Ricci curvature. Assume that $\rho \geq 0$ is a smooth d-closed $(1,1)$-form. Suppose that*

$$\int_0^r s \fint_{B_o(s)} \|\rho\|(y)\,d\mu(y)\,ds = o(\log r) \tag{1.5}$$

for some $o \in M$. Then $\rho \equiv 0$.

Note that there exists an algebraic curvature [22] which has positive holomorphic sectional curvature, positive orthogonal bisectional curvature (hence positive Ricci curvature), but with negative bisectional curvature for some pair of vectors. Hence the Liouville-type result can be viewed complementary to the case of [9]. In the last part we show that the perturbation technique of Huang and Tam [8] based on the unitary construction of Wu and Zheng [24] can be adapted to construct examples of complete Kähler metrics with unitary symmetry such that its curvature has (NOB), but not nonnegative bisectional curvature. More recently, in [20], unitary symmetric complete Kähler metrics have been constructed on $\mathbb{C}^m$ which has (NOB) and nonnegative Ricci curvature, but holomorphic sectional curvature can be negative. This shows that the (NOB) condition is completely independent of $\mathrm{Ric} \geq 0$ and the result proved here is completely independent of the result of G. Liu [9].

As partially explained above, there have been two approaches towards the gap theorem and the Liouville-type theorem for plurisubharmonic functions on manifolds up to very recently. The first approach of proving the Liouville-type theorem was developed in [17], which is based on the induction on the dimension of the manifold, and a result of the first author in [11] for splitting. Then the gap theorem can be derived via solving the Poincaré–Lelong equation along the original work of Mok, Siu and Yau [10]. The second approach is via the study of the large asymptotics of the solution to parabolic equations. For gap theorem it is via the Hodge–Laplace heat equation, developed by the first author in [13, 14] and further applied in [19]. For the Liouville-type theorem it involves the heat equation deformation of a plurisubharmonic function and was developed in [12,17]. The key of this parabolic approach is a monotonicity built upon a differential Harnack estimates of both authors [15]. In view of [23], such a estimate seems elusive under the positivity of isotropic curvature (as well as (NOB)) here we follow the first approach towards both results. Even though the parabolic three-circle theorem has been explicit in [12], it is only in a recent article of Liu [9], the elliptic version was proved, which supplies the third approach towards the Liouville-type theorem.

It remains an interesting question if the gap theorem still holds under condition (NQOB) and $\mathrm{Ric} \geq 0$.

2. Proof of the Liouville and the gap theorems

In [17], the authors proved a Liouville-type theorem for plurisubharmonic functions on a Kähler manifold with nonnegative holomorphic bisectional curvature. A key ingredient of the proof is a maximum principle for Hermitian symmetric tensor satisfying the Lichnerowicz heat equation. In fact, the same results still holds on a Kähler manifold with the weaker conditions of (NOB) and nonnegativity of the Ricci curvature. We first recall the maximum principle for Hermitian symmetric tensor $\eta(x,t)$ satisfying the Lichnerowicz heat equation

$$\left(\frac{\partial}{\partial t} - \Delta\right)\eta_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}\gamma\bar{\delta}}\eta_{\delta\bar{\gamma}} - \frac{1}{2}\left(R_{\alpha\bar{p}}\eta_{p\bar{\beta}} + R_{p\bar{\beta}}\eta_{\alpha\bar{p}}\right). \tag{2.1}$$

Here the expressions above are with respect to a unitary frame $\{e_\alpha\}$ and the repeated indices are summed from 1 to m.

Theorem 2.1. *Let (M^m, g) be a complete noncompact Kähler manifold with nonnegative holomorphic orthogonal bisectional curvature and nonnegative Ricci curvature. Let $\eta(x,t)$ be a Hermitian symmetric $(1,1)$ tensor satisfying (2.1) on $M \times [0,T]$ with $0 < T < \frac{1}{40a}$ such that $\|\eta\|$ satisfies*

$$\int_M \|\eta\|(x,0)\exp(-ar^2(x))\,dx < +\infty \tag{2.2}$$

and

$$\liminf_{r\to\infty}\int_0^T\int_{B_o(r)}\|\eta\|^2(x,t)\exp(-ar^2(x))\,dx\,dt < +\infty. \tag{2.3}$$

Suppose at $t=0$, $\eta_{\alpha\bar\beta}\geq -bg_{\alpha\bar\beta}$ for some constant $b\geq 0$. Then there exists T_0, $0<T_0<T$, depending only on T and a so that the following are true:

(i) *$\eta_{\alpha\bar\beta}(x,t)\geq -bg_{\alpha\bar\beta}(x)$ for all $(x,t)\in M\times[0,T_0]$.*

(ii) *For any $T_0>t'\geq 0$, suppose there is a point x' in M^m and there exist constants $\nu>0$ and $R>0$ such that the sum of the first k eigenvalues $\lambda_1,\dots,\lambda_k$ of $\eta_{\alpha\bar\beta}$ satisfies*

$$\lambda_1+\cdots+\lambda_k\geq -kb+\nu k\phi_{x',R}$$

for all x at time t', where $\phi:[0,\infty)\to[0,1]$ is a smooth cut-off function such that $\phi\equiv 1$ on $[0,1]$ and $\phi\equiv 0$ on $[2,\infty)$, $\phi_{x',R}(x)=\phi(\frac{d(x,x')}{R})$, the eigenvalues of η are of ascending order. Then for all $t>t'$ and for all $x\in M$, the sum of the first k eigenvalues of $\eta_{\alpha\bar\beta}(x,t)$ satisfies

$$\lambda_1+\cdots+\lambda_k\geq -kb+\nu k f_{x',R}(x,t-t'),$$

where $f_{x',R}$ is the solution of $(\frac{\partial}{\partial t}-\Delta)f=-f$ with initial value $\phi_{x',R}(x)$.

Proof. The proof is to observe that the argument of [17, Theorem 2.1] only requires the nonnegativity of the orthogonal bisectional curvature and nonnegativity of the Ricci curvature. For the sake of the completeness we include some details of the argument here and pay special attention on the places where the nonnegativity of the orthogonal bisectional curvature is needed. By (2.1), one has

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\|\eta\|^2 &= -\|\eta_{\alpha\bar\beta s}\|^2-\|\eta_{\alpha\bar\beta\bar s}\|^2+2R_{\alpha\bar\beta p\bar q}\eta_{q\bar p}\eta_{\beta\bar\alpha}-2R_{\alpha\bar p}\eta_{p\bar\beta}\eta_{\beta\bar\alpha}\\ &\leq -\|\eta_{\alpha\bar\beta s}\|^2-\|\eta_{\alpha\bar\beta\bar s}\|^2,\end{aligned}$$

where we choose $\{e_\alpha\}$ so that $\eta_{\alpha\bar\beta}=\lambda_\alpha\delta_{\alpha\bar\beta}$. Thus

$$\begin{aligned}2R_{\alpha\bar\beta p\bar q}\eta_{q\bar p}\eta_{\beta\bar\alpha}-2R_{\alpha\bar p}\eta_{p\bar\beta}\eta_{\beta\bar\alpha} &= 2\Big(\sum R_{\alpha\bar\alpha\beta\bar\beta}\lambda_\alpha\lambda_\beta-\sum_\alpha R_{\alpha\bar\alpha}\lambda_\alpha^2\Big)\\ &= -\sum_{\alpha,\beta}R_{\alpha\bar\alpha\beta\bar\beta}(\lambda_\alpha-\lambda_\beta)^2\leq 0,\end{aligned}$$

provided that (M,g) has nonnegative quadratic orthogonal bisectional curvature (which is a weaker condition than nonnegativity of the orthogonal bisectional curvature).

Combining with the inequality

$$2|\nabla\|\eta\||^2\leq \|\eta_{\alpha\bar\beta s}\|^2+\|\eta_{\alpha\bar\beta\bar s}\|^2$$

it implies (as in [17]) that

$$\left(\frac{\partial}{\partial t}-\Delta\right)\|\eta\|\leq 0.$$

With [17, Lemma 1.2] which holds on the manifold with nonnegative Ricci curvature and (2.2), then

$$h(x,t) = \int_M H(x,y,t)\|\eta\|(y)\,dy$$

is a solution to the heat equation on the set $M \times [0, \frac{1}{40a}]$ with initial value $\|\eta\|(x)$. With assumption (2.3) and [18, Theorem 1.2], there exists $0 < T_0 < T$ such that

$$\|\eta\|(x,t) \le h(x,t)$$

on $M \times [0, T_0]$.

For any $r_2 > r_1$, let $A_o(r_1, r_2)$ denote the annulus $B_o(r_2) \setminus B_o(r_1)$. For any $R > 0$, let σ_R be the cut-off function which is 1 on $A_o(\frac{R}{4}, 4R)$ and 0 outside $A_o(\frac{R}{8}, 8R)$. We define

$$h_R(x,t) = \int_M H(x,y,t)\sigma_R(y)\|\eta\|(y,0)\,dy.$$

Then h_R satisfies the heat equation with initial value $\sigma_R\|\eta\|$. Moreover, [17, Lemma 2.2] holds when Ricci curvature is nonnegative. That is, there exists $0 < T_0 < T$ depending only on a such that:

(1) There exists a function $\tau = \tau(r) > 0$ with the property that $\lim_{r\to\infty} \tau(r) = 0$ such that for all $R \ge \max\{\sqrt{T_0}, 1\}$ and for all $(x,t) \in A_o(\frac{R}{2}, 2R) \times [0, T_0]$,

$$h(x,t) \le h_R(x,t) + \tau(R).$$

(2) For any $r > 0$,

$$\lim_{R\to\infty} \sup_{B_o(r)\times[0,T_0]} h_R = 0.$$

By [17, Lemma 1.2 and Corollary 1.1], we can find a solution $\phi(x,t)$, $(\frac{\partial}{\partial t} - \Delta)\phi = \phi$ such that

$$\phi(x,t) \ge \exp(c(r^2(x) + 1))$$

for some $c > 0$ for all $0 \le t \le T$.

As in [17] we only need to prove (ii) by assuming (i) since the proof of (i) is similar, but easier. Without the loss of generality, we assume that $t' = 0$ and there exist $x' \in M$, $\nu > 0$ and $R_0 > 0$ such that the first k eigenvalues $\lambda_1, \dots, \lambda_k$ of $\eta_{\alpha\bar\beta}$ satisfy

$$\lambda_1 + \cdots + \lambda_k \ge -kb + \nu k \phi_{x',R_0}$$

for all x in M at time $t = 0$. For simplicity, we assume that $\nu = 1$.

Let $\epsilon > 0$, for any $R > 0$, define

$$\psi(x,t,\epsilon,R) = -f_{x',R_0}(x,t) + \epsilon\phi(x,t) + h_R(x,t) + \tau(R) + b$$

and let

$$(\eta_R)_{\alpha\bar\beta} = \eta_{\alpha\bar\beta} + \psi g_{\alpha\bar\beta}.$$

Then at $t = 0$, at each point the sum of the first k eigenvalues of η_R is positive. We want to prove that for any $T_0 \ge t > 0$ and $R > 0$, the sum of the first k eigenvalues of η_R in the set $B_o(R) \times [0, T_0]$ is positive, provided R is large enough. Then one can argue by contradiction similarly as in [17, proof of Theorem 2.1]. The only attention to pay is [17, proof of (2.14)].

That is, if η_R has eigenvectors $v_p = \frac{\partial}{\partial z^p}$ for $1 \leq p \leq m$ with eigenvalues λ_p, then

$$
\begin{aligned}
&\sum_{\alpha,\beta=1}^{k} [R_{\delta\bar{\gamma}\alpha\bar{\beta}}(\eta_{\gamma\bar{\delta}} + \psi g_{\gamma\bar{\delta}}) - \frac{1}{2}R_{\alpha\bar{p}}(\eta_{p\bar{\beta}} + \psi g_{p\bar{\beta}}) - \frac{1}{2}R_{p\bar{\beta}}(\eta_{\alpha\bar{p}} + \psi g_{\alpha\bar{p}})]g^{\alpha\bar{\beta}} \\
&\quad = \sum_{\alpha=1}^{k}\sum_{\gamma=1}^{m} R_{\gamma\bar{\gamma}\alpha\bar{\alpha}}\lambda_\gamma - \sum_{\alpha=1}^{k} R_{\alpha\bar{\alpha}}\lambda_\alpha \\
&\quad = \sum_{\alpha=1}^{k}\sum_{\gamma=1}^{m} R_{\gamma\bar{\gamma}\alpha\bar{\alpha}}\lambda_\gamma - \sum_{\alpha=1}^{k}\sum_{\gamma=1}^{m} R_{\gamma\bar{\gamma}\alpha\bar{\alpha}}\lambda_\alpha \\
&\quad = \sum_{\alpha=1}^{k}\sum_{\gamma=k+1}^{m} R_{\gamma\bar{\gamma}\alpha\bar{\alpha}}\lambda_\gamma - \sum_{\alpha=1}^{k}\sum_{\gamma=k+1}^{m} R_{\gamma\bar{\gamma}\alpha\bar{\alpha}}\lambda_\alpha \\
&\quad = \sum_{\alpha=1}^{k}\sum_{\gamma=k+1}^{m} R_{\gamma\bar{\gamma}\alpha\bar{\alpha}}(\lambda_\gamma - \lambda_\alpha) \\
&\quad \geq 0,
\end{aligned}
$$

where we only used the fact that M has nonnegative orthogonal bisectional curvature and $\lambda_\gamma \geq \lambda_\alpha$ for $\gamma \geq \alpha$.

The rest of the proof is the same as that of [17, Theorem 2.1]. □

The next observation is that one still has the following corollary, as [17, Corollary 2.1], under the weaker condition (NOB).

Corollary 2.2. *Let M and η be as in the Theorem* 2.1 *with $b = 0$. That is, $\eta(x, 0) \geq 0$ for all $x \in M$. Let T_0 be such that the conclusions of the theorem are true. For $0 < t < T_0$, let*

$$\mathcal{K}(x,t) = \{w \in T_x^{1,0}(M) : \eta_{\alpha\bar{\beta}}(x,t)w^\alpha = 0 \text{ for all } \beta\}$$

be the null space of $\eta_{\alpha\bar{\beta}}$. Then there exists T_1, $0 < T_1 < T_0$, such that for any $0 < t < T_1$, $\mathcal{K}(x,t)$ is a smooth distribution on M.

Proposition 2.1. *Let $\eta(x,t)$ be a Hermitian symmetric tensor satisfying* (2.1). *Assume that $\eta(x,t) \geq 0$ on $M \times (0,T)$ and M has nonnegative orthogonal bisectional curvature. Then $\mathcal{K}(x,t)$ is invariant under parallel translation. In particular, if M is simply-connected, there is a splitting $M = M_1 \times M_2$ with η being zero on M_1 and positive on M_2, and M_i has nonnegative orthogonal bisectional curvature.*

One can modify the original argument in [17] for this slightly more general result. On the other hand the strong maximum principle of Bony adapted by Brendle and Schoen [2] (cf. Bony [1]) can be applied to obtain this result. In this case one formulates everything on the principle $\mathsf{U}(n)$-bundle. For any unitary frame $e = \{e_i\}_{i=1}^m$ define $u(e) = \eta(e_1, e_{\bar{1}})$. Let $\widetilde{Y}$ be the horizontal lifting of the vector field $\frac{\partial}{\partial t}$ on $M \times (0,T)$. At e, let $\widetilde{X_i}$ be the horizontal lifting of e_i. Similar computation as in [2] yields that

$$\Big(\widetilde{Y} - \sum \widetilde{X}_i\widetilde{X}_i\Big)u = \sum_{s\geq 2} R_{1\bar{1}s\bar{s}}(\eta_{s\bar{s}} - \eta_{1\bar{1}}) \geq -Ku,$$

where K is a local constant depending only on M. Here we have used that $R_{1\bar{1}s\bar{s}}\eta_{s\bar{s}} \geq 0$ for $s \geq 2$. This is enough to conclude that $\mathcal{K}(x,t)$ is invariant under the parallel translation. In fact, the following slight general version of Bony's strong maximum principle holds.

Theorem 2.3 (Bony, Brendle–Schoen). *Let Ω be an open subset of $\mathbb{R}^n$. Let $\{X_i\}_{i=1}^k$ be smooth vector fields on Ω. Assume that $u : \Omega \to \mathbb{R}$ is a nonnegative smooth function satisfying*

$$\sum_{i=1}^{k}(D^2u)(X_i, X_i) \leq -K \min\Big\{0, \inf_{|\xi|=1}(D^2u)(\xi,\xi)\Big\} + K|\nabla u| + Ku,$$

with K being a positive constant. Let $Z = \{x : u(x) = 0\}$ be the zero set. Let $\gamma(s) : [0,1] \to \Omega$ be a smooth curve such that $\gamma(0) \in Z$ and $\gamma'(s) = \sum_{i=1}^k a_i(s)X_i(\gamma(s))$ with $a_i(s)$ being smooth functions. Then $\gamma(s) \in Z$ for all $s \in [0,1]$.

Since $\mathcal{K}(x,t)$ is invariant under the parallel translation and clearly it is a clear subspace, the decomposition follows from the De Rham decomposition theorem.

Proof of Theorem 1.3 *under* (NOB) *and the nonnegativity of the Ricci curvature.* The proof is similar to that of [17, Theorem 3.2].

Without loss of generality, we may assume that M is simply connected (by lifting the function to the universal cover, the growth condition clearly is preserved). For any fixed constant c, we let $u_c = \max\{u, c\}$. It is well known that u_c is plurisubharmonic and u_c satisfies

$$|u_c(x)| \leq C \exp(ar^2(x)) \tag{2.4}$$

for some constant $C > 0$ and $a > 0$. By adding a constant, we can also assume that $u_c \geq 0$. Then u_c is a nonnegative continuous plurisubharmonic function satisfying (2.4). Now we consider the heat equation

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \Delta\right) v_c(x,t) = 0, \\ \qquad\qquad v_c(x,0) = u_c(x). \end{cases}$$

Then the above Dirichlet boundary problem has a solution $v_c(x,t)$ on $M \times [0, \frac{1}{40a}]$, obtained in [17, Lemma 1.2] which holds on the manifold with nonnegative Ricci curvature.

Since Theorem 2.1 and Corollary 2.2 (and Proposition 2.1) hold on the manifold with (NOB) and nonnegative Ricci curvature, one can go through [17, proof of Theorem 3.1] to see that there exists $0 < T_0 < T$ such that $v_c(x,t)$ is a smooth plurisubharmonic function on $M \times (0, T_0]$. Moreover, there exists $0 < T_1 < T_0$ such that the null space of $(v_c)_{\alpha\bar{\beta}}(x,t)$,

$$\mathcal{K}(x,t) = \{w \in T_x^{1,0}(M) : (v_c)_{\alpha\bar{\beta}} w^\alpha = 0 \text{ for all } \beta\},$$

is a distribution on M for any $0 < t < T_1$. Moreover, the distribution is invariant under parallel translation. Then by Proposition 2.1, for any $t_0 > 0$ small enough, $M = M_1 \times M_2$ isometrically and holomorphically such that when restricted on M_1, $(v_c)_{\alpha\bar{\beta}}$ is zero, and $(v_c)_{\alpha\bar{\beta}}$ is positive everywhere when restricted on M_2 by the De Rham decomposition. We want to conclude that M_2 factor does not exist. By [17, Corollary 1.1], we have

$$\limsup_{x\to\infty} \frac{v_c(x,t_0)}{\log r(x)} = 0. \tag{2.5}$$

Hence when restricted on M_2 (if the factor exists), (2.5) still holds. This contradicts with the fact that $(v_c)_{\alpha\bar{\beta}}$ is positive when restricted on M_2, since by [11, Proposition 4.1], which asserts that if a plurisubharmonic function $p(x)$ on a Kähler manifold with nonnegative Ricci curvature satisfies the growth condition (2.5), then $(\partial\bar{\partial}p)^m = 0$, where m is the complex-dimension of the manifold. Hence $(v_c)_{k\bar{l}}(x, t_0) \equiv 0$ on M for all t_0 small enough. By the gradient estimate of Cheng and Yau [4] and (2.5), we can conclude that $v_c(x, t_0)$ is a constant, provided t_0 is small enough. Hence u_c is a constant. Since c is arbitrary, it shows that $u(x)$ is also a constant. □

Proof of Theorem 1.1 *under the assumptions of* (NOB) *and nonnegativity of the Ricci curvature.* Let ρ be the Ricci form, which is a smooth nonnegative closed real $(1,1)$-form on M. It is easy to check that for any $y \in M$,

$$\|\rho\|(y) \leq S(y) \leq \sqrt{n}\|\rho\|(y).$$

Then

$$k_{\|\rho\|}(r) = o(r^{-2}) \tag{2.6}$$

and (1.2) follows when (1.1) holds for some fixed point $o \in M$. Since the curvature condition (NOB) is stronger than (NQOB), Theorem 1.2 still holds with the assumptions of (NOB) and nonnegativity of the Ricci curvature. Moreover, the solution u to the Poincaré–Lelong equation $\rho = \sqrt{-1}\partial\bar{\partial}u$ is a plurisubharmonic function and (1.3) holds. In fact, (1.3) implies that

$$\lim_{x\to\infty} \frac{u(x)}{\log r(x)} = 0,$$

since (2.6) implies that

$$\int_{2r}^{\infty} k_{\|\rho\|}(s)\, ds = o(r^{-1}),$$
$$\int_{0}^{2r} s k_{\|\rho\|}(s)\, ds = o(\log r),$$
$$\int_{0}^{\epsilon r} s k_{\|\rho\|}(s)\, ds = o(\log r).$$

By the generalization of the Liouville theorem proved above, we conclude that u must be constant. This implies that $\mathrm{Ric} \equiv 0$. For any unitary frame $\{e_\alpha\}_{\alpha=1}^m$, (NOB) implies that for any $\alpha \neq \beta$, by considering $\tilde{e}_\alpha = \frac{1}{2}(e_\alpha + e_\beta)$, $\tilde{e}_\beta = \frac{1}{2}(e_\alpha - e_\beta)$,

$$R(\tilde{e}_\alpha, \bar{\tilde{e}}_\alpha, \tilde{e}_\beta, \bar{\tilde{e}}_\beta) \geq 0,$$

which is equivalent to that

$$R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} + R_{\beta\bar{\beta}\beta\bar{\beta}} - R_{\alpha\bar{\beta}\alpha\bar{\beta}} - R_{\beta\bar{\alpha}\beta\bar{\alpha}} \geq 0. \tag{2.7}$$

If we replace e_β by $\sqrt{-1}e_\beta$ in $\tilde{e}_\alpha$ and $\tilde{e}_\beta$, then

$$R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} + R_{\beta\bar{\beta}\beta\bar{\beta}} + R_{\alpha\bar{\beta}\alpha\bar{\beta}} + R_{\beta\bar{\alpha}\beta\bar{\alpha}} \geq 0. \tag{2.8}$$

By summing (2.7) and (2.8), we obtain the following inequality:

$$R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} + R_{\beta\bar{\beta}\beta\bar{\beta}} \geq 0.$$

But

$$R_{\alpha\bar{\alpha}\beta\bar{\beta}} \geq 0,$$

$$R_{\alpha\bar{\alpha}} = R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} + \sum_{\gamma\neq\alpha} R_{\alpha\bar{\alpha}\gamma\bar{\gamma}} = 0,$$

$$R_{\beta\bar{\beta}} = R_{\beta\bar{\beta}\beta\bar{\beta}} + \sum_{\gamma\neq\beta} R_{\beta\bar{\beta}\gamma\bar{\gamma}} = 0.$$

Then $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 0$ for any α, which implies that M is flat and the generalization of Theorem 1.1 follows. □

Proof of Theorem 1.4. By Theorems 1.2 and 1.3, it suffices to establish the estimate

$$r \int_{2r}^{\infty} k_{\|\rho\|}(s)\, ds = o(\log r). \tag{2.9}$$

From (1.5), we know

$$\int_{\frac{1}{2}r}^{r} s k_{\|\rho\|}(s)\, ds = o(\log r).$$

For any $\frac{1}{2}r \leq s \leq r$, by volume comparison,

$$\begin{aligned} 2^{-2m} k_{\|\rho\|}\left(\frac{r}{2}\right) &\leq \frac{\mathrm{Vol}(B_o(\frac{r}{2}))}{\mathrm{Vol}(B_o(r))} k_{\|\rho\|}\left(\frac{r}{2}\right) \\ &\leq k_{\|\rho\|}(s) = \frac{1}{\mathrm{Vol}(B_o(s))} \int_{B_o(s)} \|\rho\|(y)\, d\mu(y) \\ &\leq \frac{\mathrm{Vol}(B_o(r))}{\mathrm{Vol}(B_o(\frac{r}{2}))} k_{\|\rho\|}(r) \leq 2^{2m} k_{\|\rho\|}(r). \end{aligned}$$

From this and

$$\int_{\frac{r}{2}}^{r} s k_{\|\rho\|}(s)\, ds = o(\log r),$$

we derive

$$k_{\|\rho\|}(r) = o\left(\frac{\log r}{r^2}\right).$$

This implies (2.9), for

$$\lim_{r\to\infty} \frac{\int_{2r}^{\infty} k_{\|\rho\|}(s)\, ds}{r^{-1}\log r} = 2 \lim_{r\to\infty} \frac{r^2 k_{\|\rho\|}(2r)}{\log r - 1} = 0,$$

and completes the proof of the theorem. □

We note that the proof of the gap theorem via the Liouville theorem and the solution of the Poincaré–Lelong equation also suggests the following gap theorem, in view of the Liouville theorem proved by Liu.

Theorem 2.4. *Let (M, g) be a complete Kähler manifold with nonnegative quadratic orthogonal bisectional curvature and nonnegative holomorphic sectional curvature. Assume that $\rho \geq 0$ is a smooth d-closed $(1,1)$-form. Suppose that*

$$\int_0^r s \fint_{B_o(s)} \|\rho\|(y)\, d\mu(y)\, ds = o(\log r)$$

for some $o \in M$. Then $\rho \equiv 0$. In particular, if (1.1) *holds,* (M, g) *is flat.*

Proof. By [8, Proposition 3.1] (see also [21]) the nonnegativity of the quadratic orthogonal bisectional curvature implies that

$$R_{\alpha\bar{\alpha}} - R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \geq 0,$$

which together with the nonnegativity of the holomorphic sectional curvature implies that the Ricci curvature is nonnegative. Hence Theorem 1.2 can be applied. Namely for the d-closed real $(1,1)$-form ρ, there exists a function u such that $\sqrt{-1}\partial\bar{\partial}u = \rho$. The proof of the above theorem shows that the solution provided by Theorem 1.2 satisfies that $u(x) = o(\log(r(x)))$ as $x \to \infty$. Now we apply the Liouville theorem of Liu, and conclude that $u(x)$ must be a constant. Hence $\rho \equiv 0$.

Now we prove the last part regarding the flatness of (M, g). Apply the first part to ρ being the Ricci form. We conclude that (M, g) has vanishing Ricci curvature. In particular, (M, g) has vanishing scalar curvature. On the other hand, by a result of Berger (cf. [5, Lemma E.6.3]), for every point $p \in M$,

$$S(p) = \frac{m(m+1)}{\mathrm{Vol}(\mathbb{S}^{2m-1})} \int_{|X|=1,\, X \in T^{1,0}M} H_p(X)\, d\theta(X),$$

where H_p denotes the holomorphic sectional curvature at p. Hence the vanishing of the scalar curvature implies that the holomorphic sectional curvature (which is assumed to be nonnegative) vanishes everywhere. This proves the last claim. □

Remark 2.5. As pointed out before, algebraically even the (NOB) condition (which implies (NQOB)) together with the nonnegativity of the bisectional curvature does not implies the nonnegativity of the bisectional curvature. It would be interesting to construct a complete metric with (NOB), nonnegative holomorphic sectional curvature, but with negative bisectional curvature somewhere.

3. Examples

In this section we always assume that $m \geq 3$. In [24], Wu and Zheng considered the $\mathsf{U}(m)$-invariant Kähler metrics on $\mathbb{C}^m$ and obtained necessary and sufficient conditions for the nonnegativity of the curvature operator, nonnegativity of the sectional curvature, as well as the nonnegativity of the bisectional curvature respectively. In [25], Yang and Zheng later proved that the necessary and sufficient condition in [24] for the nonnegativity of the sectional curvature holds for the nonnegativity of the complex sectional curvature under the unitary symmetry. In [8], the authors obtained the necessary and sufficient conditions for (NOB) and (NQOB) respectively. Moreover, they constructed a $\mathsf{U}(m)$-invariant Kähler metric on $\mathbb{C}^m$, which is of (NQOB), but does not have (NOB) nor nonnegativity of the Ricci curvature. In this section, we will construct a $\mathsf{U}(m)$-invariant Kähler metric on $\mathbb{C}^m$ which has (NOB) but does not have nonnegative bisectional curvature. The existence of such metric was pointed out in [8, Remark 4.1], and the construction below is a modification of the perturbation construction therein.

We follow the same notations as in [24, 25]. Let $(z_1, \ldots, z_m)$ be the standard coordinate on $\mathbb{C}^m$ and $r = |z|^2$. A $\mathsf{U}(m)$-invariant metric on $\mathbb{C}^m$ has the Kähler form

$$\omega = \frac{\sqrt{-1}}{2}\partial\bar{\partial}P(r),$$

where $P \in C^{\infty}([0, +\infty))$. Under the local coordinates, the metric has the components

$$g_{i\bar{j}} = f(r)\delta_{ij} + f'(r)\bar{z}_i z_j. \tag{3.1}$$

We further denote

$$f(r) = P'(r), \quad h(r) = (rf)'.$$

It is easy to check that ω will give a complete Kähler metric on $\mathbb{C}^m$ if and only if

$$f > 0, \quad h > 0, \quad \int_0^{\infty} \frac{\sqrt{h}}{\sqrt{r}}\, dr = +\infty.$$

If $h > 0$, then $\xi = -\frac{rh'}{h}$ is a smooth function on $[0, \infty)$ with $\xi(0) = 0$. On the other hand, if ξ is a smooth function on $[0, \infty)$ with $\xi(0) = 0$, one can define

$$h(r) = \exp\left(-\int_0^r \frac{\xi(s)}{s}\, ds\right) \quad \text{and} \quad f(r) = \frac{1}{r}\int_0^r h(s)\, ds$$

with $h(0) = 1$. It is easy to see that $\xi(r) = -\frac{rh'}{h}$. Then (3.1) defines a $\mathsf{U}(m)$-invariant Kähler metric on $\mathbb{C}^m$.

The components of the curvature operator of a $\mathsf{U}(m)$-invariant Kähler metric under the orthonormal frame $\{e_1 = \frac{1}{\sqrt{h}}\partial_{z_1}, e_2 = \frac{1}{\sqrt{f}}\partial_{z_2}, \dots, e_m = \frac{1}{\sqrt{f}}\partial_{z_m}\}$ at $(z_1, 0, \dots, 0)$ are given as follows, see [24]:

$$A = R_{1\bar{1}1\bar{1}} = -\frac{1}{h}\left(\frac{rh'}{h}\right)' = \frac{\xi'}{h}, \tag{3.2}$$

$$B = R_{1\bar{1}i\bar{i}} = \frac{f'}{f^2} - \frac{h'}{hf} = \frac{1}{(rf)^2}\left[rh - (1-\xi)\int_0^r h(s)\, ds\right], \ i \geq 2, \tag{3.3}$$

$$C = R_{i\bar{i}i\bar{i}} = 2R_{i\bar{i}j\bar{j}} = -\frac{2f'}{f^2} = \frac{2}{(rf)^2}\left(\int_0^r h(s)\, ds - rh\right), \ i \neq j,\ i, j \geq 2. \tag{3.4}$$

The other components of the curvature tensor are zero, except those obtained by the symmetric properties of curvature tensor.

The following result was proved in [24], which plays an important role in the construction.

Theorem 3.1 (Wu–Zheng). *The following statements hold:*

(1) *If* $0 < \xi < 1$ *on* $(0, \infty)$*, then* g *is complete.*

(2) g *is complete and has positive bisectional curvature if and only if* $\xi' > 0$ *and* $0 < \xi < 1$ *on* $(0, \infty)$*, where* $\xi' > 0$ *is equivalent to* $A > 0$, $B > 0$ *and* $C > 0$.

(3) *Every complete* $\mathsf{U}(m)$*-invariant Kähler metric on* $\mathbb{C}^m$ *with positive bisectional curvature is given by a smooth function* ξ *in* (2).

Using the above notations and formulations, Huang and Tam [8] proved the following:

Theorem 3.2 (Huang–Tam). *A* $\mathsf{U}(m)$*-invariant Kähler metric on* $\mathbb{C}^m$ *has nonnegative orthogonal bisectional curvature if and only if* $A + C \geq 0$, $B \geq 0$ *and* $C \geq 0$.

Let ξ be a smooth function on $[0,\infty)$ with $\xi(0)=0, \xi'(r)>0$ and $0<\xi(r)<1$ for $0<r<\infty$. Let $a=\lim_{r\to\infty}\xi(r)$. Then $0<a\le 1$. By the above discussion this gives a complete $\mathsf{U}(m)$-invariant metric on $\mathbb{C}^m$ with positive bisectional curvature. The strategy of [8] is to perturb this metric by adding a perturbation term to ξ to obtain the one with needed property. In particular, the work of [8] produces metric with (NQOB), but does not satisfy (NOB) nor nonnegativity of the Ricci curvature. To achieve this, in [8, 24] the following estimates (cf. [8, Lemma 4.1]) were obtained.

Lemma 3.1. *Let ξ be as above with $\lim_{r\to\infty}\xi = a$ $(\in (0,1])$. We have the following:*

(1) *For $r>0$, $(rh-(1-\xi)\int_0^r h)'>0$, and*

$$\lim_{r\to\infty}\int_0^r h=\infty, \quad \lim_{r\to\infty} h=0, \quad \lim_{r\to\infty}\frac{rh}{\int_0^r h}=1-a.$$

(2) *For any $\epsilon>0$, and for any $r_0>0$, there is $R>r_0$ such that*

$$\xi'(R)-\epsilon h(R)C(R)<0.$$

(3) $\lim_{r\to\infty} h(r)C(r)=0$.

(4) *For all $\epsilon>0$, there exists $\delta>0$ such that if $R\ge 3$, $\delta\ge\eta\ge 0$ is a smooth function with support in $[R-1,R+1]$, then for all $r\ge 0$,*

$$h(r)\le \bar{h}(r)\le (1+\epsilon)h(r) \quad \textit{and} \quad \int_0^r h\le \int_0^r \bar{h}\le (1+\epsilon)\int_0^r h,$$

where $\bar{h}(r)=\exp(-\int_0^r \frac{\bar{\xi}}{t}\,dt)$ and $\bar{\xi}=\xi-\eta$.

Let ϕ be a smooth cut-off function on $\mathbb{R}$ as in [8] such that

(i) $0\le\phi\le c_0$ with c_0 being an absolute constant,

(ii) $\mathrm{supp}(\phi)\subset[-1,1]$,

(iii) $\phi'(0)=1$ and $|\phi'|\le 1$.

The construction is to perturb ξ into $\bar{\xi}(r)=\xi(r)-\alpha h(R)C(R)\phi(r-R)$ for suitable choice of R,α. Note that this only changes the value of ξ on a compact set. Once $\bar{h}$ is defined, equations (3.2)–(3.4) define the corresponding curvature components $\bar{A},\bar{B},\bar{C}$ of the perturbed metric.

Theorem 3.3. *There is $1>\alpha>0$ such that for any $r_0>0$ there is $R>r_0$ satisfying the following: If $\bar{\xi}(r)=\xi(r)-\alpha h(R)C(R)\phi(r-R)$, then $\bar{\xi}$ determines a complete $\mathsf{U}(m)$-invariant Kähler metric on $\mathbb{C}^m$ such that*

(i) $\bar{A}(R)<0$,

(ii) $\bar{A}+\bar{C}>0$ *on* $[R-1,R+1]$,

(iii) $\bar{B}(r)>0$ *for all* r,

(iv) $\bar{C}(r)>0$ *for all* r.

Then $\bar{\xi}$ will give a compete $\mathsf{U}(m)$-invariant Kähler metric which satisfies (NOB) but does not have nonnegative bisectional curvature, nor nonnegative holomorphic sectional curvature.

Proof. Let $\beta = \alpha h(R)C(R)$. Let $\epsilon > 0$, a suitable constant, and $R \geq 3$, a large number, to be chosen later. Since $C(R)h(R) \to 0$ as $R \to \infty$, β will be small constant if R is chosen to be large. Then by Lemma 3.1 (3), for any δ with $a > \delta > 0$, for R sufficiently large we have $\beta > 0$ with $\beta c_0 < \delta$. Recall that c_0 is the upper bound of the cut-off function $|\phi|$. Let $\bar{\xi}(r) = \xi(r) - \beta\phi(r - R)$. Such chosen $\bar{\xi} \in (0, 1)$ determines a complete $\mathsf{U}(m)$-invariant complete Kähler metric on $\mathbb{C}^m$. Moreover, for all r,

$$h(r) \leq \bar{h}(r) \leq (1+\epsilon)h(r), \quad \int_0^r h \leq \int_0^r \bar{h} \leq (1+\epsilon)\int_0^r h.$$

We shall prove that for suitable chosen α and R, $\bar{\xi}$, $\bar{h}$, $\bar{f}$ will define a complete unitary symmetric Kähler metric on $\mathbb{C}^m$ satisfying (1)–(4). Assuming (1)–(4) in the theorem, (1) implies that the metric does not have nonnegative holomorphic sectional curvature (hence cannot have nonnegative bisectional curvature). Equation (3.2) together with (4) implies that (2) is sufficient to conclude that $\bar{A} + \bar{C} > 0$ for all r. Hence by Theorem 3.2 the perturbed metric has (NOB).

By Lemma 3.1 (2), for any $\epsilon > 0$ (sufficiently small), there is $R > r_0$ such that

$$\bar{\xi}'(R) = \xi'(R) - \beta \leq (\epsilon - \alpha)h(R)C(R).$$

Hence for (1), it suffices to choose $\alpha > \epsilon$.

By formula (3.4) and [8, proof of Lemma 4.2] (precisely [8, (4.6)]), we may choose a large r_1 so that if $R > r_1$ and for $r \in [R-1, R+1]$,

$$\bar{C}(r) \geq \frac{2}{(1+\epsilon)^2 \int_0^R h}(a - 2\epsilon + a\epsilon - \epsilon^2)$$

provided $a - 2\epsilon + a\epsilon - \epsilon^2 > 0$. We choose $\epsilon > 0$ so that it satisfies this condition. Here a is the constant from Lemma 3.1. On the other hand,

$$C(R) \leq \frac{2}{\int_0^R h}(a + \epsilon)$$

if r_1 is large enough depending only on ϵ and $R > r_1$. Hence, if ϵ and r_1 satisfy the above conditions, then for $r \in [R-1, R+1]$,

$$\bar{C}(r) \geq \frac{a - 2\epsilon + a\epsilon - \epsilon^2}{(a+\epsilon)(1+\epsilon)^2} C(R).$$

Therefore, if $\epsilon > 0$ satisfies $a > \epsilon$ and $a - 2\epsilon + a\epsilon - \epsilon^2 > 0$, we can find $r_1 > r_0$ such that if $R > r_1$, then it holds for $r \in [R-1, R+1]$,

$$\begin{aligned}
\bar{A}(r) + \bar{C}(r) &\geq \frac{\xi'(r) - \beta}{\bar{h}} + \bar{C}(r) \\
&\geq \frac{-\beta}{\bar{h}(r)} + \frac{a - 2\epsilon + a\epsilon - \epsilon^2}{(a+\epsilon)(1+\epsilon)^2} C(R) \\
&\geq -\frac{\beta}{(1-\epsilon)h(R)} + \frac{a - 2\epsilon + a\epsilon - \epsilon^2}{(a+\epsilon)(1+\epsilon)^2} C(R) \\
&= \frac{1}{(1-\epsilon)h(R)}\left[-\beta + (1-\epsilon)\frac{a - 2\epsilon + a\epsilon - \epsilon^2}{(a+\epsilon)(1+\epsilon)^2} h(R)C(R)\right].
\end{aligned}$$

In the third line we have used the fact

$$\lim_{r\to\infty} \frac{h(r)}{h(r+r_0)} = 1$$

established in [8, Lemma 4.1 (i)]. If we can choose α and ϵ with $0 < \epsilon < \alpha < 1$ such that

$$\epsilon < \alpha < (1-\epsilon)\frac{a - 2\epsilon + a\epsilon - \epsilon^2}{(a+\epsilon)(1+\epsilon)^2}, \tag{3.5}$$

then

$$\bar{A}(r) + \bar{C}(r) > 0$$

on $[R-1, R+1]$. To achieve the requirement above, for any fixed $a \in (0,1)$ we pick an α with $0 < \alpha < 1$, then there exists an ϵ sufficiently small such that the above claimed estimate (3.5) holds. Hence (1) and (2) follow.

To prove (3), we can appeal [8, Lemma 4.3 (i)], for r_1 is large enough and $R > r_1$. First $B(r) = \bar{B}(r)$ for $r \le R-1$. Hence we only need to prove (3) for $r \ge R-1$. By formula (3.4) and Lemma 3.1 it can be seen that

$$\lim_{r\to\infty} C(r) \int_0^r h = 2 \lim_{r\to\infty} \left(1 - \frac{rh}{\int_0^r h}\right) = 2a,$$

which implies that $\beta(R) = \alpha C(R) h(R)$ satisfies

$$\beta(R) \int_0^R h \to 0 \quad \text{as } R \to \infty$$

since $h(r) \to 0$ as $r \to \infty$. On the other hand, for any $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that if $\beta c_0 \le \delta_1$ and R sufficiently large the conclusion in (4) of Lemma 3.1 holds with ϵ replaced with $\epsilon_1 > 0$. Here $c_0 > 0$ is the constant in the definition of the cut-off function ϕ. The computation in [8, proof of Lemma 4.3] shows that

$$(r\bar{f})^2 \bar{B}(r) = r\bar{h} - (1 - \bar{\xi}(r)) \int_0^r \bar{h} = \int_0^r (\bar{\xi}(r) - \bar{\xi}(t))\bar{h}(t)\, dt. \tag{3.6}$$

Using $\xi' > 0$ and $h' < 0$ the above gives, for $r \ge R-1$,

$$(r\bar{f})^2 \bar{B}(r) \ge \int_0^r (\xi(r) - \xi(t))h(t)\, dt - 2\delta_1(1+\epsilon_1)h(R-1) - c_0\beta(R)\int_0^R h(t).$$

Using the second part of (3.6) again, we have

$$\int_0^r (\xi(r) - \xi(t))h(t)\, dt = rh(r) - (1-\xi(r))\int_0^r h(t)\, dt \ge \underline{\xi} > 0$$

for some $\underline{\xi}$ by Lemma 3.1 (1). This shows that $\bar{B}(r) > 0$ if $R \ge r_1$ for some r_1 large. Since $\bar{h}' = -\frac{1}{r}\bar{h}\bar{\xi} < 0$ when $r > 0$, it follows that

$$\int_0^r \bar{h} > \bar{h} r \quad \text{when } r > 0,$$

which implies that (4) holds, by formula (3.3). This provides a simplification of [8, proof of Lemma 4.3 (ii)]. □

The computation of [8] also implies the following result.

Theorem 3.4. *A* $\mathsf{U}(m)$*-invariant Kähler metric on* $\mathbb{C}^m$ *has nonnegative orthogonal bisectional curvature and nonnegative Ricci curvature if and only if*

$$A + C \geq 0, \quad A + (m-1)B \geq 0, \quad B \geq 0, \quad C \geq 0.$$

In fact, algebraically one can construct a curvature of (NOB) and nonnegative Ricci curvature but does not have nonnegative holomorphic bisectional curvature. Recently in [20], using the method of [24], an example of unitary symmetry as the above with (NOB) and nonnegative Ricci curvature, but not nonnegative bisectional curvature has been constructed. This shows that (NOB) is completely independent of Ricci curvature.

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