

# Notes on Transnormal Functions on Riemannian Manifolds

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## 1 Introduction.

Let  $M$  be a connected complete Riemannian manifold,  $f$  be a smooth function defined on  $M$ . If  $f$  is not a constant function and there is a smooth function  $b : J := f(M) \rightarrow \mathbb{R}$  such that

$$(1.1) \quad \|\nabla f\|^2 = b(f),$$

then we call  $f$  a transnormal function. This equation was first studied by Elie Cartan in [C] where he first began the project of classifying the isoparametric hypersurfaces in space forms. Later on, this equation appeared in the series of papers [M1] [M2] [F] [FKM] [N]. By studying the whole family of hypersurfaces defined by the level sets of the corresponding transnormal functions, [M1] [M2] gave surprising restriction on the isoparametric hypersurfaces in sphere. On general Riemannian manifolds this equation was first studied in [W]. The major result which was proved there is;

(1) There is no critical value in  $\text{int}(J)$ . So the focal varieties, i.e. the singular level sets of  $f$ , are only the level sets corresponding to the maximum or the minimum point of  $J$  (we denote them by  $V_+$  or  $V_-$ ).

(2) **Theorem A** in [W1]. If  $M$  is a connected complete Riemannian manifold, and  $f$  is a transnormal function on  $M$ , then

a) The focal varieties of  $f$  are smooth submanifolds of  $M$ .

b) Each regular level set of  $f$  is a tube over either of the focal varieties.

This is a generalization of the geometry provided by the isoparametric family in [M1] and [M2]. In this paper, we will show that the existence of transnormal functions puts very strong restriction on the topology and geometry of the manifolds. In particular, we can prove that if a simply-connected compact three manifold supports a transnormal function then this manifold has to be a three-sphere. We can also show that the level hypersurfaces of transnormal function in  $S^n$  are all isoparametric hypersurfaces(i.e. All principle curvatures are constant on the hypersurface). The interesting point is the interaction between topology and geometry, i.e. that the geometry of transnormal functions restrict the topology of the manifold where they are defined and the topological structure, on the other hand, helps us to get more geometric information of the leaves of the foliation provided by the functions. On complete manifold, there are plenty of transnormal functions (See next section for examples). But we can give a complete classification of transnormal functions in  $R^n$  and  $H^n$ .

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## 2 Transnormal foliations.

**Examples.** Before we study the general theory let's start with some concrete examples of transnormal functions;

Example 1.  $M = R^n$ ,  $f = x_1^2 + x_2^2 + \dots + x_k^2$ .

Example 2.  $M = S^n$ ,  $f = x_1^2 + x_2^2 + \dots + x_k^2 - x_{k+1}^2 - \dots - x_{n+1}^2$ .

Example 3.  $M = S^n$ , all the polynomials constructed using Clifford algebra representations in [F-K-M].

Example 4.  $M$  is complete manifold with nonnegative Ricci curvature and a line, then the Buseman fuction is a transnormal function (See [SY]). In this case, since the Buseman function satisfies  $|\nabla B|^2 = 1$  we know that all the level sets are regular and we can have, at least topologically, splitting of the manifold by the structure theorem A of Wang in the introduction.

**Definition 2.1** (See [B]). *A partition  $F$  of a complete Riemannian manifold  $M$  is called **transnormal foliation**(or **transnormal system**) if every geodesic of  $M$  cuts the leaves (the connected elements) of  $F$  orthogonally at none or all of its points. And a transnormal foliation is called **regular** if all leaves has same codimension. Otherwise it's called **singular**.*

**Proposition 2.1.** *Let  $M$  be a complete manifold,  $f$  be a transnormal function over  $M$ . Then the level sets of  $f$  yields a transnormal (might be singular) foliation of  $M$ .*

*Proof.* The proof follows directly from the geometry described in [W], majorly the Theorem A there. In fact, in our case we have codimension one foliation i.e. generic leaves are codimension 1.

**Lemma 2.1.** *Let  $M$  be a compact manifold,  $F$  be a singular transnormal foliation of  $M$  with only one singular leave  $S$ . Then there exists a double cover  $\pi : \tilde{M} \rightarrow M$ .*

*Proof.* By the Lemma 1 in [B], we know that the exponential map  $\exp_S : N_S \rightarrow M$  preserves the leaves of the foliations (Over the  $N_S$ , the normal bundle of  $S$  in  $M$ , the foliation is given by the sections of constant length.) and it must have conjugate locus of  $S$ . Otherwise, by the Theorem 2 in [B] we know that  $M$  is diffeomorphic to a vector bundle over  $S$ , which is contradictory to the compactness of  $M$ . Let's denote the first conjugate locus of  $\exp_S$  is  $C(S)$ . While  $S$  is the only singular leave of  $F$  we know  $\exp_S(C(S)) = S$ . And we can assume that  $N_{2t}S = \{(s, y) \in N_S \mid \|y\| = 2t\}$  is the first conjugate locus. By the above, we know that the cut locus of  $S$  is  $N_tS$  and we denote  $H_t = \exp_S(N_tS)$ . Furthermore we have  $\exp_S(N_tS) \rightarrow H_t$  is a double cover. If we denote the deck transformation of this double cover by  $h$  we can construct  $\tilde{M}$  by gluing two copies of  $N_{\leq t}S$  along their boundary through  $h$ . From the construction it's quite clear that we have double cover  $\pi : \tilde{M} \rightarrow M$ .

**Theorem 2.1.** *If  $M$  is a simply-connected compact 3-manifold and with a Riemannian metric  $g$  and a smooth function  $f$  such that  $f$  is transnormal with respect to  $g$ . Then  $M$  is a three sphere.*

*Proof.* Let  $F$  be the transnormal foliation of  $M$  provided by  $f$ . By Lemma 2.1 and the simply-connectness of  $M$  we know that  $F$  has more than one singular leaves. By the Theorem 3 of [B] we know that  $F$  has exactly two singular leaves. Let  $S_1$  and  $S_2$  to be the two singular leaves and  $d_i (i = 1, 2)$  to be the codimension of them in  $M$ . For  $(d_1, d_2)$  we have the following three cases;

i)  $(d_1, d_2) = (3, 3)$ . In this case, we know  $S_1$  and  $S_2$  are all points then  $M = N_{\leq t}\{p\} \cup N_{\leq t}\{q\}$ , is a three sphere by Brown's theorem ([BW]).

ii)  $(d_1, d_2) = (2, 2)$ . In this case  $S_1 = S_2 = S^1$ , and  $M$  is so-called lens space  $L(p, q)$  (See, for example, [R] Page 234-235) and we know that  $L(1, q) = S^3$  and  $\pi_1(L(p, q)) = Z_p$ . By the simply-connectness we also know that  $M$  is a three sphere.

iii)  $(d_1, d_2) = (3, 2)$ . But this case can't happen in our situation by some Mayer-Vietoris argument and it was essentially proved in corollary 1 of [B]. For the sake of completeness we include a simple argument here. We know that  $S_1 = p$  and  $S_2 = S^1$  and  $M = N_{\leq t}\{p\} \cup N_{\leq t}S^1$ . In particular we have  $H_t = S^2 = \exp_{S^1}(N_tS^1)$ . But it's a contradiction since  $\pi_1(S^2) = 0$  and  $\pi_1(N_tS^1) = Z \oplus Z$ .

### 3 Goemetric constrains of the transnormal foliation.

In this section we will see that if  $M$  happen to be space form with nonnegative curvature the level set of a transnormal function has some interesting geometric properties. Before we prove our results we need to set up some preliminary results.

From (1.1) we know that  $\frac{\text{grad}f}{\sqrt{b(f)}} (b(f) \neq 0)$  is a self-parallel vector field (cf.[C-R]), and the integral curve is geodesic perpendicular to  $f^{-1}(\alpha)$ . We give the following definition according to this observation.

**Definition 3.1.** *A geodesic segment  $\sigma : (\alpha, \beta) \rightarrow M$  is called an **f-segment**, if  $f(\sigma(t))$  is increasing and  $\dot{\sigma}(t) = \frac{\text{grad}f}{\sqrt{b(f)}} (b(f) \neq 0)$ .  $\sigma(t)$  is called **inverse f-segment** if  $\sigma(-t)$  is a f-segment(cf. [W]).*

To give the description which Wang provided in [W], we need to define the following map  $\Phi(t, p)$ . Suppose  $\alpha \in \text{int}(J)$ ,  $M_\alpha := f^{-1}(\alpha)$  is a hypersurface. Let  $p \in M_\alpha$ , and  $\xi(p)$  be the unit normal vector pointing to the  $f$ -increasing direction, then  $\text{exp}_p(t\xi(p))$  is the arc-length geodesic starting from p. This is a  $f$ -segment. We can define a smooth map  $\Phi(t, p) := \text{exp}_p(t\xi(p))$ , and we set  $\phi_t(p) := \Phi(t, p)$ . We know from lemmas of [W] that  $\phi_t(M_\alpha)$  belongs to the level set of  $f$ . When  $d\phi_t$  is nondegenerate for  $0 \leq t \leq r$ , it's a hypersurface of  $f$ , and  $d(M_\alpha, \phi_r(M_\alpha)) = r$ . The  $f$ -segment  $\phi_t(p)$  is the minimizing geodesic which joins  $M_r$  to  $\phi_r(M_\alpha)$ .

**Remark 1** *From [W] we know that  $V_+ = \phi_r(M_\alpha)$ , if  $r$  is the first degenerate point of  $d\phi_t$ .*

**Definition 3.2.** *Let  $K$  be a submanifold of  $M$ ,  $p \in K$ ,  $\sigma(t)$  be a geodesic starting from  $p$ ,  $\dot{\sigma}(0) \perp T_p(K)$ , and  $Y(t)$  be the Jacobi field along  $\sigma(t)$ . We call  $Y(t)$  **K-Jacobi field** provided it satisfies:  $Y(0) \in T_p(K)$ , and  $S_{\dot{\sigma}(0)}Y(0) + \dot{Y}(0) \in T_p(K)^\perp$ , where  $S$  is the second fundamental form of  $K$ (cf. [B-C]).*

The relation between the K-Jacobi fields and the function  $f$  is given in the following proposition.

Let  $M_\alpha$  be a level set. We know from the theorem A of [W] that it's always a manifold, and we can assume that it is connected. Let  $q$  be a point of  $M$ , but  $q \notin M_\alpha$ , we can assume  $\beta = f(q) > \alpha$ ,  $\sigma(t)$  is a  $f$ -segment joining  $p := \sigma(0)$  to  $q = \sigma(l)$ . We know from [B-C] that for any two vector  $X, Y \in T_q(M_\beta)$ , there exists two  $M_\beta$ -Jacobi fields  $J_i, i = 1, 2$  such that  $J_1(l) = X, J_2(l) = Y$ . We have the following proposition.

**Proposition 3.1.**

$$(3.2) \quad D^2 f(X, Y) = \sqrt{b(f)} < J_1(l), J_2(l) > .$$

*Proof.* We divide the proof into two cases. Case 1).  $M_\beta$  is a hypersurface. In this case we can compute directly by using the concrete construction of  $J_i$ . Let  $\gamma_i(s)$  be two curves in  $M_\beta$  such that  $\gamma_i(0) = q$ ,  $\dot{\gamma}_1(0) = X$ ,  $\dot{\gamma}_2(0) = Y$ , then by using the exponential map, we can get two families of geodesics  $\gamma_i(t, s)$ . These two families give two geodesic variations of  $\sigma(t)$ . Therefore, the variational vector fields are the  $M_\beta$ -Jacobi fields (they are  $M_\alpha$ -Jacobi fields too). From the construction we know  $J_1(l) = X$ ,  $J_2(l) = Y$ , and we can do the following calculation:

$$\begin{aligned}
D^2 f(X, Y) &= D^2 f(Y, X) = XYf - D_X Yf \\
&= X \langle Y, \text{grad} f \rangle - \langle D_X Y, \text{grad} f \rangle \\
&= \langle Y, D_X \text{grad} f \rangle = \langle Y, D_X \sqrt{b(f)} \dot{\sigma} \rangle \\
&= \langle Y, \sqrt{b(f)} D_X \dot{\sigma} \rangle + \frac{1}{2} \frac{b'(f)}{\sqrt{b(f)}} \langle Y, \langle X, \text{grad} f \rangle \dot{\sigma} \rangle \\
&= \sqrt{b(f)} \langle J_2, D_{J_1} \dot{\sigma} \rangle = \sqrt{b(f)} \langle \dot{J}_1, J_2 \rangle .
\end{aligned}$$

Case 2).  $M_\beta$  is a focal submanifold. We can construct Jacobi fields as Case 1). From [W] we know that  $\sigma(t)$  belongs to the hypersurface, provided  $l - \epsilon < t < l$ . So Case 2) follows Case 1) by continuity.

**Remark 2.** *If  $\sigma(t)$  is the inverse  $f$ -segment, we can get the similar result:*

$$(3.3) \quad D^2 f(X, Y) = -\sqrt{b(f)} \langle \dot{J}_1, J_2 \rangle .$$

The above proposition relates the Hessian of  $f$  to the  $M_\alpha$ -Jacobi fields. However, we have known the following equation on hypersurface  $M_\alpha$  (See proof in [C-R]).

$$(3.4) \quad \langle S_\xi X, Y \rangle = -\frac{D^2 f(X, Y)}{\sqrt{b(f)}} ,$$

where  $S$  is the second fundamental form of the level hypersurface,  $\xi = \frac{\text{grad} f}{\sqrt{b(f)}}$ .

From the equation (3.2) and the Proposition 3.1 we can calculate the principal curvatures of the level set of  $f$ . In the case that  $M_\alpha$  is a hypersurface, let  $\sigma(t)$  be an  $f$ -segment joining  $M_\alpha$  to another level hypersurface  $M_\beta$ . Then the  $M_\alpha$ -Jacobi fields are the vector fields  $J(t)$  satisfying

$$(3.5) \quad \begin{cases} \ddot{J}(t) + R_t J = 0 \\ J(0) \in T_p(M_\alpha), S_{\dot{\sigma}} J(0) = -J(0), \end{cases}$$

where  $R_t J = R(\dot{\sigma}(t), J(t)) \dot{\sigma}(t)$ . If we can solve (3.5) we can get some information about the principal curvatures of parallel hypersurfaces.

Combining the global geometrical structure of transnormal function and the calculation given above, we can prove the following result.

**Theorem 3.2** *Let  $M = M(c)$  be a Riemannian manifold with nonnegative constant sectional curvature  $c$  and  $f$  be a transnormal function on  $M$ . Then all the regular leaves of the related transnormal foliation are isoparametric hypersurfaces, i.e. all the principal curvatures are constant on the hypersurface.*

*Proof.* When  $c > 0$ , we can assume  $c=1$ , and we give the proof only for the case  $c=1$ . For  $c=0$  one can do similarly. We divide the proof into two cases.

Case 1).  $V_+$  and  $V_-$  have codimension greater than 1. In this case we can apply the Theorem A in [W] to prove that the regular level set of  $f$ , say  $M_\alpha$ , is an isoparametric hypersurface. The map  $\phi_r(p)$  defined as before is our starting point. We know from [B-C] that  $d\phi_t(X)$  is the Jacobi fields  $J(t)$  along the  $f$ -segment  $\phi_t(p)$ , which have the initial value  $X$ . Whether  $d\phi_r$  is degenerate is determined by whether there is a Jacobi-field  $J(t)$  with nonzero initial value, but degenerate at  $\phi_r(p)$ .

We denote  $l_1 := d(M_\alpha, V_+)$ ,  $l_2 := d(V_+, V_-)$ . From the fact we describe in the introduction and the assumption that  $V_+$  and  $V_-$  have codimension greater than 1, we know that when  $r \neq l_1 + kl_2, k = 0, 1, \dots$ ,  $\phi_r$  is a diffeomorphism from  $M_\alpha$  to another hypersurface, and  $d\phi_r$  degenerates if and only if  $r = l_1 + kl_2$ . However, we can solve the equation (1.3) explicitly. Let the principal curvatures at  $p$  be  $\lambda_{1,1} = \lambda_{1,2} = \dots = \lambda_{1,m_1} > \lambda_{2,1} = \lambda_{2,2} = \dots > \lambda_{g,1} = \lambda_{g,2} = \dots = \lambda_{g,m_g}$ , and the principal vectors be  $X_{1,1}, X_{1,2}, \dots, X_{1,m_1}, X_{2,1}, X_{2,2}, \dots, X_{g,1}, X_{g,2}, \dots, X_{g,m_g}$ , where  $g$  is the number of distinct principal curvatures. Solving the Jacobi equations

$$\begin{cases} \ddot{J}_{i,j}(t) + J_{i,j}(t) = 0, \\ J_{i,j} = X_{i,j}, \\ \dot{J}_{i,j}(0) = -\lambda_{i,j} X_{i,j}, \end{cases}$$

we get the solutions

$$(3.6) \quad \{J_{i,j}(t) = (\cos t - \lambda_{i,1} \sin t) \widetilde{X}_{i,j}(t),$$

where  $\widetilde{X}_{i,j}(t)$  is the parallel transport of  $X_{i,j}$ . So we conclude  $J_{i,j} = 0$  if and only if  $t = \arccos(\lambda_{i,1})$ . While  $\phi_r$  is diffeomorphism if  $0 < r < l_1$ , hence we have:  $\lambda_{1,j} = \text{ctg} l_1, j = 1, \dots, m_1$ .

Similarly, because  $\phi_r$  is a diffeomorphism if  $l_1 < r < l_1 + l_2$  and degenerates at  $r = l_1 + l_2$ , we get  $\lambda_{2,j} = \text{ctg}(l_1 + l_2)$ . Inductively, we can conclude  $\lambda_{i,j} = \text{ctg}(l_1 + (i-1)l_2)$ . So we complete the proof in Case 1).

Case 2). Since  $\phi_r$  must degenerate for some  $r$  (cf [B-C]),  $V_+$  and  $V_-$  can not be hypersurfaces together. We might as well assume that  $V_+$  has codimension greater than 1. In this case,  $\phi_t(p)$  reaches  $V_+$ , at  $t=l_1$ , then  $f(\phi_t(p))$  begins to decrease until  $\phi_t$  reaches  $V_-$ , but  $\phi_r : M_\alpha \rightarrow V_-$  is a diffeomorphism, so  $d\phi_{l_1+l_2}$  does not degenerate. When  $t=l_1 + 2l_2$ ,  $\phi_t(p)$  reaches  $V_+$ , and  $d\phi_t$  degenerates. By the same way as Case 1) we can get  $\lambda_{i,j} = \text{ctg}(l_1 + 2(i-1)l_2)$ . In both cases we get  $M_\alpha$  is isoparametric.

**Remark 3** *From the above proof we can conclude that  $gl_2 = \pi$  for Case 1) and  $2gl_2 = \pi$  for Case 2). This is just the geometry which Münzner described in [M1]. And this result might be helpful to the classification of the isoparametric hypersurfaces in spheres.*

**Remark 4** *In [W] there is a claim saying that the transnormal functions on  $S^n$  are all isoparametric functions. In fact the claim is not correct and our theorem is the right version of the claim.*

**Theorem 3.3** *If  $M = M(0)$ , then any transnormal function on  $M$  is the function of distance to a totally geodesic submanifold.*

To prove theorem 2 we will use the following lemma which is a corollary of Cartan's fundamental equation(See [F], Proposition 4).

**Lemma 2** (See [F], Theorem 5). *If  $N$  is an isoparametric hypersurface in  $M = M(c)$  of constant curvature  $c \leq 0$ , then the number of distinct principal curvatures is  $g \leq 2$ . For  $g = 1$ ,  $N$  is totally umbilic. For  $g = 2$ , the two distinct curvatures satisfy*

$$(3.7) \quad \lambda_1 \lambda_2 = c.$$

*Proof* of theorem 3.3. From Theorem 3.2 we know that if  $f$  is a transnormal function on  $M$ , the level hypersurface  $M_\alpha := f^{-1}(\alpha)$  is isoparametric. However, from Lemma 2. we know  $g$ , the number of distinct principal curvature,  $\leq 2$ , and at least one of the two principal curvatures is zero. If  $M_\alpha$  is totally geodesic then the principal curvatures are all zero. From the proof of theorem 3.2, we know there is no level set of  $f$  with codimension greater than one. While  $t = \int_\alpha^f \frac{1}{\sqrt{b(x)}} dx$ , we can conclude  $f = f(t)$ , where  $t$  is the distance from  $M_\alpha$ . In this case  $M$  has topological type  $M_\alpha \times R$  or  $M_\alpha \times S^1$ . If  $M_\alpha$  is not totally geodesic, from the proof of theorem we know the focal manifold always exists. If  $\lambda_i$  are the principal curvatures of the focal manifold and  $M_\alpha$  is the  $l$ -tube of the focal manifold  $V$ , we can assume this focal manifold  $V$  is  $V_-$ , and  $\dim V_- = m$ . Then through the modified calculation of Lemma 2, we can get the principal curvatures of  $M_\alpha$ :  $\frac{\lambda_i}{\lambda_i l + 1}$ , and  $\frac{1}{l}$  with

multiplicity  $n - m - 1$ . From Corollary 1, we know  $\lambda_i = 0$ , i.e. the focal manifold is totally geodesic. In the case we also have the theorem.

**Definition 3.3.** (See [W] also) *If  $f$  is a transnormal function on  $M$  and the second Beltrami differential is also a continuous function of itself, i.e.  $\Delta f = a(f)$  for some continuous function  $a$ , we call  $f$  an **isoparametric function**. We call the corresponding foliation an **isoparametric foliation**.*

The further application of the sphere-bundle structure will give further properties on the singular leaves of the isoparametric foliation.

**Theorem 3.4.** *Let  $M$  be a connected Riemannian manifold,  $F$  be an isoparametric foliation on  $M$  given by an isoparametric function  $f$ . Then the singular leaves of  $F$  are minimal submanifolds in  $M$ .*

*Proof.* Before our proof we give the following notation:  $1 \leq i, j, \dots, \leq m, 1 \leq A, B, \dots, \leq n, m + 1 \leq \alpha, \beta, \dots, \leq n$ , where  $\dim M = n + 1$ ,  $m$  is the dimension of a fixed focal manifold.

We assume  $M_\alpha$  is a focal manifold, and  $\alpha$  is the maximum of  $f$ . Then we consider the  $M_\alpha$ -Jacobi fields. Let  $p \in M_\alpha$ ,  $\xi(p)$  be a normal vector at  $p$ , and  $e_1, e_2, \dots, e_{n+1}$  be a local orthonormal frame field such that  $e_1, \dots, e_m \in T_p(M_\alpha)$ , and  $e_{n+1} = \xi(p)$ ,  $S_{\xi(p)}e_i = -\lambda_i e_i$ ,  $\sigma(t)$  be an inverse  $f$ -segment starting from  $p$  with  $\dot{\sigma}(0) = e_{n+1}$ ,  $\{e_A(t)\}$  be the parallel transport of  $\{e_A\}$ . Since  $D^2 f(e_{n+1}, e_{n+1}) = \frac{1}{2}b'(f)$ , at  $\sigma(t)$  we have

$$(3.8) \quad [\Delta f - \frac{1}{2}b'(f)] = \sum_{A=1}^n D^2 f(e_A(t), e_A(t)).$$

Let  $J_A(t)$  be the Jacobi fields which satisfy

$$\begin{aligned} \ddot{J}_A(t) + R_t J_A(t) &= 0, \\ J_A(0) &= \begin{cases} e_A, & 1 \leq A \leq m, \\ 0, & m + 1 \leq A \leq n, \end{cases} \\ \dot{J}_A(0) &= \begin{cases} \lambda_A e_A, & 1 \leq A \leq m, \\ e_A, & m + 1 \leq A \leq n, \end{cases} \end{aligned}$$

then from (3.8) and the Remark 2, we have

$$[\Delta f - \frac{1}{2}b'(f)] = - \sum_{1 \leq A, B, C \leq n} g_{AB} g_{AC} \langle \dot{J}_B, \dot{J}_C \rangle \sqrt{b(f)},$$



where  $g_{AB}(t)$  is given by the equations  $e_A(t) = \sum g_{AB}(t)J_B(t)$ . If we set  $H = (h_{AB}) := (g_{AB})^{-1}$ , then we have

$$(3.9) \quad [\Delta f - \frac{1}{2}b'(f)] = -\sqrt{b(f)} \text{ trace}(H^{-1}H_t).$$

In the following, we compute  $\text{trace}(H^{-1}H_t)$ . Since  $H(t)$  satisfies

$$\ddot{H}(t) - R(t)H(t) = 0,$$

$$H(0) = \begin{pmatrix} I_{m \times m} & 0 \\ 0 & 0 \end{pmatrix},$$

$$H_t(0) = \begin{pmatrix} \lambda_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \lambda_m & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix},$$

where  $R(t) = (R_{AB}(t))_{n \times n}$ ,  $R_{AB}(t) = -\langle R(\dot{\sigma}(t), e_A(t)) \dot{\sigma}(t), e_B(t) \rangle$ . Now we have the expansion

$$H^{-1}(t) = \begin{pmatrix} \frac{1}{1+t\lambda_1} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \frac{1}{1+t\lambda_m} & 0 & \dots & 0 \\ 0 & \dots & 0 & \frac{1}{t} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \frac{1}{t} \end{pmatrix} [I - \frac{t^2}{2} \begin{pmatrix} \frac{R_{11}(0)}{1+t\lambda_1} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \frac{R_{mm}(0)}{1+t\lambda_m} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} + o(t^2)];$$

$$H_t(t) = \begin{pmatrix} \lambda_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \lambda_m & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix} + t \begin{pmatrix} R_{11}(0) & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & R_{mm}(0) & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} + o(t).$$

Hence we obtain

$$(3.10) \quad H^{-1}H_t = \begin{pmatrix} \frac{1}{1+t\lambda_1} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \frac{1}{1+t\lambda_m} & 0 & \dots & 0 \\ 0 & \dots & 0 & \frac{1}{t} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \frac{1}{t} \end{pmatrix} o(t),$$

$$(3.11) \quad \text{trace}(H^{-1}H_t) = \sum_{i=1}^m \frac{\lambda_i}{1+t\lambda_i} + \frac{n-m}{t} + o(t).$$

Combining (3.10) (3.11) we have

$$-[a(f) - \frac{1}{2}b'(f)] = \sqrt{b(f)} \left( \sum_{i=1}^m \frac{\lambda_i}{1+t\lambda_i} + \frac{n-m}{t} \right) + \sqrt{b(f)}o(t).$$

Finally we obtain

$$\sum_{i=1}^m \frac{\lambda_i}{1+t\lambda_i} = \frac{1}{\sqrt{b(f)}} \left[ \frac{1}{2}b'(f) - a(f) - \sqrt{b(f)} \frac{n-m}{t} \right] + o(t).$$

Taking  $t \rightarrow 0$ , we have

$$\begin{aligned} \sum_{i=1}^m \lambda_i &= \lim_{t \rightarrow 0} \frac{1}{\sqrt{b(f)}} \left[ \frac{1}{2}b'(f) - a(f) - \sqrt{b(f)} \frac{n-m}{t} \right] \\ &= \lim_{f \rightarrow \alpha} \frac{1}{\sqrt{b(f)}} \left[ (m-n) \frac{\sqrt{b(f)}}{\int_f^\alpha \frac{1}{\sqrt{b(x)}} dx} - a(f) + \frac{1}{2}b'(f) \right]. \end{aligned}$$

But the right side of the above equation is independent of the choice of  $\xi(p)$ , so  $\sum_{i=1}^m \lambda_i = 0$ , i.e. the mean curvature is zero. Then the focal manifold is a minimal submanifold.

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